Random Matrix Theory (RMT) and Many Body Localization (MBL)

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Outline

1. 随机矩阵
2. 高斯系综
3. 能级间距分布，能级间距比分布
4. 多体局域化

随机矩阵

- Random Matrices
  Define：A random matrix is a matrix-valued random variable—that is, a matrix in which some or all elements are random variables.（Wiki）
  例如：(a, b... 为随机数)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} 2 & 3.4 & a & -2 \\ \sqrt{5} & b & 1 & c \\ -1 & 0 & 2+5i & \pi \\ d & 3 & -2.6 & e \end{bmatrix}$$

- Gaussian Ensembles 高斯系综
  矩阵 A 的元素独立同分布，每个元素满足 Gaussian 分布 $N(0,1)$

$$f(A_{ij}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{A_{ij}^2}{2}}$$

Gaussian Ensembles 高斯系综

- Gaussian Orthogonal Ensemble (GOE,$\beta = 1$): real symmetric matrices $H_{N\times N} = (A + A^T)/2$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

- Gaussian Unitary Ensemble (GUE,$\beta = 2$): complex Hermitian matrices $H_{N\times N} = (A + A^\dagger)/2$

$$\begin{bmatrix} a & b_1 + ib_2 \\ b_1 - ib_2 & c \end{bmatrix}$$

- Gaussian Symplectic(辛) Ensemble (GSE,$\beta = 4$): quaternionic(四元数) Hermitian matrices $H_{2N\times 2N} = (A + A^\dagger)/2$

$$\begin{bmatrix} a & 0 & c+id & e+if \\ 0 & a & -e+if & c-id \\ c-id & -e-if & b & 0 \\ e-if & c+id & 0 & b \end{bmatrix}$$

$(2N \times 2N)$ matrices cut into N^2 blocks of (2×2), for example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}(a+d)I - \frac{i}{2}(a-d)e_1 - \frac{1}{2}(b-c)e_2 + \frac{i}{2}(b+c)e_3$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

Gaussian Orthogonal Ensemble

- GOE
- 变换不变:

$$H' = U^T H U$$

U 是实正交矩阵

$$U^T U = U U^T = 1$$

概率分布不变

$$P(H')dH' = P(H)dH$$

- H_{kj} 统计独立, 概率密度函数

$$P(H) = \prod_{k\leq j} f_{kj}(H_{kj})$$

独立矩阵元素 $N(N+1)/2$

Mehta, M. L. (1970). Random Matrices and the Statistical Theory of Energy Levels. Journal of the American Statistical Association, 65(332).

GUE，GSE

- GUE
  - 变换不变:

$$H' = U^{-1}HU$$

U 是幺正矩阵

$$U^{-1}U = UU^{-1} = 1$$

  - $H_{kj}^{(0)}$ 和 $H_{kj}^{(1)}$ 分别是 H_{kj} 的实部和虚部且统计独立，概率密度函数:

$$P(H) = \prod_{k\le j} f_{kj}^{(0)}(H_{kj}^{(0)}) \prod_{k<j} f_{kj}^{(1)}(H_{kj}^{(1)})$$

独立矩阵元素 N^2

- GSE
  - 变换不变:

$$H' = U^R HU$$

U 是辛矩阵,$U^T\Omega U = \Omega$

$$\Omega = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$$

  - 概率密度函数:

$$P(H) = \prod_{k\le j} f_{kj}^{(0)}(H_{kj}^{(0)}) \prod_{\lambda=1}^{3} \prod_{k<j} f_{kj}^{(\lambda)}(H_{kj}^{(\lambda)})$$

独立矩阵元素 $N(2N-1)$

Joint Probability Density Function (J.P.D.F) for Eigenvalues

- GOE
  H 实对称矩阵，$N(N+1)/2$ 个独立元素 H_{kj}
  N 个本征值 θ_k
  $N(N-1)/2$ 个本征态正交约束条件 $\{p_1, p_2, ..., p_{N(N-1)/2}\}$
- 定理（Wishart,1928）：$P(H) = \exp\left(-a\mathrm{tr}H^2 + b\mathrm{tr}H + c\right)$，$a$ 是正实数，b, c 是实数

$$\mathrm{tr}(H^2) = \sum_k \theta_k^2, \qquad \mathrm{tr}(H) = \sum_k \theta_k$$

$$P(H)dH = exp(-a\sum_k \theta_k^2 + b\sum_k \theta_k + c)dH \equiv P(\varepsilon)d\varepsilon$$

$$dH = dH_{11}dH_{12}...dH_{NN}, \quad d\varepsilon = d\theta_1...d\theta_N dp_1...dp_{N(N-1)/2}$$

- Jacobian 变换：$dH = Jd\varepsilon$

$$P(\varepsilon)d\varepsilon = J(\theta, p)\exp\left(-a\sum_k \theta_k^2 + b\sum_k \theta_k + c\right)d\varepsilon$$

$$J(\theta, p) = \left|\frac{\partial(H_{11}, H_{12}, ..., H_{NN})}{\partial(\theta_1...\theta_N p_1...p_{N(N-1)/2})}\right|$$

J.P.D.F Gaussian Orthogonal Ensemble

- Jacobian 矩阵（$N(N+1)/2 \times N(N+1)/2$）:

$$[J(\theta, p)] = \begin{bmatrix} \frac{\partial H_{jj}}{\partial \theta_\gamma} & \frac{\partial H_{jk}}{\partial \theta_\gamma} \\ \frac{\partial H_{jj}}{\partial p_\mu} & \frac{\partial H_{jk}}{\partial p_\mu} \end{bmatrix}$$

两列对应 N,$N(N-1)/2$ 实际的列,$1 \leq j < k \leq N$
两行对应 N,$N(N-1)/2$ 实际的行,$\gamma = 1, 2, ..., N$,$\mu = 1, 2, ..., N(N-1)/2$

- 对角化:

$$H = U\Theta U^T, \ \ UU^T = U^T U = 1$$

Θ diagonal matrix:$\theta_1 \leq \theta_2 \leq ... \leq \theta_N$
U: $N(N-1)/2$ parameters p_μ

- $U^T U = 1$ 偏微分:

$$\frac{\partial U^T}{\partial p_\mu} U + U^T \frac{\partial U}{\partial p_\mu} = 0$$

$$S^{(\mu)} = U^T \frac{\partial U}{\partial p_\mu} = -\frac{\partial U^T}{\partial p_\mu} U$$

J.P.D.F Gaussian Orthogonal Ensemble

- $H = U\Theta U^T$ 对 p_μ 偏微分:

$$\frac{\partial H}{\partial p_\mu} = \frac{\partial U}{\partial p_\mu}\Theta U^T + U\Theta\frac{\partial U^T}{\partial p_\mu}$$

左乘 U^T, 右乘 U:

$$U^T\frac{\partial H}{\partial p_\mu}U = S^{(\mu)}\Theta - \Theta S^{(\mu)}$$

$$\sum_{j,k}\frac{\partial H_{jk}}{\partial p_\mu}U_{j\alpha}U_{k\beta} = S^{(\mu)}_{\alpha\beta}(\theta_\beta - \theta_\alpha)$$

- $H = U\Theta U^T$ 对 θ_γ 偏微分：

$$\frac{\partial H}{\partial \theta_\gamma} = U\frac{\partial \Theta}{\partial \theta_\gamma}U^T$$

$$U^T\frac{\partial H}{\partial \theta_\gamma}U = \frac{\partial \Theta}{\partial \theta_\gamma}$$

$$\sum_{j,k}\frac{\partial H_{jk}}{\theta_\gamma}U_{j\alpha}U_{k\beta} = \frac{\partial \Theta_{\alpha\beta}}{\partial \theta_\gamma} = \delta_{\alpha\beta}\delta_{\alpha\gamma}$$

- 构造 $[V]$:

$$[V] = \begin{bmatrix}(U_{j\alpha}U_{j\beta})\\(U_{j\alpha}U_{k\beta})\end{bmatrix}$$

其中 $1 \le j < k \le N, \quad 1 \le \alpha \le \beta \le N$

J.P.D.F Gaussian Orthogonal Ensemble

- 相乘 $[J][V]$:

$$[J][V] = \begin{bmatrix} \delta_{\alpha\beta}\delta_{\alpha\gamma} \\ S_{\alpha\beta}^{(\mu)}(\theta_\beta - \theta_\alpha) \end{bmatrix}$$

- 取行列式：

$$J(\theta, p) \det V = \prod_{\alpha<\beta} |\theta_\beta - \theta_\alpha| \det \begin{bmatrix} \delta_{\alpha\beta}\delta_{\alpha\gamma} \\ S_{\alpha\beta}^{(\mu)} \end{bmatrix}$$

$$J(\theta, p) = \prod_{\alpha<\beta} |\theta_\beta - \theta_\alpha| f(p)$$

带入 $P(\varepsilon)d\varepsilon = J(\theta, p) \exp\left(-a\sum_k \theta_k^2 + b\sum_k \theta_k + c\right)d\varepsilon$, 并对 p 积分得到

- 本征值联合概率密度分布：

$$P(\theta_1, ..., \theta_N) = \exp\left[\sum_k(-a\theta_k^2 + b\theta_k + c)\right] \prod_{j<k} |\theta_k - \theta_j|$$

$\theta_k \to (1/\sqrt{2a})x_k + b/2a$

$$P_{N1}(x_1, ..., x_N) = C_{N1} \prod_{j<k} |x_k - x_j| e^{-\frac{1}{2}\sum_{j=1}^N x_j^2}$$

J.P.D.F Gaussian Unitary Ensemble

- GUE

$$J(\theta, p) = \frac{\partial(H_{11}^{(0)}, ..., H_{NN}^{(0)}, H_{12}^{(0)}, H_{12}^{(1)}, ..., H_{N-1,N}^{(0)}, H_{N-1,N}^{(1)})}{\partial(\theta_1, ..., \theta_N, p_1, ..., p_{N(N-1)})}$$

$$\sum_{j,k} \frac{\partial H_{jk}}{\partial p_\mu} U_{j\alpha}^* U_{k\beta} = S_{\alpha\beta}^{(\mu)}(\theta_\beta - \theta_\alpha)$$

$$\sum_{j,k} \frac{\partial H_{jk}}{\partial \theta_\gamma} U_{j\alpha}^* U_{k\beta} = \frac{\Theta_{\alpha\beta}}{\partial \theta_\gamma} = \delta_{\alpha\beta}\delta_{\alpha\gamma}$$

$$\begin{bmatrix} \frac{\partial H_{jj}^{(0)}}{\partial \theta_\gamma} & \frac{\partial H_{jk}^{(0)}}{\partial \theta_\gamma} & \frac{\partial H_{jk}^{(1)}}{\partial \theta_\gamma} \\ \frac{\partial H_{jj}^{(0)}}{\partial p_\mu} & \frac{\partial H_{jk}^{(0)}}{\partial p_\mu} & \frac{\partial H_{jk}^{(1)}}{\partial p_\mu} \end{bmatrix} \cdot \begin{bmatrix} v & w \\ A^{(0)} & B^{(0)} \\ A^{(1)} & B^{(1)} \end{bmatrix} = \begin{bmatrix} \rho_{\gamma,\alpha\alpha} & \sigma_{\gamma,\alpha\beta}^{(0)} & \sigma_{\gamma,\alpha\beta}^{(1)} \\ \epsilon_{\alpha\alpha}^{\mu} & S_{\alpha\beta}^{(0\mu)}(\theta_\beta - \theta_\alpha) & S_{\alpha\beta}^{(1\mu)}(\theta_\beta - \theta_\alpha) \end{bmatrix}$$

$$1 \le j < k \le N, \qquad 1 \le \alpha < \beta \le N$$

$$1 \le \mu \le N(N-1), \qquad 1 \le \gamma \le N$$

J.P.D.F Gaussian Unitary Ensemble

- Jacobian

$$J(\theta, p) = \prod_{\alpha<\beta} (\theta_\beta - \theta_\alpha)^2 f(p)$$

- 本征值联合概率密度分布：

$$P(\theta_1, ..., \theta_N) = \exp\left[\sum_k (-a\theta_k^2 + b\theta_k + c)\right] \prod_{j<k} |\theta_k - \theta_j|^2$$

$\theta_k \to (1/\sqrt{a})x_k + b/2a$

$$P_{N2}(x_1, ..., x_N) = C_{N2} \prod_{j<k} |x_k - x_j|^2 \exp\left(-\sum_{j=1}^{N} x_j^2\right)$$

J.P.D.F Gaussian Symplectic Ensemble

- GSE

$$H = U \Theta U^R$$

Θ consists of N blocks of the form:

$$\begin{bmatrix} \theta_j & 0 \\ 0 & \theta_j \end{bmatrix}$$

$$\mathrm{tr} H^2 = 2 \sum_1^N \theta_k^2, \qquad \mathrm{tr} H = 2 \sum_1^N \theta_k$$

$$P(\theta, p) = \exp\left[-\sum_1^N (2a\theta_k^2 - 2b\theta_k - c) \right] J(\theta, p)$$

$$J(\theta, p) = \frac{\partial(H_{11}^{(0)}, \ldots, H_{NN}^{(0)}, H_{12}^{(0)}, \ldots, H_{12}^{(3)}, \ldots, H_{N-1,N}^{(0)}, \ldots, H_{N-1,N}^{(3)})}{\partial(\theta_1, \ldots, \theta_N, p_1, \ldots, p_{2N(N-1)})}$$

$$H_{jk} = H_{jk}^{(0)} + H_{jk}^{(1)} e_1 H_{jk}^{(2)} e_2 + H_{jk}^{(3)} e_3$$

$$S_{\alpha\beta}^{(\mu)} = S_{\alpha\beta}^{(0\mu)} + S_{\alpha\beta}^{(1\mu)} e_1 + S_{\alpha\beta}^{(2\mu)} e_2 + S_{\alpha\beta}^{(3\mu)} e_3$$

J.P.D.F Gaussian Symplectic Ensemble

$$
\begin{bmatrix}
\frac{\partial H_{jj}^{(0)}}{\partial \theta_\gamma} & \frac{\partial H_{jk}^{(0)}}{\partial \theta_\gamma} & \frac{\partial H_{jk}^{(1)}}{\partial \theta_\gamma} \cdots & \frac{\partial H_{jk}^{(3)}}{\partial \theta_\gamma} \\
\frac{\partial H_{jj}^{(0)}}{\partial p_\mu} & \frac{\partial H_{jk}^{(0)}}{\partial p_\mu} & \frac{\partial H_{jk}^{(1)}}{\partial p_\mu} \cdots & \frac{\partial H_{jk}^{(3)}}{\partial p_\mu}
\end{bmatrix}
\cdot
\begin{bmatrix}
v & w \\
A^{(0)} & B^{(0)} \\
\cdots & \cdots \\
\cdots & \cdots \\
A^{(3)} & B^{(3)}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\rho_{\gamma,\alpha\alpha} & \sigma_{\gamma,\alpha\beta}^{(0)} & \cdots & \sigma_{\gamma,\alpha\beta}^{(3)} \\
\epsilon_{\alpha\alpha}^{\mu} & S_{\alpha\beta}^{(0\mu)}(\theta_\beta - \theta_\alpha) & \cdots & S_{\alpha\beta}^{(3\mu)}(\theta_\beta - \theta_\alpha)
\end{bmatrix}
$$

$$
1 \le j < k \le N, \qquad 1 \le \alpha < \beta \le N
$$

$$
1 \le \mu \le 2N(N-1), \qquad 1 \le \gamma \le N
$$

- Jacobian

$$
J(\theta, p) = \prod_{\alpha<\beta} (\theta_\beta - \theta_\alpha)^4 f(p)
$$

- 本征值联合概率密度分布：

$$
P_{N4}(x_1, ..., x_N) = C_{N4} \prod_{j<k} |x_k - x_j|^4 \exp\left(-2 \sum_{j=1}^{N} x_j^2\right)
$$

J.P.D.F Gaussian Ensemble

- J.P.D.F Gaussian Ensemble

$$P_{N\beta}(x_1, ..., x_N) = C_{N\beta} \prod_{j<k} |x_k - x_j|^\beta \exp\left(-\frac{1}{2}\beta \sum_{j=1}^{N} x_j^2\right)$$

$\beta = 1$: GOE
$\beta = 2$: GUE
$\beta = 4$: GSE

- Get $C_{N\beta}$:

$$\int_{-\infty}^{\infty} \cdots \int P_{N\beta}(x_1, ..., x_N) dx_1 \cdots dx_N = 1$$

- $N \to \infty$ Semicircle Rule 半圆率 (Wigner):

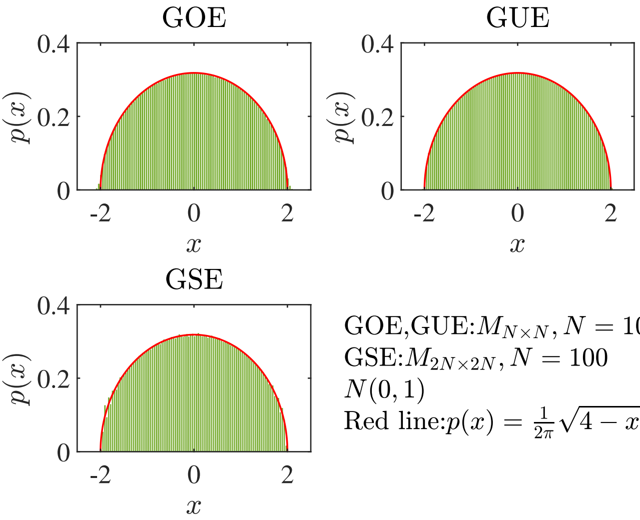
$$\rho(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

ref: Random Matrices and the Statistical Theory of Energy Levels,M. L. Mehta

MATLAB, Mathematica

- MATLAB：
GOE: A = randn(N); X = A + A';
GUE: A = randn(N) + 1i*randn(N); X = A + A';
GSE: e0 = eye(2); e1 = [1i 0; 0 -1i]; e2 = [0 1; -1 0]; e3 = [0 1i; 1i 0];
A = kron(randn(N),e0) + kron(randn(N), e1) + kron(randn(N),e2) + kron(randn(N),e3); X = A + A';
- Mathematica:
GOE: RandomVariate[GaussianOrthogonalMatrixDistribution[N]];
GUE: RandomVariate[GaussianUnitaryMatrixDistribution[N]];
GSE: RandomVariate[GaussianSymplecticMatrixDistribution[N]];
- Matlab 其他一些随机数：
(1) 均值为 μ，方差为 σ 的正态分布 $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
A = normrnd(μ,σ,N,N);
(2) a-b 之间的均匀分布
A = unifrnd(a,b,N,N);
(3) 二项分布 (Bernoulli), $P\{X = k\} = C_n^k p^k (1-p)^{n-k}, \quad (k = 0, 1, \cdots, n)$
A = binornd(n,p,N,N);
(4) λ 的泊松分布 $P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$
A = poissrnd(λ,N,N);

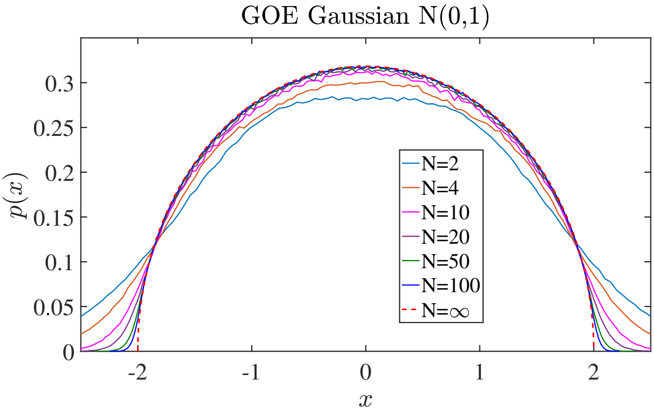
Semicircle Rule



GOE,GUE:$M_{N\times N}, N = 100$
GSE:$M_{2N\times 2N}, N = 100$
$N(0,1)$
Red line:$p(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$

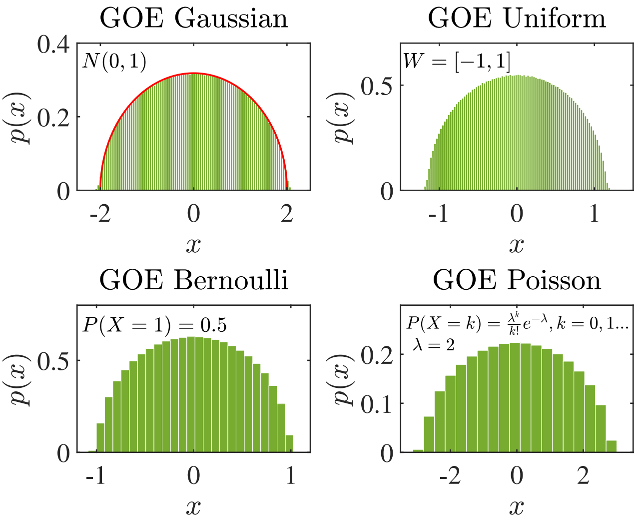
- GOE,GUE,GSE：Semicircle rule are same

Semicircle Rule



- $N \to \infty \Rightarrow$ Semicircle Rule: $p(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$

Semicircle Rule



- 不同分布的 Semicircle Rule 形状相同

Wigner 假设-能级间距分布

- 考虑 2×2 矩阵，有两个能级 λ_1, λ_2，得到：

$$P(\lambda_1, \lambda_2) = \frac{1}{Z_\beta(\sigma)} |\lambda_1 - \lambda_2|^\beta e^{-\frac{\lambda_1^2 + \lambda_2^2}{2\sigma^2}}$$

- 归一化：

$$\int\int P(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 = 1$$

$$Z_\beta(\sigma) = \int\int |\lambda_1 - \lambda_2|^\beta e^{-\frac{\lambda_1^2 + \lambda_2^2}{2\sigma^2}}$$

$(x_1, x_2) = (\lambda_1 + \lambda_2, \lambda_1 - \lambda_2)$

$$\begin{aligned} Z_\beta(\sigma) &= \int\int |x_2|^\beta e^{-\frac{x_1^2 + x_2^2}{4\sigma^2}} dx_1 dx_2 \\ &= 2\sigma\sqrt{\pi} \int_0^\infty x_2^\beta e^{-\frac{x_2^2}{4\sigma^2}} dx_2 \\ &= 2\sigma\sqrt{\pi} \frac{(2\sigma)^{\beta+1}}{2} \int_0^\infty t^{\beta/2 - 1/2} e^{-t} dt \\ &= \frac{\sqrt{\pi}}{2} (2\sigma)^{\beta+2} \Gamma(\beta/2 + 1/2) \end{aligned}$$

Wigner 假设-能级间距分布

- 能级间距 $S = |\lambda_1 - \lambda_2|$
  (1). $\lambda_1 > \lambda_2 \Rightarrow S = \lambda_1 - \lambda_2$

$$P(\lambda_2 + S, \lambda_2) = \frac{1}{Z_\beta(\sigma)} S^\beta e^{-\frac{\lambda_2^2 + (\lambda_2 + S)^2}{2\sigma^2}}$$

  (2). $\lambda_1 < \lambda_2 \Rightarrow S = \lambda_2 - \lambda_1$

$$P(\lambda_1, \lambda_1 + S) = \frac{1}{Z_\beta(\sigma)} S^\beta e^{-\frac{\lambda_1^2 + (\lambda_1 + S)^2}{2\sigma^2}}$$

$$P(S) = 2\int_{-\infty}^{\infty} \frac{1}{Z_\beta(\sigma)} S^\beta e^{-\frac{x^2 + (x+S)^2}{2\sigma^2}} dx = \frac{2\sigma\sqrt{\pi}}{Z_\beta(\sigma)} S^\beta e^{-\frac{S^2}{4\sigma^2}}$$

- 平均间距：

$$\mathbb{E}(S) = \int_0^\infty SP(S)dS = \int_0^\infty \frac{(2\sigma)^{\beta+1}dt}{2} t^{\beta/2} e^{-t} = \frac{2\sigma\sqrt{\pi}}{Z_\beta(\sigma)} \Gamma(\beta/2+1)(2\sigma)^{\beta+2}$$

$$\mathbb{E}(S) = 1 \Rightarrow \sigma = \frac{\Gamma(\beta/2 + 1/2)}{2\Gamma(\beta/2 + 1)}$$

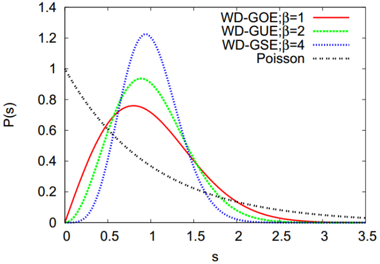
Wigner 假设

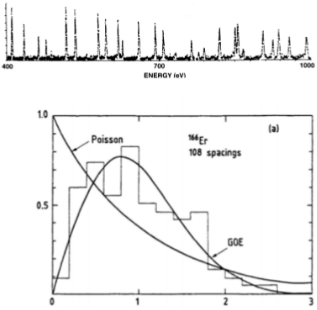
- Wigner surmise：

$$P(s) = C_\beta s^\beta e^{-a_\beta s^2}$$

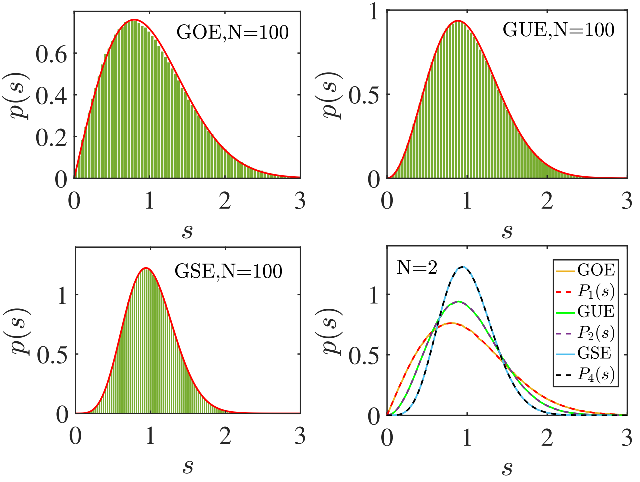
$$C_\beta = \frac{2\sigma\sqrt{\pi}}{Z_\beta(\sigma)}, \qquad a_\beta = -\frac{1}{4\sigma^2}$$

symmetry class	β	a_β	C_β
GOE	1	$\pi/4$	$\pi/2$
GUE	2	$4/\pi$	$32/\pi^2$
GSE	4	$64/9\pi$	$262144/729\pi^3$





Wigner 假设

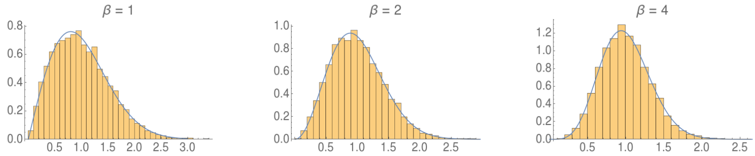


- Wigner surmise: 高斯 $N(0,1)$ 分布随机数，$N=2$ 时数值解析严格一致

Wigner 假设-mathematica

```
gaussiandists = {GaussianOrthogonalMatrixDistribution[2], GaussianUnitaryMatrixDistribution[2],
                 高斯正交矩阵分布                          高斯酉矩阵分布
    GaussianSymplecticMatrixDistribution[2]};
    高斯辛矩阵分布
spacingdists = MatrixPropertyDistribution[{-1, 1}.MinMax[Eigenvalues[x]], x ≈ #] & /@ gaussiandists;
               矩阵属性分布                           最小…  特征值
gaps = Normalize[RandomVariate[#, 10000], Mean] & /@ spacingdists;
       正规化    伪随机变数              平均值
WignerSurmisePDF[x_, β : 1] := Pi (x / 2) Exp[-Pi (x / 2) ^2];
                               圆周率      指…  圆周率
WignerSurmisePDF[x_, β : 2] := 2 (4 x / Pi) ^2 Exp[(-4 / Pi) x^2];
                                        圆周率   指数形式 圆周率
WignerSurmisePDF[x_, β : 4] := (64 / (9 Pi)) ^3 x^4 Exp[(-64 / (9 Pi)) x^2];
                                        圆周率        指数形式       圆周率
histogram = Histogram[#, {0.1}, PDF] & /@ gaps;
            直方图             概率密度函数
plot = Plot[WignerSurmisePDF[x, #], {x, 0, 3}] & /@ {1, 2, 4};
       绘图
GraphicsRow@MapThread[Show[#1, #2, ImageSize → 180, PlotLabel → Row[{"β = ", #3}]] &,
按行画出图形  映射线程    显示         图像尺寸              绘图标签       行
  {histogram, plot, {1, 2, 4}}]
```

Out[11]=



能级间距比 r 分布

- 定义 r 函数：

$$r_n = \frac{\min(S_n, S_{n+1})}{\max(S_n, S_{n+1})}$$

$S_n = E_{n+1} - E_n$

- 对于 Poisson distribution: $P(S) = \rho e^{-\rho S} \Rightarrow P(r)$, 令 $x = S_n, y = S_{n+1}$

$$\begin{aligned}
P(r) &= \langle \delta(r - \frac{\min(x,y)}{\max(x,y)}) \rangle \\
&= \rho^2 \int_0^\infty dx \int_0^\infty dy \delta(r - \frac{\min(x,y)}{\max(x,y)}) e^{-\rho x} e^{-\rho y} \\
&= \rho^2 \int_0^\infty dx \int_0^x dy \delta(r - \frac{y}{x}) e^{-\rho x} e^{-\rho y} + \rho^2 \int_0^\infty dy \int_0^y dx \delta(r - \frac{x}{y}) e^{-\rho x} e^{-\rho y} \\
&= 2\rho^2 \int_0^\infty dx \int_0^x dy \delta(r - \frac{y}{x}) e^{-\rho x} e^{-\rho y} = 2\rho^2 \int_0^\infty x e^{-\rho x r} e^{-\rho x} dx \\
&= \frac{2}{(1+r)^2}
\end{aligned}$$

$$\langle r \rangle = \int_0^1 r P(r) dr = \int_0^1 \frac{2r}{(1+r)^2} dr = 2\ln 2 - 1 \approx 0.386294$$

GOE P(r) distribution

- 3x3-matrix, 假设本征值 $e_1 \le e_2 \le e_3$，由

$$P_{N\beta}(x_1, ..., x_N) = C_{N\beta} \prod_{j<k} |x_k - x_j|^\beta \exp\left(-\frac{1}{2}\beta \sum_1^N x_j^2\right)$$

得到

$$P(r) \propto \int_{-\infty}^{\infty} de_2 \int_{-\infty}^{e_2} de_1 \int_{e_2}^{\infty} de_3 P(e_1, e_2, e_3) \delta(r - \frac{\min(e_3 - e_2, e_2 - e_1)}{\max(e_3 - e_2, e_2 - e_1))}$$

- 变换：$x = e_2 - e_1$，$y = e_3 - e_2$

$$P(r) \propto 2 \int \int_0^\infty dxdy \delta(rx - y) x^{\beta+1} y^\beta (x + y)^\beta e^{-(x^2+y^2)/2 + (x-y)^2/6}$$

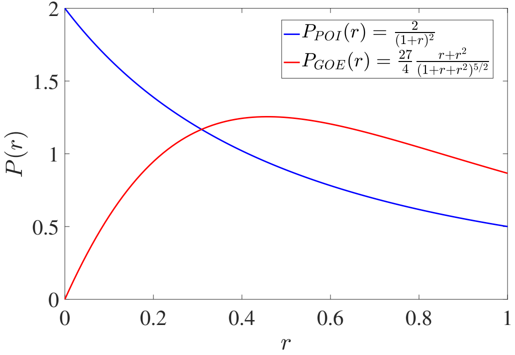
$$P(r) = \frac{1}{Z_\beta} \frac{(r + r^2)^\beta}{(1 + r + r^2)^{1+3\beta/2}}$$

Z_β 为归一化系数，由 $\int_0^1 P(r)dr = 1$ 给出。

GOE P(r) distribution

$$P_{GOE}(r) = \frac{27}{4}\frac{r + r^2}{(1 + r + r^2)^{5/2}}$$

$$\langle r \rangle = \int_0^1 rP_{GOE}(r)dr = 4 - 2\sqrt{3} \approx 0.535898$$



Anderson model

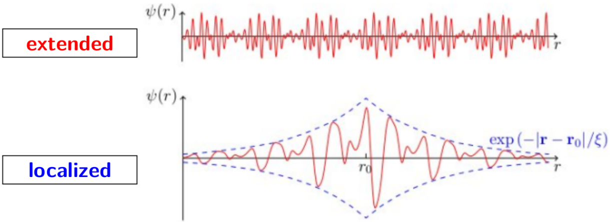
$$[-\frac{\nabla^2}{2m} + U(r)]\psi(r) = \epsilon\psi(r)$$

- $U(r)$ Uniform, Periodic lattice Bloch function:

$$\psi_k(r) = u_k(r)e^{ikr}$$

- Anderson model: $U(r)$ disorder

$$\psi(r) \sim e^{-r/\xi}$$

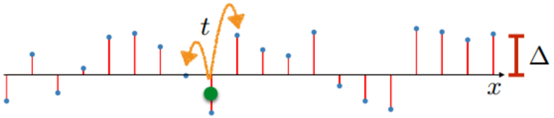


Anderson model as a random matrix

- Tight binding Anderson model:

$$H_{AL} = -t\sum_i (c_i^\dagger c_{i+1} + h.c.) + \sum_i \epsilon_i c_i^\dagger c_i, \qquad \epsilon_i \in [-\Delta, \Delta]$$

$$H_{AL} = \begin{bmatrix} \epsilon_1 & -t & 0 & 0 \\ -t & \epsilon_2 & -t & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & -t & \epsilon_N \end{bmatrix}$$



$\Delta > \Delta_c$

Insulator

All eigenstates are localized

Localization length ξ

The eigenstates, which are localized at different places will not repel each other

Poisson spectral statistics

$\Delta < \Delta_c$

Metal

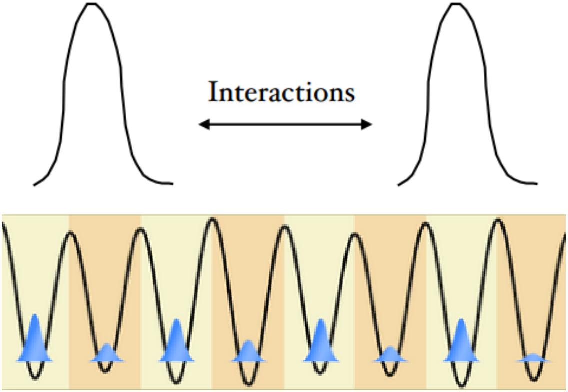
There appear states extended all over the whole system

Any two extended eigenstates repel each other

Wigner – Dyson spectral statistics

Many Body Localization (MBL)

- Single particle + Disorder → Anderson Localization(AL)
  AL + weak interaction → Many Body Localization(MBL)



- Poisson statistics
  Eigenstate Thermalization Hypothesis (ETH) fails

Ergodic Hypothesis

- Boltzmann, 1871
- Microcanonical ensemble:

$$\rho_{mc}(p,q) = \begin{cases} const. & \text{if}(p,q) \in \Gamma_{N,V,E} \\ 0 & \text{else} \end{cases}$$

  Phase space volume $\Gamma(E)$:

$$\Gamma(E) = \int_\Gamma d^{3N}p d^{3N}q \rho_{mc}(p,q)$$

- Let $A(p,q)$ be an integrable function, time average:

$$\overline{A} = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt A(p(t),q(t))$$

- Microcanonical averages of A:

$$\langle A \rangle_{mc} = \frac{1}{\Gamma(E)} \int_\Gamma d^{3N}p d^{3N}q A(p,q) \rho_{mc}(p,q)$$

- Then the system is ergodic if:

$$\overline{A} = \langle A \rangle_{mc}$$

Eigenstate Thermalization Hypothesis（ETH）

- Quantum System：

$$H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle$$

Initial state:

$$|\psi(t=0)\rangle \equiv |\psi(0)\rangle$$

$$|\psi(0)\rangle = \sum_\alpha C_\alpha|\psi_\alpha\rangle, \quad C_\alpha = \langle\psi_\alpha|\psi(0)\rangle, \quad \sum_\alpha |C_\alpha|^2 = 1$$

Total energy:

$$\langle E\rangle = \langle\psi(0)|H|\psi(0)\rangle = \sum_\alpha |C_\alpha|^2 E_\alpha$$

- Time Evolution

$$|\psi(t)\rangle = \sum_\alpha C_\alpha e^{-\frac{i}{\hbar}E_\alpha t}|\psi_\alpha\rangle$$

$$\langle A(t)\rangle = \langle\psi(t)|A|\psi(t)\rangle = \sum_{\alpha,\beta} C_\alpha^* C_\beta e^{\frac{i}{\hbar}(E_\alpha - E_\beta)t} A_{\alpha\beta}$$

where $\langle\psi_\alpha|A|\psi_\beta\rangle \equiv A_{\alpha\beta}$.

Eigenstate Thermalization Hypothesis（ETH）

- The infinite-time average：

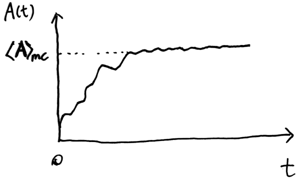
$$\overline{A} \equiv \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \langle A(t)\rangle dt = \sum_\alpha |C_\alpha|^2 A_{\alpha\alpha}$$

Quantum Ergodicity:

$$\overline{A} = \langle A\rangle_{mc}(\langle E\rangle)$$

- Mark Srednicki in 1994: ETH, individual many-body eigenstates are thermal.

$$A_{\alpha\alpha} = \langle A\rangle_{mc}(E_\alpha), \qquad \forall\alpha$$

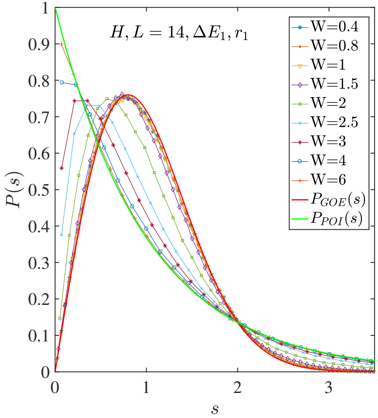


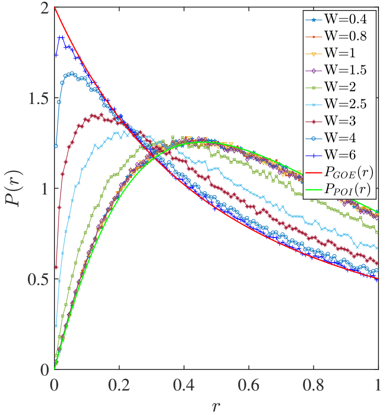
XXZ-disorder model

- Model:

$$H = \sum_{i}^{L} J(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_z S_i^z S_{i+1}^z + \sum_i h_i S_i^z, \qquad h_i \in [-W, W]$$

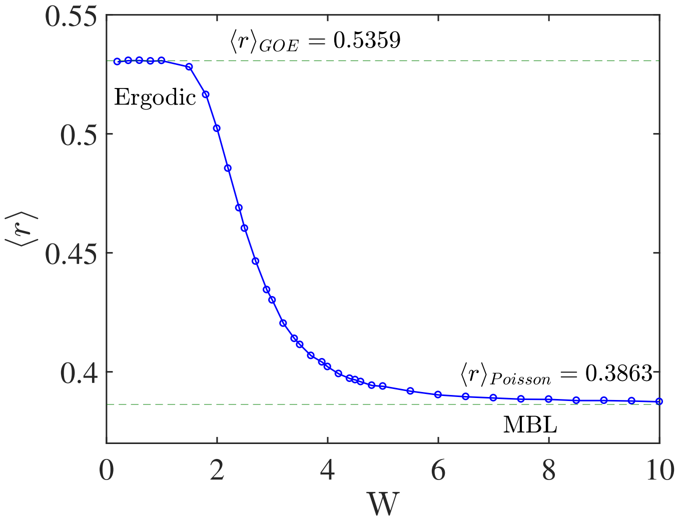
L 为一维晶格长度，J, J_z 为耦合系数，h_i 为格点无序，W 代表无序强度。
例如参数选择:$L = 14, J = J_z = 1$



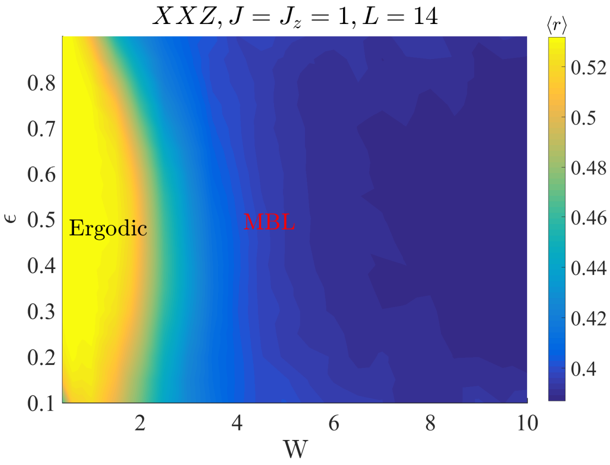


XXZ-disorder model

- Ergodic(遍历)-Many body localization(MBL，多体局域化) transition:



GOE: XXZ-disorder model



- Many-body mobility edge

Summary

1. 随机矩阵理论 RMT
2. Many-body localization
3. 本征态热化假说（ETH）
4. Ergodic/Thermalization - GOE 分布，Extended
MBL - Poisson 分布，Localized