hw23 求解 Brown motion $m\dot{v} = -\eta v + \xi$

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1 思想

思路和 hw22 完全一样，加入一项阻尼项 $-\eta v$。

2 分析与具体实验结果

方程 $m\dot{v} = -\eta v + \xi$ 的解为：

$$v(t) = e^{-\frac{\eta}{m}t} v_0 + \int_0^t e^{-\frac{\eta}{m}(t-t')} \xi(t') \mathrm{d}t' \tag{1}$$

和 hw22 中做的事情一样，这次的均值和方差变为：

$$\begin{aligned} \overline{v(t)} &= v_0 e^{-\frac{\eta}{m}t} \\ \overline{v^2(t)} &= \overline{v(t)}^2 + A \int_0^t e^{-2\frac{\eta}{m}(t-t')} \mathrm{d}t' \\ \sigma_t^2 &= \overline{v^2(t)} - \overline{v(t)}^2 = A \int_0^t e^{-2\frac{\eta}{m}(t-t')} \mathrm{d}t' \\ &= A\frac{m}{2\eta}\left(1 - e^{-2\frac{\eta}{m}t}\right) \end{aligned} \tag{2}$$

做 Euler 法迭代即可：$v_{n+1} = v_n - \eta v_n \mathrm{d}t + \sqrt{A}\mathrm{d}w_t$，数值模拟结果见 fig.1，数值模拟曲线和理论曲线几乎完全重合，可见和理论预言符合地非常好。

3 Code

```
A = 1;
sA = sqrt(A);
eta = 0.1;
dt = 0.04;
sdt = sqrt(dt);
T = 60;
N = T/dt + 1;
v0 = 10;

R = 1000000; %repeat

```

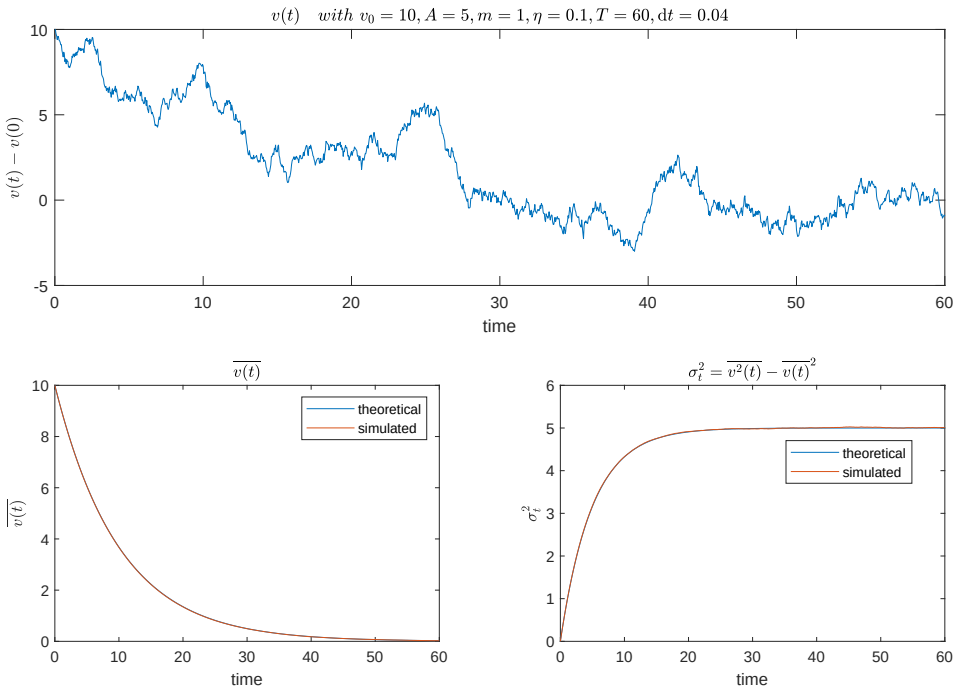

图 1: 数值计算结果，$A = 1$，重复 10^6 次。

```
    v = zeros(R,N);

    for i = 1:R
        v(i,1) = v0;
        for j = 2:N
            v(i,j) = v(i,j-1) - eta*v(i,j-1)*dt + sA*randn()*sdt;
        end
    end

    vbar = zeros(1,N);

    for i = 1:N
        vbar(i) = sum(v(:,i))/R;
    end

    vdev = zeros(1,N);

    for i = 1:N
        vdev(i) = sum(v(:,i).*v(:,i))/R - vbar(i)^2;
    end
```

```
32
33      figure
34
35      x = linspace(0,T,N);
36
37      subplot(2,1,1);
38      plot(x,v(R/2,:));
39      xlabel('time');
40      ylabel('$$v(t)-v(0)$$','Interpreter','latex');
41      title('$$v(t) \quad with \ v_0=10,A=5,m=1,\eta=0.1,T=60,\mathrm{d}t
           =0.04$$','Interpreter','latex');
42
43      subplot(2,2,3);
44      tbar = v0*exp(-eta*x);
45      plot(x,[tbar;vbar]);
46      legend('theoretical','simulated');
47      xlabel('time');
48      ylabel('$$\overline{v(t)}$$','Interpreter','latex')
49      title('$$\overline{v(t)}$$','Interpreter','latex');
50
51      subplot(2,2,4);
52      tdev = A*(1-exp(-2*eta*x))/(2*eta);
53      plot(x,[tdev;vdev]);
54      legend('theoretical','simulated','Location','best');
55      xlabel('time');
56      ylabel('$$\sigma^2_t$$','Interpreter','latex')
57      title('$$\sigma^2_t = \overline{v^2(t)} - \overline{v(t)}^2$$','
           Interpreter','latex');
58      drawnow
```