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Zurek-Kibble Causality Bounds in Time-Dependent Ginzburg-Landau Theory and Quantum Field Theory

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Zurek's and Kibble's causal constraints for defect production at continuous transitions are encoded in the field equations that condensed matter systems and quantum fields satisfy. In this article we highlight some of the properties of the solutions to the equations and show to what extent they support the original ideas.

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1. INTRODUCTION

There is a tradition of condensed matter physicists and quantum field theorists (particle physicists) exploring ideas and techniques in common. The development of *equilibrium* renormalisation group methods in the '70s and onwards by both communities to describe phase transitions has been one of the major successes of that period. More recently, the adoption of simple causal bounds to constrain the *non-equilibrium* dynamics of continuous phase transitions has led to a renewal of this dialogue, reflected in the title *Cosmological Experiments in Condensed Matter Systems* of the review article by Zurek[1], the first extensive presentation of these ideas.

Although subsequent work by several authors (including Zurek himself[2,3]) has refined these original proposals[4] they are, at heart, very simple: after a continuous transition the order parameter field (or fields) cannot adapt to a single uniform ground state value immediately. The reason is straightforward. Although the adiabatic (equilibrium) correlation length $\xi_{eq}(t)$ diverges at the transition the true correlation length $\xi(t)$ does not, since there is not enough time for it to do so. Causality imposes a maximum rate at which the

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correlation length can grow and hence a maximum correlation length, $\bar{\xi}$ say, at the onset of the transition. At the same time, causality imposes a horizon outside which the fields are uncorrelated. A consequence of a correlation length that is always finite is the creation of topological defects (monopoles, vortices, walls, etc.) that mark 'domain' boundaries. These defects then self-interact and annihilate, leading to larger and larger regions over which the field assumes one of the possible ground state values.

If the simple relationship $\bar{\xi} = O(\bar{\xi}_{def})$ between the correlation length $\bar{\xi}$ and the separation length $\bar{\xi}_{def}$ of defects suggested by this picture is valid then the density of defects is bounded, and calculable, at their moment of formation. If their evolution is known, the density of defects at late times is similarly constrained.

Since causality is as equally embodied in quantum field theory (QFT) as in condensed matter, these ideas had been posed independently by Kibble[5,6] in the context of phase transitions of QFT in the very early universe. One of the great successes of QFT has been the unification of the electroweak forces through spontaneous symmetry breaking. There is every reason to believe that this and other symmetries were not always broken but that, in the very early and hot universe, they were restored. Kibble[7,8] and others[9] have observed that the same causal arguments put useful constraints on the density of defects at the time of their formation, which could have consequences today. Unfortunately, because of our lack of understanding of the details of the early universe it is impossible to make reliable predictions[10].

It was Zurek who argued that the same causal bounds could be tested directly in condensed matter systems, whose phase transitions produce defects whose densities can be measured fairly readily. In the final section of this paper we review most of the experimental evidence for Zurek's bounds which, on several occasions[11–13], are satisfied at a good qualitative level.

Prior to that, in the next two sections we reiterate and rephrase the Zurek-Kibble bounds before providing an alternative scenario for the way defects appear after a transition. This is based on the fact that the simple defects that we shall consider have 'false' ground-state or vacuum at their cores where the field *vanishes*. Under suitable circumstances the separation length $\bar{\xi}_{def}$ is more sensibly derived by counting *zeroes* of the field as $\bar{\xi}_{def} = O(\bar{\xi}_{zero})$, where $\bar{\xi}_{zero}$ measures the separation of field zeroes. However, in principle, $\bar{\xi}_{zero}$ and $\bar{\xi}$ are *different* correlation lengths, exploring different attributes of the fluctuation spectrum.

We then apply these ideas in turn to condensed matter systems, using the empirical time-dependent Ginzburg Landau (TDGL) theory as specifying the dynamics, and to relativistic Quantum Field Theory (QFT). In the latter case, there is a further complication in that the system needs to decohere (i.e. lose its quantum mechanical interference) before we can use classical probabilities to identify defects, and we devote Section 6 to that.

Nonetheless, subject to certain conditions, we find that $\bar{\xi}_{def}$ is, indeed, $O(\bar{\xi})$, as predicted, both for condensed matter and QFT. The Kibble-Zurek bounds are qualitatively valid, albeit for somewhat different reasons than simple causality. A byproduct is that the failure of ^4He experiments to give agreement[14] may lie in an incorrect assumption of the decay rate of defects.

We have not attempted to be exhaustive in this article and several important topics have not been included (e.g. temperature inhomogeneities). The reader is referred to the references for more details.

2. CAUSAL CONSTRAINTS ON THE PRODUCTION OF DEFECTS

2.1. Condensed Matter

The role of defects in our preliminary discussion was to both to infer and to predict the finiteness of the correlation length after the transition. Both condensed matter systems (^3He in particular[15]) and QFT[16] permit a prodigious variety of defects, and we shall report on some of them later. Rather than try to be generic we use the fact that most (but not all) experiments involve vortices and, with this in mind, we take an idealised superfluid as the prototypical condensed matter system in which to display these arguments. An essential property of superfluids is that their symmetries are *global.* For simplicity, we assume a single complex order parameter field $\phi(\mathbf{x},t)$, which permits the most simple vortices[17].

Most simply, the transition from normal fluid to superfluid is achieved through a pressure or temperature quench. The rate at which $\phi(\mathbf{x},t)$ can adjust, and the field become correlated, is the speed of (second) sound $\bar{c}(\epsilon)$, where $\epsilon = (T-T_c)/T_c$, which slows critically to zero as $\epsilon \to 0$. If we know the time dependence $\epsilon(t)$ of $(T-T_c)/T_c$ in the vicinity of the transition, the diameter of the 'sonic horizon' at time t is

$$h(t) = 2\int_0^t dt'\,\bar{c}(t'), \tag{1}$$

where $\bar{c}(t) = \bar{c}(\epsilon(t))$, and we have assumed that the transition began at time $t=0$, for which $\epsilon(0)=0$.

This immediately leads to a 'weak' causal constraint that, at time t after the transition has been completed, the correlation length $\xi(t)$ satisfies the horizon bound

$$\xi(t) < h(t). \tag{2}$$

i.e. the field is uncorrelated outside its causal horizon. For long times (2) becomes $\xi(t) \leq 2\bar{c}(\epsilon_f)$, where ϵ_f is the final relative temperature.

At his stage we need to be more explicit in that, for our prototypical case of a single complex field, there are two candidate correlation lengths, one for the field magnitude (e.g. superfluid density) and one for the phase correlation (e.g. superflow). Once the transition is complete it is the latter, associated with Goldstone modes, that determines the horizon.

However, for the purpose of experiment the more relevant bound invoked by Zurek (and by Kibble in Ref.6) is the following 'strong' causal bound, for which this delineation is not necessary, in the first instance: Zurek argued that the correlation length freezes in *before* the transition, at $t = -\bar{t}$, when its growth rate reaches its causal bound $\dot{\xi}_{eq}(-\bar{t}) = +\bar{c}(-\bar{t})$, and only unfreezes at a comparable time $+\bar{t}$ *after* the transition.

The major difference with the weak bound is that, before the transition, there is a single correlation length $\xi(t)$, since the symmetry is unbroken. Further,as we shall see, the time $\bar{t}$ after the transition occurs early, just before the order parameter has reached its equilibrium value. At this time the real and imaginary parts of ϕ are still effectively independent, and we still have only one correlation length.

We reach the same conclusion from the viewpoint of the causal horizon. There is a time $\bar{t}$,

$$h(\bar{t}) = \xi_{eq}(\bar{t}), \tag{3}$$

when the horizon is big enough to accommodate a defect. Before this time, $0 < t < \bar{t}$, the adiabatic approximation has completely broken down, and the field is frozen in. Up to factors $O(1)$, this matches the value of $\bar{t}$ given in (3). It is argued that, at best, the system will emerge at time $\bar{t}$ with a correlation length $\bar{\xi} = \xi_{eq}(\bar{t})$ that characterises identifiable domains.

More specifically, in this and the other approaches, it is the longest wavelength modes, which control the field ordering, that freeze in first and unfreeze last. It must be stressed that these are rough estimates for the earliest time and the largest correlation length when the system permits an adiabatic description after the onset of the transition and we would not be surprised if, in practice, there was a difference by a factor of a few.

Whatever, in a second assumption, that is independent of causality, Zurek further identified $\bar{\xi} = \xi_{eq}(\bar{t})$ with $\xi_{def}(\bar{t})$, the defect (vortex) separation at the earliest time, $\bar{t}$, at which they could be produced. To be specific, and simple, we assume that

$$\frac{d\epsilon}{dt} = -\frac{1}{\tau_Q} \tag{4}$$

in the vicinity of $t = 0$, and adopt mean field critical indices for the equilibrium correlation length $\xi_{eq}(t)$ and the speed of sound $\bar{c}(t)$,

$$\xi_{eq}(t) = \xi_0 \left|\frac{t}{\tau_Q}\right|^{-1/2}, \ \bar{c}(t) = c_0 \left|\frac{t}{\tau_Q}\right|^{1/2}, \tag{5}$$

at this time, for suitable (zero-temperature) parameters ξ_0 and $c_0 = \xi_0/\tau_0$.

From (3) it follows that

$$\bar{t} \approx \sqrt{\tau_Q \tau_0} = \bar{t}_Z \tag{6}$$

and

$$\bar{\xi} = \xi_{eq}(\bar{t}_Z) \approx \xi_0 \left(\frac{\tau_Q}{\tau_0}\right)^{1/4} = \bar{\xi}_Z. \tag{7}$$

In practice, even for the fastest transitions, we have $\tau_Q \gg \tau_0$, whereby $\bar{\xi}_Z \gg \xi_0$. The variant of the causal argument that has the field frozen in with this correlation length at $t = -\bar{t}_Z$ suggests that $\xi(t) \approx \bar{\xi}_Z$ for $-\bar{t}_Z \geq t \leq \bar{t}_Z$. In particular $\xi(0) \approx \bar{\xi}_Z$.

From the second assumption we have that the *initial* vortex density $\bar{n}_{def}$ $= n_{def}(\bar{t})$ is

$$\bar{n}_{def} = O\left(\frac{1}{\bar{\xi}_Z^2}\right) = \frac{1}{f^2 \xi_0^2}\left(\frac{\tau_0}{\tau_Q}\right)^{1/2}, \tag{8}$$

where $f = O(1)$ estimates the fraction of defects per 'domain'. Equivalently, the length of vortices in a box volume v is $O(\bar{n}_{def} v)$.

2.2. Relativistic Field Theory

Let us assume global symmetry breaking again. We note that if there had been no critical slowing down, as in relativistic QFT, and $\bar{c}(t) = c$ is constant, then the 'sonic' horizon $h(t)$ of (1) is replaced by the usual causal 'light' horizon, to give

$$\xi(t) \leq h(t) \approx 2ct \tag{9}$$

for the phase correlation length. This constraint was used to bound the production of monopoles[7] and cosmic strings[9] (vortices) in the early universe.

On the other hand the strong causal bound (3) now gives

$$\bar{t}_K = \bar{t} \approx (\tau_Q \tau_0^2)^{1/3} \tag{10}$$

and

$$\bar{\xi}_K = \bar{\xi} = \xi_{eq}(\bar{t}_K) \approx \xi_0 \left(\frac{\tau_Q}{\tau_0}\right)^{1/3}. \tag{11}$$

In (11) is the common correlation length for early time, leading to the estimate $\bar{n}_{def} = O(1/\bar{\xi}_K^2)$ for the defect density at the time of their production.

We stress that critical slowing down is not necessary for the existence of causal bounds. Its importance lies in the ameliorating effect it has on temperature inhomogeneities, whose effect is to diminish defect production[18]. In the early universe temperature inhomogeneity is slight, unlike in condensed matter physics experiments, where it is often substantial, and critical slowing down is crucial.

This is all for flat space-time.

2.3. Non-causal Mechanisms: The Ginzburg Regime

In particular, what strikes us about the causal predictions is that they are as universal as the mean field approximation is valid. That, of course, is their main attraction. In the first instance the basic distance and time scales ξ_0 and τ_0 do not use any information about the magnitude of the order parameter (i.e. the strength of the interactions). Even the adoption of non-mean-field critical indices (necessary for ^4He, unnecessary for ^3He) hardly changes the picture.

In stressing the primacy of causal horizons Zurek downplayed the importance of thermal fluctuations, whose characteristic scale is set by the Ginzburg temperature T_G through $\epsilon(T_G)$, independent of the quench rate and dependent on the microscopic parameters of the theory. This is in contrast to the earlier suggestion (by Kibble[5] and others[9]) that thermal fluctuations in the Ginzburg regime might also lead to the production of vortices, again at early times. The reason why the Ginzburg regime might be important is the following: once we are below T_c, the Ginzburg temperature $T_G < T_c$ signals the temperature above which there is a significant probability for thermal fluctuations between one degenerate groundstate and another on the scale of the correlation length at that temperature. That is, the thermal energy in such a fluctuation matches the energy required to pass over the hump of the unstable minimum.

Whereas, above T_G one might anticipate a population of 'domains', fluctuating in and out of existence, at temperatures below T_G fluctuations from one minimum to the other become increasingly unlikely. Thus field configurations with non-trivial topology formed above T_G could stabilise. Since the Ginzburg regime does depend on the value of the order parameter, $\epsilon(T_G)$ can vary hugely from system to system. It had been suggested[5] that we identify

$$\xi_{eq}(T_G) = \frac{\xi_0}{\sqrt{1 - T_G/T_c}} \tag{12}$$

with the scale $\bar{\xi}$ at which stable domains begin to form. This is still invoked in cosmology[9].

We shall see later why this is not the case. However, even if thermal fluctuations are not the mechanism for the formation of larger domains, they cannot be ignored completely. At the least they are relevant to the formation of small 'domains', and to wiggles in the boundaries of larger domains where defects are to be found. As such, they can make the definition of defect density scale invariant.

3. AN ALTERNATIVE APPROACH: INSTABILITIES AND (LINE) ZEROES

In the several years since these simple bounds were first proposed we have acquired a much better understanding of the way in which transitions occur. These does not mean that these bounds have lost their relevance, but that they need to be qualified.

To be more quantitative we assume that our *single* complex scalar field ϕ in three spatial dimensions has a wine-bottle potential. The transition is continuous. That is, we assume that the qualitative dynamics are conditioned by the field's *equilibrium* free energy, of the form

$$F(T) = \int d^3x \,\Big(|\nabla\phi|^2 + \epsilon(T)|\phi|^2 + \lambda|\phi|^4\Big). \tag{13}$$

where we have scaled the field to be as generic as possible.

On rescaling ϕ could represent a relativistic quantum field, with free energy

$$F(T) = \int d^3x \,\Big(|\nabla\phi|^2 + m^2(T)|\phi|^2 + \lambda|\phi|^4\Big). \tag{14}$$

where $m^2(T) = M^2\epsilon(T)$ at one-loop level. On the other hand, on scaling, F could be the Ginzburg-Landau free energy

$$F(T) = \int d^3x \,\left(\frac{\hbar^2}{2m}|\nabla\phi|^2 + \alpha(T)|\phi|^2 + \beta|\phi|^4\right) \tag{15}$$

in which the chemical potential $\alpha(T) = \alpha_0\epsilon(T)$ vanishes at the critical temperature T_c. With minor qualifications (15) is the free energy adopted for simplified ^3He in Ref.19 and Bose-Einstein condensation[20] (BEC). It is not a reliable representation of ^4He, but may still be useful in this case.

3.1. Time-scales: The Spinodal Time

Any dynamical equations for the onset of a continuous transition will embody causality, by definition. However, the transition cannot be said to have happened before the order parameter has achieved its equilibrium value $|\phi|^2 = 1/\lambda$, (or M^2/λ or α_0/β, depending on how we rescale (13)). If $\langle ...\rangle_t$ denotes ensemble averaging at time t then a lower bound on the first time from which we can start counting defects is $t = t_{sp}$, for which $\langle|\phi|^2\rangle_t = 1/\lambda$.

Ensemble averaging is taken with respect to the relative probability $p_t[\phi]$ that the order parameter field ϕ takes the value $\phi(\mathbf{x})$ at time t. The way in which $\langle|\phi|^2\rangle_t$ builds up to its final value is by the growth of the amplitudes of the unstable long-wavelength modes, which are unstable because of the upturned parabolic free energy at initial times. These long wavelength modes order the field on increasingly larger scales. The time t_{sp} is, crudely, the time for these modes to roll from the top of the hill to the groundstates at the bottom, and we have termed it the spinodal time. This is a very different picture from that of a field freezing in by, or before, the transition that was proposed in Section 2.

A priori, t_{sp} is not related to the causal $\bar{t}_K$ or $\bar{t}_Z$, but it is not difficult to see why they might be comparable. Unstable modes grow exponentially fast. As long as dimensional analysis makes $\bar{t}_K$ or $\bar{t}_Z$ the natural unit in which to measure time, any exponentially growing term will achieve values that are not exponentially large at times $t = O(1)$ in these units. We thus anticipate only a logarithmic sensitivity to the microscopic parameters of the theory (and the quench rate).

3.2. Quantum Field Theory: The Decoherence Time

Although the spinodal time is equally important for QFT, there are additional problems in that the evaluation of ensemble averages $\langle ...\rangle_t$ is not enough. In particular, in QFT we need to consider the whole density matrix $\langle\phi^+|\hat{\rho}(t)|\phi^-\rangle$ rather than just the diagonal elements $p_t[\phi] = \langle\phi|\hat{\rho}(t)|\phi\rangle$ that determine the probability functional.

The reason is that probabilities are only useful when there is no, or little, quantum interference between adjacent configurations (cf. the two-slit problem). Such interference is measured by the magnitude of the off-diagonal density matrix elements.

Before we can think of identifying particular field configurations like vortices in QFT we must have that the reduced density matrix is approximately diagonal on coarse-graining.

By coarse-graining we now mean the separation of the whole into the 'system', the long wavelength modes of ϕ which establish the field ordering, and its 'environment' whose degrees of freedom are integrated over to produce the reduced matrix. The environment comprises the other fields with which our scalar is interacting, together with the short wavelength modes of the scalar field itself [21,22]. This coarse-graining is more sophisticated than the effective short-distance cutoff that is essentially all that is needed in condensed matter.

The reduced density matrix is non-unitary (dissipative) and the effect of this environment is to introduce a new time scale, the 'decoherence' time t_D such that quantum interference can be ignored at times later than it. Only for $t > t_D$ does it make sense to think of classical defects. However, dissipation will also grow exponentially initially since it is driven by the same long wavelength unstable modes. Thus, although t_D depends on interaction strengths as well as the quench rate, we anticipate that it is insensitive to them, for the same reasons that $\bar{t}$ is insensitive. For the moment we assume that $t_D \leq t_{sp}$ to keep the discussion simple.

3.3. Length-scales: Vortices as Line Zeroes

As we noted earlier, this dynamic picture, in which modes build up their amplitudes from their initial small Boltzmann values when the field is sitting at the top of the hill, by rolling down it, or spreading over it, seems totally at variance with that given in Section 2. There, it was assumed that the field freezes in at, or even prior to the transition, with a correlation length $\bar{\xi}$ ($\bar{\xi}_K$ or $\bar{\xi}_Z$) that is already huge, comparatively.

In fact, even at early times when we can get by with a single correlation length for the ϕ-field, we are talking about two *different* lengths. The length $\bar{\xi}$ is the usual correlation length, obtained from the long-distance behaviour of the correlation function at time t,

$$\langle\phi(\mathbf{x})\phi^*(\mathbf{0})\rangle_t = G(r,t) = \int d\!\!/^3k\, e^{i\mathbf{k}.\mathbf{x}} P(k,t), \tag{16}$$

($r = |\mathbf{x}|$) in which the $P(k,t)$ is the power spectrum of the fluctuations. To obtain $\bar{\xi}$ we take $r \to \infty$, and look for behaviour of the form $\exp -(r/\bar{\xi})^{\gamma}$. That is, $\bar{\xi}$ is determined by the nearest singularity of $P(k,t)$ in the k-plane.

This is not the length that characterises the separation of defects, which are formed from the phase separation as the field falls from the hill in different directions at different spatial points. Vortices form because these phases do not match up smoothly and are frustrated by zeroes in the field.

More specifically, at long times after the transition, we expect to find widely separated classical vortices. The classical vortices of this simple theory, solutions to the equation $\delta F/\delta\phi = 0$ are tubes of false groundstate, with line-zeroes $\phi = 0$ at their cores, with width $O(\xi_{eq}(T))$. If we write $\phi = (\phi_1 + \phi_2)/\sqrt{2}$ these line-zeroes are the intersections of the surfaces $\phi_1 = \phi_2 = 0$. From the earliest numerical simulations onwards[23,24] simple vortices (and defects in lower dimension) have been identified by counting zeroes of the fields.

That is, the empirical measure of defect separation is ξ_{zero}, the typical separation of line-zeroes. If, for the sake of argument, we assume Gaussian field fluctuations, then we shall see that, at their time of formation, $\xi_{zero}(t)$ is given by[25,26]

$$\xi^2_{zero}(t) = O\left(\frac{-G(0,t)}{G''(0,t)}\right). \tag{17}$$

Primes denote differentiation with respect to r. Empirically[27], (17) has been known to be valid until the spinodal time, even though the field has ceased to be Gaussian by then.

This suggests that a more realistic length-scale for calculating defect densities at the onset of defect production is $\bar{\xi}_{zero} = \xi_{zero}(t_{sp})$. In this case there is no immediate reason to relate $\bar{\xi}_{zero}$ to $\bar{\xi}$ since, unlike $\xi_{eq}(t)$, $\xi_{zero}(t)$ is given entirely by the *short*-distance behaviour of $G(r,t)$ or equivalently, by the dominant wavenumbers in $P(k,t)$.

It is not difficult to extend the picture of vortices as line zeroes to other defects like global monopoles and domain walls in three dimensions, or kinks in one dimension. No novelty arises, and we shall not do so.

3.4. When are Line Zeroes Vortices?

We have rather assumed that, even if we cannot identify $\bar{\xi}_{zero}$ with $\xi_{eq}(\bar{t})$ ($\bar{t}_Z$ or $\bar{t}_K$), we can identify $\bar{\xi}_{zero}$ with $\bar{\xi}_{def}$.

This cannot be the case exactly in that, whereas all such defects can be identified by their line-zeroes (or zeroes), not all line zeroes (or zeroes) are candidate defects, since zeroes occur on all scales. A starting-point for counting vortices is to count line zeroes of an appropriately coarse-grained field, in which structure on a scale smaller than ξ_0, the classical vortex size, is not present[28]. Lattice-based numerical simulations do this automatically (but see Ref.29). For the moment, we put in a cutoff $l = O(\xi_0)$ by hand. We note that the inclusion of a cut-off does not affect the long-distance correlation $\bar{\xi}$.

Suppose that, at time t_{sp}, the typical separation of line-zeroes for this coarse-graining is $\xi_{zero,l}(t_{sp})$. Before line zeroes can be identified with classical (global) vortex cores, and $\xi_{zero,l}(t_{sp})$ with $\xi_{def}(t_{sp})$, the following conditions are necessary.

- Although the correlation length $\xi(t)$ is insensitive to a short-distance cutoff l, this is not the case for $\xi_{zero,l}(t)$. Thermal fluctuations will give structure on small scales which will lead to ambiguity in its definition and in the subsequent density. Only when $\partial\xi_{zero,l}/\partial l$ is small in comparison to $\xi_{zero,l}/l$ at $l = \xi_0$ will the line-zeroes have the small-scale non-fractal nature of classical defects, although defects may behave like random walks on larger scales. Among other things, this will depend on the Ginzburg regime.

- As a final, related, check, the energy in field gradients should be commensurate with the energy in classical vortices with the same density as that of line zeroes.

In fact, most (but not all[30]) numerical lattice simulations cannot distinguish between line-zeroes and classical vortices.

We shall see that, roughly, fluctuations separate into thermal fluctuations controlled by the current temperature $T(t)$, which determine the small-scale structure of line-zeroes, which compete with large-amplitude long-wavelength fluctuations whose role is to provide large-scale order. It is these latter with which classical defects are associated.

3.5. Counting Zeroes

Suppose, at some time, that the field has line zeroes $\mathbf{R}_n(s)$, where $n = 1, 2, ..$ labels the zero, and s measures the length along it. As a result the *topological line density* of zeroes $\vec{\rho}(\mathbf{r})$ can be defined[25,26] by

$$\vec{\rho}(\mathbf{x}) = \sum_n \int ds \frac{d\mathbf{R}_n}{ds} \delta^3[\mathbf{x} - \mathbf{R}_n(s)]. \tag{18}$$

In (18) ds is the incremental length along the line of zeroes $\mathbf{R}_n(s)$ and $d\mathbf{R}_n/ds$ is a unit vector pointing in the direction which corresponds to positive winding number.

It follows that, in terms of the zeroes of the field $\phi(\mathbf{x})$, $\rho_i(\mathbf{x})$ can be written as

$$\rho_i(\mathbf{x}) = \delta^2[\phi(\mathbf{x})]\epsilon_{ijk}\partial_j\phi_1(\mathbf{x})\partial_k\phi_2(\mathbf{x}), \tag{19}$$

where $\delta^2[\phi(\mathbf{x})] = \delta[\phi_1(\mathbf{x})]\delta[\phi_2(\mathbf{x})]$, where $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. The coefficient of the δ-function in (19) is the Jacobian of the more complicated transformation from line zeroes to field zeroes. What we want is not this, but the *total line density* $\bar{\rho}(\mathbf{x})$,

$$\bar{\rho}_i(\mathbf{x}) = \delta^2[\phi(\mathbf{x})]|\epsilon_{ijk}\partial_j\phi_1(\mathbf{x})\partial_k\phi_2(\mathbf{x})|. \tag{20}$$

Let us suppose that, at time t, the probability density that the field takes functional form $\phi(\mathbf{x})$ is $p_t[\phi]$. The ensemble average of $K[\phi]$ at time t is

$$\langle K[\phi]\rangle_t = \int \mathcal{D}\phi_1\mathcal{D}\phi_2\, p_t[\phi]\, K[\phi]. \tag{21}$$

The vanishing field expectation value and the independence of the field and its derivatives

$$\langle\phi_a(\mathbf{x})\rangle_t = 0 = \langle\phi_a(\mathbf{x})\partial_j\phi_b(\mathbf{x})\rangle_t, \tag{22}$$

imply $\langle\rho_j(\mathbf{x})\rangle_t = 0$ *i.e.* an equal likelihood of a string line-zero or an anti-string line-zero passing through an infinitesimal area. However,

$$n_{zero}(t) = \langle\bar{\rho}_i(\mathbf{x})\rangle_t > 0 \tag{23}$$

and measures the *total* line-zero density in the direction i, without regard to string orientation. The isotropy of the initial state guarantees that n is independent of the direction i.

To get a basic idea as to what this means let us assume that the field fluctuations are Gaussian. This has been the starting assumption for networks of cosmic strings[23,24]. Then everything is given in terms of the two-field correlator (16) at time t,

$$\langle\phi_a(\mathbf{x})\phi_b(\mathbf{0})\rangle_t = \delta_{ab}G(\mathbf{r},t). \tag{24}$$

We coarse-grain the field by putting in a cutoff $l = O(\xi_0)$ by hand, as

$$G_l(r,t) = \int d\!\!{}^-{}^3k\, e^{i\mathbf{k}.\mathbf{x}}P(k,t)\, e^{-k^2l^2}, \tag{25}$$

where we assume $l = O(\xi_0)$ or $O(\xi_{eq}(T_{final}))$, as appropriate.

As we indicated earlier, in this Gaussian approximation $n_{zero,l}(t)$ is determined completely[25,26] by the *short-distance* behaviour of $G(r,t)$ as

$$n_{zero,l}(t) = \frac{1}{2\pi\xi_{zero,l}(t)^2} = \frac{-1}{2\pi}\frac{G_l''(0,t)}{G_l(0,t)}, \tag{26}$$

where we have used (26) to define $\xi_{zero}(t)$. That is, $n_{zero}(t)$ is determined by the ratio of the fourth to second moments of $P(k,t)$.

For non-Gaussian fields the situation is much more complicated. However, as long as there is a dominant wavenumber $k_0(t)$ in $P(k;t)$ this sets a length scale $\xi \approx k_0(t)^{-1}$ that characterises vortex separation. As we said earlier, (26) can be approximately valid until the spinodal time, and we shall assume that to be the case.

We note that a line-zero is invariant under gauge transformations $\phi \to \phi\, e^{i\alpha}$. Whereas a simple spatial cutoff breaks gauge-invariance, we have learned from lattice gauge theory how to coarse-grain a gauge field. However, for a local theory, taking ϕ_1 and ϕ_2 independently Gaussian is a gauge-dependent statement. We shall not pursue this further.

We conclude with a brief discussion of topological charge through a surface, for which the topological density (19), rather than the total density (20), is appropriate, again in the Gaussian approximation. Consider a circular path in the bulk material (in the 1-2 plane), circumference C, the boundary of a surface S. For given field configurations $\phi_a(\mathbf{x})$ the phase change θ_C along the path can be expressed as the surface integral

$$\theta_C = 2\pi \int_{\mathbf{x}\in S} d^2x\ \rho_3(\mathbf{x}), \tag{27}$$

Again we quench from an initial state satisfying (22) (i.e. no rotation).

It is not difficult to show[31,32] that, if $f(r,t) = G(r,t)/G(0,t)$, then

$$(\Delta\theta_C)^2 = \frac{C}{\xi_s(t)} = 2C \int_0^\infty dr \frac{f\prime_l^2(r,t)}{1 - f_l^2(r,t)}. \tag{28}$$

The linear dependence on C is purely a result of Gaussian fluctuations. We note that $(\Delta\theta_C)^2$ requires more than the very short distance behaviour of f_l. Nonetheless, if $P(k,t)$ is strongly peaked at $k = k_0$, then $\xi_s = O(k_0^{-1})$, as is ξ_{zero}. Further, if we removed all material except for a strip from the neighbourhood of the contour C we would still have the same result. This supports the assertion by Zurek that the correlation length for phase variation in bulk fluid is also appropriate for annular flow.

That $(\Delta\theta_C)^2$ varies in time is a consequence of defects migrating through the boundary from the inside to the outside and vice-versa, and annihilating. For a finite system, where such migration is impossible and topological charge is conserved, we would think of adopting (28) at the moment that defects appeared.

4. WHEN DO THE APPROACHES AGREE? CONDENSED MATTER

Let us summarise the situation for our idealised condensed matter, as far as the 'strong' causal bounds of Zurek and Kibble are concerned.. The time and length scales of Zurek's analysis are $\bar{t}_Z$ of (6) and $\bar{\xi}_Z = \xi_{eq}(\bar{t}_Z)$

of (7). In our analysis of zeroes the relevant time and length scales are t_{sp} and $\bar{\xi}_{zero} = \xi_{zero}(t_{sp})$. Above and beyond that, thermal fluctuations may complicate the identity of $\bar{\xi}_{zero}$ and $\bar{\xi}_{def}$.

We have already suggested that $\bar{t}_Z$ and t_{sp} may be comparable, qualitatively, but the situation for length scales is less simple, not least because of thermal fluctuations. To make further progress requires a concrete model in which correlation functions can be calculated explicitly.

4.1. The Time-Dependent Ginzburg-Landau (TDGL) Equation

Not surprisingly, because the Time-Dependent Ginzburg-Landau (TDGL) equation lends itself so directly to numerical analysis, it has been the basis of many papers, largely by Zurek[2,3] and co-workers (but not exclusively e.g. see Refs.19,30). By definition it encodes causality. In Ref.19 a stripped-down global $U(1)$ model was used to mimic the Helsinki ^3He experiment[11]. A more realistic TDGL model for ^3He, that takes the full order-parameters into account has been used elsewhere[33]. The TDGL equation is also useful in BEC[20].

In the last resort, only numerical methods permit us to get long-time solutions, but in this article we pick out some properties of simplified analytic solutions that we think show the relevance of the causal bounds.

We assume that, for the condensed matter systems of interest to us, the dynamics of the transition can be derived from the explicitly *time-dependent* Landau-Ginzburg free energy ($\xi_0 = 1$)

$$F(t) = \int d^3x \left(\frac{1}{2}(\nabla\phi_a)^2 + \frac{1}{2}\epsilon(t)\phi_a^2 + \frac{1}{4}\lambda(\phi_a^2)^2\right). \tag{29}$$

in which we substitute $T(t)$ for T directly in (13), where $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ ($a = 1, 2$) is the complex order-parameter field,

For a simple dissipative system the time-dependent Landau-Ginzburg (TDLG) equation is

$$\frac{1}{\Gamma}\frac{\partial\phi_a}{\partial t} = -\frac{\delta F}{\delta\phi_a} + \eta_a, \tag{30}$$

where η_a is Gaussian thermal noise, satisfying

$$\langle\eta_a(\mathbf{x}, t)\eta_b(\mathbf{y}, t')\rangle = 2\delta_{ab}T(t)\Gamma\delta(\mathbf{x} - \mathbf{y})\delta(t - t'). \tag{31}$$

The basic timescale τ_0, the relaxation time of the long wavelength modes, is Γ^{-1} in our units.

In terms of the dimensionful constants of (15) the fundamental length scale ξ_0 is $\xi_0^2 = \hbar^2/2m\alpha_0$ whereby $\xi_{eq}(T) = \xi_0(T/T_c - 1)^{-1/2}$ as before. With

$\tau_0 = 1/\Gamma\alpha_0$ it follows that the equilibrium correlation length $\xi_{eq}(t)$ and the speed of sound behave when t vanishes as in (5),

$$\xi_{eq}(t) = \xi_0\left|\frac{t}{\tau_Q}\right|^{-1/2}, \; \bar{c}(t) = \frac{\xi_0}{\tau_0}\left|\frac{t}{\tau_Q}\right|^{1/2}. \tag{32}$$

from which the causal bounds (6) and (7) follow.

4.2. Freezing-in Happens (and is Irrelevant)

It is relatively simple to show that the TDGL equation (30) embodies Zurek's causality bound. At early times $|t| \leq \bar{t}_Z$ the effective potential $V(\phi, T)$ is still roughly quadratic and the self-interaction term can be neglected ($\lambda = 0$).

In space, time and temperature units in which $\xi_0 = \tau_0 = k_B = 1$, Eq.30 then becomes

$$\dot{\phi}_a(\mathbf{x}, t) = -[-\nabla^2 + \epsilon(t)]\phi_a(\mathbf{x}, t) + \bar{\eta}_a(\mathbf{x}, t). \tag{33}$$

where $\bar{\eta}$ is the renormalised noise. The solution of the 'free'-field linear equation is straightforward, giving a Gaussian equal-time correlation function

$$G(\mathbf{r}, t) = \int d\!\!{}^-{}^3k\, e^{i\mathbf{k}.\mathbf{r}} P(k, t). \tag{34}$$

in which the power spectrum $P(k, t)$ has a representation in terms of the Schwinger proper-time τ as

$$P(k, t) = \int_0^\infty d\tau\, \bar{T}(t - \tau/2)\, e^{-\tau k^2}\, e^{-\int_0^\tau ds\, \epsilon(t - s/2)}, \tag{35}$$

where $\bar{T}$ is the renormalised temperature. In turn, this gives[34,35]

$$G(r, t) = \int_0^\infty d\tau\, \bar{T}(t - \tau/2)\left(\frac{1}{4\pi\tau}\right)^{3/2} e^{-r^2/4\tau}\, e^{-\int_0^\tau ds\, \epsilon(t - s/2)}. \tag{36}$$

For $\epsilon(t)$ of Eq.4 a saddle-point calculation shows that, at time $t = 0$, when the transition begins,

$$G(r, 0) \approx \frac{T_c}{4\pi r}\, e^{-a(r/\bar{\xi}_Z)^{4/3}}, \tag{37}$$

on rescaling, where $a = O(1)$, confirming Zurek's result of a frozen field correlated over many cold defect thicknesses.

It is immediately apparent that $\bar{\xi}_Z)$ is not a measure of the separation of line zeroes. Putting in the momentum cutoff $k^{-1} > l = \bar{l}\xi_0 = O(\xi_0)$ of Eq.36 by hand corresponds to damping the singularity in $G(r,t)$ at $\tau = 0$ as[34]

$$G_l(0,t) = \int_0^\infty \frac{d\tau\,\bar{T}(t-\tau/2)}{[4\pi(\tau+\bar{l}^2)]^{3/2}} e^{-r^2/4\tau}\, e^{-\int_0^\tau ds\,\epsilon(t-s/2)}, \tag{38}$$

making $G_l(0,t)$ finite. We stress that, for $t \approx 0$, the correlation length ξ remains $O(\bar{\xi})$, *independent* of l.

At $t = 0$ both $G_l(0,t)$ and

$$-G_l''(0,t) = 2\pi \int_0^\infty d\tau\, \frac{\bar{T}(t-\tau/2)}{[4\pi(\tau+\bar{l}^2)]^{5/2}}\, e^{-\int_0^\tau ds\,\epsilon(t-s/2)}. \tag{39}$$

are dominated by the short wavelength fluctuations at small τ. Even though the field is correlated over a distance $\bar{\xi}_Z \gg l$ the density of line zeroes $n_{zero} = O(l^{-2})$ depends entirely on the scale at which we look. Equivalently, $\xi_{zero}(0) = O(l)$. In no way would we wish to identify these line zeroes with prototype vortices and $\xi_{zero}(0)$ has nothing to do with $\bar{\xi}_Z$ at this time.

4.3. Why $t_{sp} \approx \bar{t}_Z$ and $\bar{\xi}_{zero} \approx \bar{\xi}_{def} \approx \bar{\xi}_Z$ (in General)

As the system evolves away from the transition time, the free equation Eq.33 ceases to be strictly valid but, to a first approximation, the back-reaction does not set in until the field has sampled the groundstates. We can keep the linear approximation, with its single correlation length, until $t \approx t_{sp}$. For the linear quench of (4) we find[34]

$$\langle|\phi|^2\rangle_t = e^{2\int_0^t du\,|\epsilon(u)|} \int_0^\infty d\tau\,\bar{T}(t-\tau/2)\frac{e^{-(\tau-2t)^2|\epsilon'(t_0)|/4}}{[4\pi(\tau+\bar{l}^2)]^{3/2}}. \tag{40}$$

However, as time passes the peak of the exponential grows and n_{zero} becomes increasingly insensitive to l. How much time we have depends on the magnitude of λ^{-1}, since once $G(0,t)$ has reached this value it stops growing. Provided that the peak at $\tau = 2t$ dominates over the thermal fluctuations at $\tau \approx 0$ then

$$\langle|\phi|^2\rangle_t \approx \bar{T}_c\, e^{(t/\bar{t}_Z)^2} \int_0^\infty d\tau\, \frac{e^{-(\tau-2t)^2/4\bar{t}_Z^2}}{[4\pi(\tau+\bar{l}^2)]^{3/2}}. \tag{41}$$

Since $G(0,t) = O(\exp((t/\bar{t}_Z)^2)$ at early times the backreaction is implemented extremely rapidly, justifying the free-field approximation in the expression for $G_l(0,t)$ above.

Let us assume that the thermal fluctuations can be ignored by time $t = \bar{t}_Z$. In our units, in which $\lambda\bar{T}_c = \sqrt{1 - T_G/T_c}$, the condition that $\langle|\phi|^2\rangle_t = 1/\lambda$ at $t = t_{sp}$ gives

$$\frac{t_{sp}}{\bar{t}_Z} \approx \frac{1}{2}\sqrt{\ln\left(\frac{\tau_Q/\tau_0}{(1 - T_G/T_c)^2}\right)} > 1. \tag{42}$$

Although greater than unity, the ratio is $O(1)$, extremely insensitive to the parameters of the model, and to the quench rate. For systems and experiments as widely different as the ones for ^3He and ^4He that we shall discuss later, we have $t_{sp}/\bar{t}_Z \approx 3$.

If we continue to assume that the thermal fluctuations at $\tau \approx 0$ can be ignored, then we can use (38) and (39) to calculate the density of line zeroes. The result is

$$\bar{n}_{zero} = n_{zero}(t_{sp}) \approx \frac{\bar{t}}{8\pi t_{sp}}\frac{1}{\xi_0^2}\sqrt{\frac{\tau_0}{\tau_Q}}, \tag{43}$$

in accord with (8) of Zurek, where $f^2 \approx 8\pi t_{sp}/\bar{t}_Z = O(10^2)$. Equivalently, the line-zero separation is

$$\bar{\xi}_{zero} \approx \xi_{zero}(t_{sp}) = 2(t_{sp}/\bar{t}_Z)^{1/2}\bar{\xi} = O(\bar{\xi}_Z), \tag{44}$$

even less sensitive to the parameters of the model and the quench rate.

By assumption, $\bar{n}_{zero}$ is independent of the cutoff l ($l = O(1)$ in our units). The line-zeroes are on the verge of being classical vortices since, once $\langle|\phi|^2\rangle_t = 1/\lambda$, the Gaussian field energy, largely in field gradients, is

$$\bar{F} \approx \langle\int_V d^3x\,\frac{1}{2}(\nabla\phi_a)^2\rangle = -VG''(0, t_{sp}), \tag{45}$$

where V is the spatial volume. This matches the energy

$$\bar{E} \approx V n_{def}(t)(2\pi G(0, t_{sp})) = -VG''(0, t_{sp}) \tag{46}$$

possessed by a network of classical global strings with density $\bar{n}_{zero}$, in the same approximation of cutting off their logarithmic tails.

4.4. Why Back-reaction may not Matter (Much)

The full TDGL equation is

$$\dot{\phi}_a(\mathbf{x}, t) = -[-\nabla^2 + \epsilon(t) + \lambda|\phi(\mathbf{x}, t)|^2]\phi_a(\mathbf{x}, t) + \bar{\eta}_a(\mathbf{x}, t). \tag{47}$$

In order to retain some analytic understanding of the way that the density is such an ideal quantity to make predictions for, we adopt the approximation of preserving Gaussian fluctuations by linearising the self-interaction as

$$\dot{\phi}_a(\mathbf{x},t) = -[-\nabla^2 + \epsilon_{eff}(t)]\phi_a(\mathbf{x},t) + \bar{\eta}_a(\mathbf{x},t), \tag{48}$$

where ϵ_{eff} contains a (self-consistent) term $O(\lambda\langle|\phi|^2\rangle)$. Additive renormalisation is necessary, so that $\epsilon_{eff} \approx \epsilon$, as given earlier, for $t \leq t_0$.

Self-consistent linearisation is the *only* usable approximation in non-equilibrium QFT[37], but is not strictly necessary here, since numerical simulations that identify line zeroes of the field can be made that use the full self-interaction[2,3]. However, to date none address the questions we are posing here exactly, and until then there is virtue in analytic approximations provided they are not taken too seriously. Our discussion complements that of Ref.38 in which a self-consistently linearised TDGL theory is also examined.

The solution for $G_l(r,t)$ is a straightforward generalisation of (36),

$$G_l(r,t) = \int_0^\infty \frac{d\tau\,\bar{T}(t-\tau/2)}{[4\pi(\tau+\bar{l}^2)]^{3/2}} e^{-r^2/4\tau}\, e^{-\int_0^\tau ds\,\epsilon_{eff}(t-s/2)}, \tag{49}$$

making $G_l(0,t)$ finite.

Assuming a *single* zero of $\epsilon_{eff}(t)$ at $t=0$, at $r=0$ the exponential in the integrand peaks at $\tau = \bar{\tau} = 2t$. Expanding about $\bar{\tau}$ to quadratic order gives

$$G_l(0,t) \approx e^{2\int_0^t du\,|\epsilon_{eff}(u)|} \int_0^\infty d\tau\,\bar{T}(t-\tau/2)\frac{e^{-(\tau-2t)^2|\epsilon'(t_0)|/4}}{[4\pi(\tau+\bar{l}^2)]^{3/2}}. \tag{50}$$

The effect of the back-reaction is to stop the growth of $G_l(0,t) - G_l(0,0) = \langle|\phi|^2\rangle_t - \langle|\phi|^2\rangle_0$ at its symmetry-broken value λ^{-1} in our dimensionless units, thereby preserving Goldstone's theorem by requiring $|\epsilon_{eff}(u)| \to \infty$.

What is remarkable in this approximation is that the density of line zeroes uses *no* property of the self-mass contribution to $\epsilon_{eff}(t)$, self-consistent or otherwise. With

$$-G_l''(0,t) = 2\pi \int_0^\infty d\tau\, \frac{\bar{T}(t-\tau/2)}{[4\pi(\tau+\bar{l}^2)]^{5/2}}\, e^{-\int_0^\tau ds\,\epsilon(t-s/2)}. \tag{51}$$

$$\approx 2\pi\, e^{2\int_0^t du\,|\epsilon_{eff}(u)|} \int_0^\infty d\tau\,\bar{T}(t-\tau/2)\frac{e^{-(\tau-2t)^2|\epsilon'(t_0)|/4}}{[4\pi(\tau+\bar{l}^2)]^{5/2}} \tag{52}$$

all prefactors in n_{zero} cancel, to give[35,39]

$$n_{zero}(t) = \frac{1}{4\pi}\frac{\int_0^\infty \frac{d\tau}{(\tau+\bar{l}^2)^{5/2}}\,\bar{T}(t-\tau/2)\,e^{-(\tau-2(t-t_0))^2/4\bar{t}^2}}{\int_0^\infty \frac{d\tau}{(\tau+\bar{l}^2)^{3/2}}\bar{T}(t-\tau/2)\,e^{-(\tau-2(t-t_0))^2/4\bar{t}_Z^2}}, \tag{53}$$

thereby justifying our free-field approximation for $t < t_{sp}$. It is for this reason that simple dimensional analysis (the basis of the causal bounds) is so successful.

For $t > t_{sp}$ the equation for $n_{zero}(t)$ is not so simple since the estimate above, based on a single isolated zero of $\epsilon_{eff}(t)$, breaks down because of the approximate vanishing of $\epsilon_{eff}(t)$ for $t > t_{sp}$. A more careful analysis shows that $G_l(0,t)$ can be written as

$$G_l(0,t) \approx \int_0^\infty \frac{d\tau\, \bar{T}(t-\tau/2)}{[4\pi(\tau+\bar{l}^2)]^{3/2}} \bar{G}(\tau,t), \tag{54}$$

where $\bar{G}(\tau,t)$ has the same peak as before at $\tau = 2t$, in the vicinity of which

$$\bar{G}(\tau,t) = e^{2\int_0^t du\, |\epsilon_{eff}(u)|}\, e^{-(\tau-2t)^2/4\bar{t}_Z^2}, \tag{55}$$

but $\bar{G}(\tau,t) \cong 1$ for $\tau < 2(t-t_{sp})$. Thus, for $\tau_Q \gg \tau_0$, $G_l(0,t)$ can be approximately separated as

$$G_l(0,t) \cong G_l^{UV}(t) + G^{IR}(t), \tag{56}$$

where

$$\begin{aligned} G_l^{UV}(t) &= \int_0^\infty d\tau\, \bar{T}(t-\tau/2)\,/[4\pi(\tau+\bar{l}^2)]^{3/2} && (57)\\ &\approx \bar{T}(t) \int_0^\infty d\tau\,/[4\pi(\tau+\bar{l}^2)]^{3/2} && (58) \end{aligned}$$

describes the scale-*dependent* short wavelength thermal noise, proportional to temperature, and

$$\begin{aligned} G^{IR}(t) &= \frac{1}{(8\pi t)^{3/2}} \int_{-\infty}^\infty d\tau \bar{T}(t-\tau/2)\bar{G}(\tau,t) && (59)\\ &\approx \frac{\bar{T}_c}{(4\pi\bar{\tau}(t))^{3/2}} \int_{-\infty}^\infty d\tau \bar{G}(\tau,t) && (60) \end{aligned}$$

describes the scale-*independent*, temperature independent, long wavelength fluctuations. The integral $\int_{-\infty}^\infty d\tau \bar{G}(\tau,t)$ is naturally time-dependent, largely cancelling its prefactor so as to keep $\langle|\phi|^2\rangle_t$ constant.

A similar decomposition $G''_l(0,t) \cong G''^{UV}_l(t) + G''^{IR}(t)$ can be performed as

$$\begin{aligned} G''^{UV}_l(t) &= 2\pi \int_0^\infty d\tau\, \bar{T}(t-\tau/2)/[4\pi(\tau+\bar{l}^2)]^{5/2} && (61)\\ &\approx 2\pi\bar{T}(t) \int_0^\infty d\tau\,/[4\pi(\tau+\bar{l}^2)]^{5/2}. && (62) \end{aligned}$$

and

$$G''^{IR}(t) = \frac{4\pi}{(8\pi t)^{5/2}} \int_{-\infty}^{\infty} d\tau \bar{T}(t-\tau/2)\,\bar{G}(\tau,t) \qquad (63)$$

$$\approx \frac{4\pi\bar{T}_c}{(4\pi\bar{\tau}(t))^{5/2}} \int_{-\infty}^{\infty} d\tau\bar{G}(\tau,t). \qquad (64)$$

In particular, $G''^{IR}(t)/G^{IR}(t) = O(t^{-1})$.

Firstly, suppose that, for $t \geq t_{sp}$, $G^{IR}(t) \gg G_l^{UV}(t)$ and $G''^{IR}(t) \gg G''^{UV}_l(t)$, as would be the case for a temperature quench $\bar{T}(t) \to 0$. Then, with little thermal noise, we have widely separated line zeroes, with density $n_{zero}(t) \approx -G''^{IR}(t)/2\pi G^{IR}(t)$. With $\partial n_{zero}/\partial l$ small in comparison to n_{zero}/l at $l = \xi_0$ we identify such essentially non-fractal line-zeroes with prototype vortices, and n_{zero} with n_{def}. Of course, we require non-Gaussianity to create true classical energy profiles. Nonetheless, the Halperin-Mazenko result may be well approximated for a while even when the fluctuations are no longer Gaussian[30]. This is supported by the observation that, once the line zeroes have straightened on small scales at $t > t_{sp}$, the Gaussian field energy continues to match that possessed by a network of classical global strings with density n_{zero}.

4.5. The Role of the Ginzburg Regime

We have seen that the UV thermal fluctuations at time t come from the small τ part of the integration and are, approximately, proportional to $T(t)$. It is the thermal fluctuations that give rise to small-scale structure on the defects and prevent us from giving a scale-independent value to their density. Thus, if we reduce the temperature to absolute zero, the thermal fluctuations vanish and the density becomes well defined.

Our first observation is that, in practice, this never happens. In pressure quenches it is the critical temperature that changes, at almost constant temperature. Even in cooling, the final temperature is usually a substantial fraction of the critical temperature.

For the sake of simplicity, let us keep $T_f = O(T_c) < T_c$ and thereby take $T = T_c$ in $G_l(0,t)$ above. The necessary time-*independence* of $G^{IR}(t)$ for $t > t_{sp}$ is achieved by taking $\epsilon_{eff}(u) = O(u^{-1})$. In consequence, as t increases beyond t_{sp} the relative magnitude of the UV and IR contributions to $G_l(0,t)$ remains *approximately constant*. Further, since for $t = t_{sp}$,

$$e^{2\int_{t_0}^{t} du\,|\epsilon_{eff}(u)|}\, e^{-(\Delta t)^2/\bar{t}_Z^2} \approx 1, \qquad (65)$$

this ratio is the ratio at $t = t_{sp}$.

Nonetheless, as long as the thermal fluctuations are insignificant at $t = t_{sp}$ the density of line zeroes will remain largely independent of scale. This follows if $G''^{IR}(t_{sp}) \gg G''^{UV}_l(t_{sp})$, since $G''_l(0,t)$ becomes scale-independent later than $G_l(0,t)$. In [34] we showed that this is true provided

$$(\tau_Q/\tau_0)(1 - T_G/T_c) < C\pi^4, \tag{66}$$

where $C = O(1)$ and T_G is the Ginzburg temperature. That is, for slow quenches or quenches in which thermal fluctuations remain large, there are no identifiable defects at the spinodal time, since the line zeroes are highly fractal on small scales and the Zurek analysis breaks down.

In our self-consistent linearisation the situation never improves, although this is extreme. Of course, if we view line-zeroes through a lattice, they will be seen. Choose a different lattice, and we will see more, or less. Numerical simulations that identify vortices with line zeroes (and nothing more) on a fixed lattice are suspect until thermal fluctuations become unimportant. Certainly, for uniform quenches, defects are not formed at the end of the Ginzburg regime by thermal fluctuations freezing in, as originally suggested.

A simple way to see if defects can be produced by thermal fluctuations is to keep the system in the Ginzburg regime for a long period $\Delta t \gg t_{sp}$ and then drop out of it. Numerical simulations have been performed by Zurek and collaborators[40] that test this possibility. Empirically, the end result is not very different from the case of the linear quench that we have been discussing hitherto. The thermal fluctuations of the Ginzburg regime are *not* the source of defect production.

To get an analytic idea as to why this is the case we return to the linear approximation of the previous section. Even with more complicated temperature profiles the separation into long and short wavelength modes will still occur.

In particular, we expect the end-point behaviour (57) and (61) for the scale-dependent short wavelength fluctuations to be unchanged. As a result, $G^{UV}_l(t)$ and $G''^{UV}_l(t)$ are largely insensitive of the past history of the system, even if that history involved a long time Δt in the Ginzburg regime.

Suppose, as in Ref.40, we begin at temperature T_0, then drop to temperature T_G for a period Δt, and then drop to temperature $T = 0$. In comparison to spending the whole time at T_G, the strength of the short wavelength fluctuations in $G''^{UV}_l(t)$ is

$$O\left(\frac{1}{t^{3/2}}\left(\frac{T_0}{T_G} - \frac{\Delta t}{t}\right)\right) \approx 0 \tag{67}$$

for $t >> \Delta t$, with comparable behaviour for $G^{UV}_l(t)$. That is, if we end up at low temperature, our intermediate history is largely unimportant for the small-scale fluctuations, however that temperature is arrived at. In particular, short-distance scale-dependent fluctuations will be negligible at long times, whatever.

Similarly, provided the temperature is monotonically decreasing (since otherwise the effect on $G_l^{IR}(t)$ and $G''^{IR}_l(t)$ is dramatic) we do not expect a strong effect of past history on the IR contributions beyond a delay in the position $\bar{\tau}$ of the peak in τ by approximately $2\Delta\tau$. Since $\bar{\tau}(t)$ depends on where $\epsilon_{eff}(t-\tau/2)$ vanishes, we must have $\bar{\tau}(t) = O(t)$, largely insensitive to intermediate conditions, once t is large enough. On neglecting the (now negligible) UV terms, we have n_{zero} determined approximately by $\bar{\tau}(t)$ alone (giving the usual $n_{zero} \propto t^{-1}$ behaviour). To a good approximation this is what is seen in the numerical simulations[40]. To quantify the slight differences seen in the numerical simulations arising from different temperature profiles requires further work.

There is no way that defects can be produced by attempting to incubate them in the Ginzburg regime.

5. WHEN DO THE APPROACHES AGREE? RELATIVISTIC QFT

For QFT the situation is rather different. In the previous section, instead of working with the TDLG equation, we could have worked with the equivalent Fokker-Planck equation[41] for the probability $p_t[\Phi]$ that, at time $t > 0$, the measurement of ϕ will give the function $\Phi(\mathbf{x})$. When solving the dynamical equations for a hot quantum field it is convenient to work with probabilities from the start.

5.1. The Closed Time-path

Take $t = t_0$ as our starting time for the evolution of the complex field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. Suppose that, at this time, the system is in a pure state, in which the measurement of ϕ would give $\Phi_0(\mathbf{x})$. That is:-

$$\hat{\phi}(t=0,\mathbf{x})|\Phi_0, t=t_0\rangle = \Phi_0|\Phi_0, t=t_0\rangle. \tag{68}$$

The probability $p_t[\Phi]$ that, at time $t_f > t_0$, the measurement of ϕ will give the value Φ is $p_t[\Phi] = |\Psi_0|^2$, where Ψ_0 is the state-functional with the specified initial condition. As a path integral

$$\Psi_0 = \int_{\phi(t_0)=\Phi_0}^{\phi(t)=\Phi} \mathcal{D}\phi \, \exp\Big\{iS_t[\phi]\Big\}, \tag{69}$$

where $S_t[\phi]$ is the (time-dependent) action that describes how the field ϕ is driven by the environment and spatial and field labels have been suppressed (e.g. $\mathcal{D}\phi = \mathcal{D}\phi_1\mathcal{D}\phi_2$). Specifically, for $t > t_0$ the action for the field is taken to be

$$S_t[\phi] = \int dx\Big(\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{1}{2}\epsilon(t)M^2\phi_a^2 - \frac{1}{4}\lambda(\phi_a^2)^2\Big). \tag{70}$$

where $\epsilon(t)$ is as before. Henceforth we return to natural units in which $\xi_0 = \tau_0 = M^{-1} = 1$.

It follows that $p_t[\Phi]$ can be written in the closed time-path form

$$p_t[\Phi] = \int_{\phi^\pm(t_0)=\Phi_0}^{\phi^\pm(t)=\Phi} \mathcal{D}\phi^+\mathcal{D}\phi^- \; \exp\Big\{i\Big(S_t[\phi^+] - S_t[\phi^-]\Big)\Big\}, \tag{71}$$

where $\mathcal{D}\phi^\pm = \mathcal{D}\phi_1^\pm\mathcal{D}\phi_2^\pm$. Instead of separately integrating $\phi^\pm$ along the time paths $t_0 \leq t \leq t_f$, the integral can be interpreted as time-ordering of a field ϕ along the closed path $C_+ \oplus C_-$ where $\phi = \phi^+$ on C_+ and $\phi = \phi^-$ on C_-. When we extend the contour from t_f to $t = \infty$ either ϕ^+ or ϕ^- is an equally good candidate for the physical field, but we choose ϕ^+.

The choice of a pure state at time $t = t_0$ is too simple to be of any use. Let us assume that Φ is Boltzmann distributed at time $t = t_0$ at an effective temperature of $T_0 = \beta_0^{-1}$ according to the Hamiltonian $H[\Phi]$ corresponding to the action $S[\phi]$, in which ϕ is taken to be periodic in imaginary time with period β_0. We now have the explicit form for $p_t[\Phi]$,

$$p_t[\Phi] = \int_B \mathcal{D}\phi \, e^{iS_C[\phi]} \, \delta[\phi^+(t_f) - \Phi], \tag{72}$$

written as the time ordering of a single field along the contour $C = C_+ \oplus C_- \oplus C_3$, extended to include a third imaginary leg, where ϕ takes the values ϕ^+, ϕ^- and ϕ_3 on C_+, C_- and C_3 respectively, for which S_C is $S[\phi^+]$, $S[\phi^-]$ and $S_0[\phi_3]$. It is sufficient for our purposes to take the Boltzmann distribution on C_3 to be the *free-field* distribution with mass parameter $m^2 = M^2\epsilon_0$, corresponding to a temperature quench from an initial state at temperature $T_0 > T_c$, where $(T_0/T_c - 1) = \epsilon_0$.

As with condensed matter, it is not necessary to calculate $p_t[\Phi]$ directly since $G_{ab}(|\mathbf{x} - \mathbf{x}'|; t) = \langle\Phi_a(\mathbf{x})\Phi_b(\mathbf{x}')\rangle_t$ is given by

$$G_{ab}(|\mathbf{x} - \mathbf{x}'|; t) = \langle\phi_a(\mathbf{x}, t)\phi_b(\mathbf{x}', t)\rangle, \tag{73}$$

the equal-time thermal Wightman function with the given thermal boundary conditions.

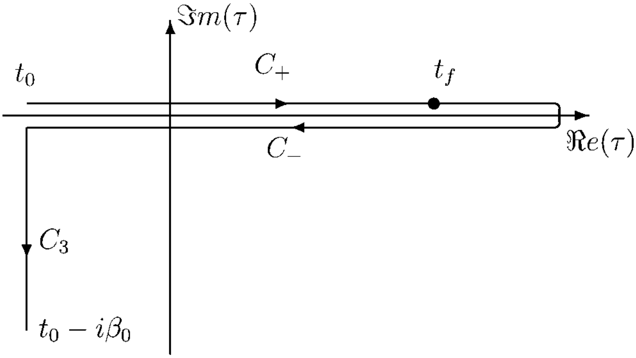

Fig. 1. The closed timepath contour $C_+ \oplus C_-$ with a third imaginary leg

5.2. The (Irrelevant) Freeze-in

Unlike for condensed matter we shall not postulate dissipative equations (with corresponding noise), but just calculate the evolution of the quantum field under its time-dependent Hamiltonian derived from (29). Insofar as the results look similar it is for totally different reasons.[For simplicity we continue to work in flat space-time.]

As for the condensed matter case, the interval $-\bar{t}_K \leq t \leq \bar{t}_K$ occurs in the *linear* regime, when the self-interactions are unimportant. The relevant equation for constructing the correlation functions of this one-field system is now the second-order equation

$$\frac{\partial^2 \phi_a}{\partial t^2} = -\frac{\delta F}{\delta \phi_a}, \tag{74}$$

for F of Eq.13. This is solvable in terms of the mode functions $\chi_k^{\pm}(t)$, identical for $a = 1, 2$, satisfying

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + \epsilon(t)\right]\chi_k^{\pm}(t) = 0, \tag{75}$$

subject to $\chi_k^{\pm}(t) = e^{\pm i\omega_{in}t}$ at large negative times, with incident frequency $\omega_{in} = \sqrt{\mathbf{k}^2 + \epsilon_0}$.

The diagonal correlation function $G(r,t)$ of Eq.34 is given as the equal-time propagator

$$G(r,t) = \int d\!\!\!{}^-{}^3k\, e^{i\mathbf{k}.\mathbf{x}} \chi_k^+(t)\chi_k^-(t)C(k) \tag{76}$$

where $C(k) = \coth(\omega_{in}(k)/2T_0)/2\omega_{in}(k)$ encodes the initial conditions.

An exact solution can be given[42] in terms of Airy functions. Dimensional analysis shows that, on ignoring the k-dependence of $C(k)$, appropriate for large r (or small k), $\bar{\xi}_K = \xi_{eq}(\bar{t}_K)$ of Eq.11 again sets the scale of the equal-time correlation function. Specifically,

$$G(r,0) \propto \int d\kappa \, \frac{\sin \kappa(r/\bar{\xi}_K)}{\kappa(r/\bar{\xi}_K)} \, F(\kappa), \tag{77}$$

where $F(0) = 1$ and $F(\kappa) \sim \kappa^{-3}$ for large κ.

Kibble's prediction for the large scale at which the field freezes in is correct but, as with condensed matter, has nothing to do with the formation of defects around $t = 0$. At $t = 0$ there are, as yet, no unstable modes and, as far as defects are concerned the field becomes ordered, as before, because of the exponential growth of long-wavelength modes, which stop growing once the field has sampled the groundstates. What matters is the relative weight of these modes in $P(k,t)$ (the 'Bragg' peak) to the fluctuating short wavelength modes, since the contribution of these latter is very sensitive to the cutoff l at which we look for defects. Only if their contribution to Eq.8 is small when field growth stops can a network of vortices be well-defined at early times. At $t = 0$ there is no Bragg peak and the density of line zeroes depends entirely on the scale at which we are looking.

5.3. The Spinodal Time and ξ_{zero}

We begin by extending the analysis from $t = 0$ to later times, still in the approximation of a *free* roll. This needs care for slow quenches since the back-reaction serves to hold the field in the vicinity of the intermediate ground-states $|\phi^2| = \phi_0^2(t)$ where, now

$$\phi_0^2(t) = \frac{-\epsilon(t)}{\lambda} = \frac{1}{\lambda}\frac{t}{\tau_Q} \tag{78}$$

for $t < \tau_Q$.

Prior to the completion of the quench at $t = \tau_Q$, the mode equation (75), now of the form

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 - \frac{t}{\tau_Q}\right]\chi_k^{\pm}(t) = 0, \tag{79}$$

is exactly solvable, as we saw earlier. As in the case of condensed matter previously, we have coarse-grained the field by introducing a simple cut-off at

$k = \Lambda = O(M)$ in dimensional units, or $l = \Lambda^{-1}$. In natural units $\bar{l} = O(1)$. The unstable exponentially growing modes appear when

$$\Omega_k^2(t) = -\mathbf{k}^2 + \frac{t}{\tau_Q} > 0. \tag{80}$$

For fixed k this occurs when $t > t_k = \tau_Q k^2$ (or $k^2 = k_t^2 = t/\tau_Q$). As before, we are not looking for vortices within vortices, but coarse-grain to the cold vortex size.

The WKB solution is adequate for our purposes. $G_l(r;t)$ can be written as $G_l(r;t) = G^{exp}(r;t) + G_l^{osc}(r;t)$ where

$$G^{exp}(r;t) \simeq T_0 \int_{|\mathbf{k}|<k_t} d\!\!\!/^3k \frac{S_k(t)}{|k_t^2 - k^2|^{1/2}}\, e^{i\mathbf{k}.\mathbf{x}} |\alpha_k^+ I_{1/3}(S_k(t)) + \alpha_k^- I_{-1/3}(S_k(t))|^2 \tag{81}$$

has exponentially growing long wavelength modes and

$$G_l^{osc}(r;t) \simeq T_0 \int_{\Lambda>|\mathbf{k}|>k_t} d\!\!\!/^3k \frac{S_k(t)}{|k_t^2 - k^2|^{1/2}}\, e^{i\mathbf{k}.\mathbf{x}} |\alpha_k^+ J_{1/3}(S_k(t)) - \alpha_k^- J_{-1/3}(S_k(t))|^2 \tag{82}$$

has short wavelength oscillatory modes. In both cases

$$S_k(t) = \int_{t_k^-}^{t} dt'\, |\Omega_k(t')| = \frac{2}{3}\frac{1}{\sqrt{\tau_Q}}|t - t_k|^{3/2}. \tag{83}$$

We stress that, at the level of a free field (or any self-consistent Gaussian) inserting a cut-off is identical to *integrating* out the short wavelength modes, for an unbiased initial state. Provided we are far from the transition we have incorporated the initial data into the $\alpha^{\pm}$ in (81) and (82), which have no λ or τ_Q dependence.

For large t the integrand in (81) will have its 'Bragg' peak at some $k_0(t) \to 0$ as $t \to \infty$, once the angular integrals have been performed. Assuming that $k_0(t) \ll k_t$ the upper bound in the integral can be dropped and $|k_t^2 - k^2|$ approximated by $|k_t^2|$, knowing that there is no singularity at $k = k_t$. With nothing to stop $|\alpha_k^+ + \alpha_k^-|^2$ behaving like a nonzero constant in the vicinity of $k = 0$, it can be treated as slowly varying and the integral approximated as[13]

$$\begin{aligned} G^{exp}(r;t) &\propto T\left(\frac{\tau_Q}{t}\right)^{1/2} e^{(4/3)(t/\bar{t}_K)^{3/2}} \int_{|\mathbf{k}|<1} d\!\!\!/^3k\, e^{i\mathbf{k}.\mathbf{x}}\, e^{-2\sqrt{t\tau_Q}k^2} \\ &\propto \frac{T}{|\epsilon(t)|^{1/2}}\left(\frac{1}{\sqrt{t\tau_Q}}\right)^{3/2} e^{(4/3)(t/\bar{t}_K)^{3/2}} e^{-r^2/2\xi^2(t)} \end{aligned} \tag{84}$$

on performing the k^2 expansion of the exponent, where

$$\xi^2(t) = \sqrt{t\tau_Q} \tag{85}$$

in natural units $\tau_0 = 1$. Although, like Eq.84, the expression Eq.85 is not supposed to be valid for small t, it does embody the Kibble freezeout condition in satisfying $\dot{\xi}(\bar{t}) = O(1)$.

There are two immediate differences between (84) and its condensed matter counterpart (36). Firstly, in (84) the length $\xi(t)$ of (85) is *both* the correlation length of the field *and* exactly the separation of line zeroes $\xi(t)_{zero}$ as defined through (26). Secondly, insofar that the time-dependence of $\xi(t)$ is a guide to early evolution, it grows as $t^{1/4}$, rather than the more customary $t^{1/2}$ of condensed matter.

The calculations above were for a free roll. Let us suppose, provisionally, that the backreaction exerts its influence over such a short time that, in effect, it is if it were an *instantaneous* brake to domain growth. The provisional freeze-in time t^* is then, effectively, the time it takes to reach the transient groundstate $\phi_0^2(t)$. That is, $G(0;t^*) = O(\phi_0^2(t^*))$, giving

$$(\sqrt{t^*\tau_Q})^{3/2}e^{-(4/3)(t^*/\bar{t}_K)^{3/2}} = O\Big(\lambda^{1/2}(\frac{\tau_Q}{t^*})^{3/2}\Big). \qquad (86)$$

This gives

$$\frac{t^*}{\bar{t}_K} \approx \frac{1}{2}\left[\ln\left(\frac{\tau_Q/\tau_0}{(1-T_G/T_c)^2}\right)\right]^{2/3}, \qquad (87)$$

and we have used $1-T_G/T_c = \lambda$, to be specific. For small coupling $t^*/\bar{t}_K > 1$, but will be $O(1)$. A comparison with (42) shows strong similarities with condensed matter and that, yet again, t^* is insensitive to the parameters of the theory. As far as the separation of scales is concerned, we have the same effect qualitatively if we had taken t^* as the time for the field to reach the final ground state as $|\phi^2| = 1/\lambda$, rather than the provisional ground states $\phi_0^2(t)$. Hence we can identify t^* with t_{sp} to a good approximation and we shall not distinguish between them.

At this qualitative level the correlation length, and line-zero separation length at the spinodal time is

$$\xi^2(t_{sp}) = \xi^2(t_{sp})_{zero} \simeq \bar{\xi}_K^2(\frac{t_{sp}}{\bar{t}_K})^{1/2} = O(\bar{\xi}_K^2). \qquad (88)$$

5.4. Fluctuations

All of this assumes that the oscillatory wavelength terms (82) can be ignored. Although we have adopted a cutoff at $l = O(\xi_0) = O(M^{-1})$, there is still a contribution from modes near M^{-1}. When we take these modes into account the density of line zeroes at t_{sp} can be written as

$$\bar{n}_{zero} = \frac{1}{f^2}\bar{n}_K(1+E). \qquad (89)$$

The error term $E = O(\lambda^{1/2}(1/\tau_Q)^{4/3}(\ln(1/\lambda))^{-1/3})$ is due to oscillatory modes, sensitive to the cut-off Λ. The condition $E^2 \ll 1$, necessary for the line-zero density to be insensitive to scale, is satisfied[39] if

$$(\tau_Q/\tau_0)^2(1 - T_G/T_c) < C, \tag{90}$$

where $C = O(1)$. This is the QFT counterpart to Eq.66. If this is the case then $f^2 = 2\pi\sqrt{t_{sp}/\bar{t}_K}$, that shows yet again that the causal estimate can be correct, but for different reasons.

5.5. Backreaction in QFT

To improve upon the free-roll result we adopt a mean-field approximation along the lines of [37,43], as we did for the condensed matter systems earlier.

$G(r;t)$ still has the mode decomposition of (76), but the modes $\chi_k^{\pm}$ now satisfy the equation

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + \epsilon(t) + \lambda\langle\Phi^2(\mathbf{0})\rangle_t\right]\chi_k^{\pm}(t) = 0, \tag{91}$$

where we have taken $N = 2$ in a large-N ($O(N)$) theory.

The end result is[37] that, on making a single subtraction at $t = 0$,

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + \epsilon(t) + \lambda\int d\!\!\!{}^-{}^3p\, C(p)[\chi_p^+(t)\chi_p^-(t) - 1]\right]\chi_k^{\pm}(t) = 0, \tag{92}$$

which we write as

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 - \epsilon_{eff}(t)\right]\chi_k(t) = 0. \tag{93}$$

On keeping just the unstable modes in $\langle\Phi^2(\mathbf{0})\rangle_t$ then, as it grows, its contribution to (92) weakens the instabilities, so that only longer wavelengths become unstable. At t^* the instabilities shut off, by definition, and oscillatory behaviour ensues.

Explicit calculation shows that the backreaction has little effect for times $t < t_{sp}$. For $t > t_{sp}$ oscillatory modes take over the correlation function and we expect oscillations in $G(k;t)$.

In practice the backreaction rapidly forces $\epsilon_{eff}(t)$ towards zero if the coupling is not too small[37]. In that case the end result is a new power spectrum, obtained by superimposing oscillatory behaviour onto the old spectrum. As a gross oversimplification, the contribution from the earlier exponential modes alone can only be to contribute terms something like

$$G(r;t) \propto \frac{T}{|\epsilon(t_{sp})|^{1/2}} e^{(4/3)(t_{sp}/\bar{t}_K)^{3/2}} \int_{|\mathbf{k}|<1} d\!\!/^3 k\, e^{i\mathbf{k}.\mathbf{x}}\, e^{-2\sqrt{t_{sp}\tau_Q}k^2} \times \left[\cos k(t-t_{sp}) + A(k)\sin k(t-t_{sp})\right]^2 \tag{94}$$

to G, where the details of $A(k)$ do not concern us.

The $k=0$ mode of Eq.94 encodes the simple solution $\chi_{k=0}(t) = a + bt$ when $\mu^2 = 0$. This has weak causality built into it. Specifically, for $r, t \rightarrow \infty$,

$$G(r,t) \approx \frac{C}{r}\theta(2t/r - 1). \tag{95}$$

It has to be said that this approximation should not be taken very seriously for large t, since we would expect rescattering to take place at times $\Delta t = O(1/\lambda)$ in a way that the Gaussian approximation precludes.

However, it demonstrates that weak causality, implemented by the Goldstone particles of the self-consistent theory, is likely to have little effect on the density of line-zeroes that we expect to mature into fully classical vortices, depending as it does largely on the behaviour of G at $r = 0$.

Finally, all of the above was predicated on the long wavelengths of the field having decohered by t_{sp}. It is to this we now turn.

6. QFT: THE DECOHERENCE TIME

The empirical TDGL condensed matter Langevin equation (30) that we adopted earlier could have been rewritten as a Fokker-Planck equation[41] for the functional probability density $p_t[\phi]$. Our discussion of QFT in the previous Section also encoded the corresponding $p_t[\phi]$. The difference is that, for QFT, we have yet to ascertain whether we can use $p_t[\phi]$ to count classical configurations, like defects. The reason was given earlier: classical defects cannot be identified until adjacent path histories decohere (i.e. until quantum interference is negligible), since only then will $p_t[\phi]$ (approximately) obey classical laws.

The mechanism for decoherence is the interaction of the field with its environment. In practice, all fields to which the ϕ-field couple will help it decohere. In addition, the short wavelength modes, which play no role in field ordering and subsequent defect production, are an important part of the environment. For exemplary purposes we consider only the effects of short wavelengths of the ϕ-field itself on decoherence[21,22]. A more complete discussion that includes the effects of additional fields has been given elsewhere[44]. At the level that we have handled QFT so far (a self-consistent Gaussian) our cut-off of short wavelength modes is the same as integrating them out, and there is no decoherence. We shall need to go beyond a free-field approximation to see any effects.

Decoherence is measured in terms of dissipation. A byproduct of establishing decoherence is that we recover a Fokker-Planck-like equation for $p_t[\phi]$ or, equivalently, a generalised Langevin stochastic equation for the classical field. Together with this should be included the explicit dissipation in the early universe as a consequence of the expansion of space-time. We shall not treat that here, despite its importance (see Refs.39,36,45).

To show the main ideas we shall consider the simpler system of a single real field (whose defects are domain walls) undergoing a very rapid quench. If we take the quench time $\tau_Q \approx \tau_0$ this is equivalent to taking an instantaneous quench, and this we do. There is no difficulty in taking slower quenches in principle, it is just that the calculations have yet to be performed.

That is, yet again we divide the field $\phi(x)$ into the long wavelength 'system'-field $\phi_<(x)$ and the short wavelength 'environment'-field $\phi_>(x)$,

$$\phi_<(x) = \int_{|\mathbf{k}|<1} d^3x\, \phi(\mathbf{x},t)\, e^{i\mathbf{k}.\mathbf{x}}, \quad \phi_>(x) = \int_{|\mathbf{k}|>1} d^3x\, \phi(\mathbf{x},t)\, e^{i\mathbf{k}.\mathbf{x}}. \tag{96}$$

$|\mathbf{k}| = 1$ is the dividing line between stable and unstable modes. With $\tau_Q \approx \tau_0$ (90) is automatically satisfied and varying the separation wavelength by a small amount will have no effect.

Since it is the system-field $\phi_<$ whose behaviour changes dramatically on taking T through T_c, we adopt an *instantaneous* quench for T from T_0 to $T_f = 0$ at time $t = 0$, in which $\epsilon(T)$ changes sign and magnitude instantly, concluding with the value $\epsilon(t) = -1, t > 0$. Meanwhile, for simplicity the $\phi_>$ mass is held at the original value ϵ_0.

The classical action separates as

$$S[\phi] = S_{\rm syst}[\phi_<] + S_{\rm env}[\phi_>] + S_{\rm int}[\phi_<, \phi_>], \tag{97}$$

where

$$S_{\rm syst}[\phi_<] = \int d^4x \left\{ \frac{1}{2}\partial_\mu\phi_<\partial^\mu\phi_< - \frac{1}{2}\epsilon(t)\phi_<^2 - \frac{\lambda}{4!}\phi_<^4 \right\},$$
$$S_{\rm env}[\phi_>] = \int d^4x \left\{ \frac{1}{2}\partial_\mu\phi_>\partial^\mu\phi_> - \frac{1}{2}\epsilon_0\phi_>^2 - \frac{\lambda}{4!}\phi_>^4 \right\},$$
$$S_{\rm int}[\phi_<, \phi_>] = -\frac{\lambda}{4}\int d^4x \phi_<^2(x)\phi_>^2(x).$$

We have not included $\phi_<^3\phi_>$ and $\phi_<\phi_>^3$ terms in $S_{\rm int}[\phi_<, \phi_>]$ since the former does not contribute to very long wavelengths and the latter only appears at higher order in λ. Our answers will have little to do with the more familiar calculations of decoherence, which are motivated by quantum mechanical systems with *linear* coupling to an environment.

Beginning from an initial thermal distribution peaked around $\phi = 0$ we follow the evolution of the system under the influence of the short wavelength environment. The spinodal time is now $t_{\rm sp} \sim \frac{1}{2}\ln[1/\lambda]$.

6.1. The Reduced Density Matrix

Whereas the field probabilities are just the diagonal elements of the density matrix $\hat{\rho}$, decoherence is determined from the full density matrix

$$\rho[\phi^+_<, \phi^+_>, \phi^-_<, \phi^-_>, t] = \langle \phi^+_< \phi^+_> | \hat{\rho} | \phi^-_< \phi^-_> \rangle. \tag{98}$$

Our Gaussian initial conditions give an uncorrelated thermal initial behaviour $\hat{\rho}[T_0] = \hat{\rho}_<[T_0]\hat{\rho}_>[T_0]$ at temperature T_0.

The relevant object is the reduced density matrix. It describes the evolution of the system under the influence of the environment, and is defined by

$$\rho_{\rm r}[\phi^+_<, \phi^-_<, t] = \int \mathcal{D}\phi_> \; \rho[\phi^+_<, \phi_>, \phi^-_<, \phi_>, t]. \tag{99}$$

The environment will have had the effect of making the system essentially classical once $\rho_r(t)$ is, effectively, diagonal.

Its temporal evolution is given by

$$\rho_{\rm r}[\phi^+_{<{\rm f}}, \phi^-_{<{\rm f}}, t] = \int d\phi^+_{<{\rm i}} \int d\phi^-_{<{\rm i}} J_{\rm r}[\phi^+_{<{\rm f}}, \phi^-_{<{\rm f}}, t | \phi^+_{<{\rm i}}, \phi^-_{<{\rm i}}, t_0] \;\; \rho_{\rm r}[\phi^+_{<{\rm i}} \phi^-_{<{\rm i}}, t_0], \tag{100}$$

where $J_{\rm r}$ is the reduced evolution operator

$$J_{\rm r}[\phi^+_{<{\rm f}}, \phi^-_{<{\rm f}}, t | \phi^+_{<{\rm i}}, \phi^-_{<{\rm i}}, t_0] = \int_{\phi^+_{<{\rm i}}}^{\phi^+_{<{\rm f}}} \mathcal{D}\phi^+_< \int_{\phi^-_{<{\rm i}}}^{\phi^-_{<{\rm f}}} \mathcal{D}\phi^-_< \; e^{i\{S[\phi^+_<] - S[\phi^-_<]\}} \; F[\phi^+_<, \phi^-_<]. \tag{101}$$

The Feynman-Vernon[46] influence functional $F[\phi^+, \phi^-]$ is defined as

$$\begin{aligned} F[\phi^+_<, \phi^-_<] &= \int d\phi^+_{>{\rm i}} \int d\phi^-_{>{\rm i}} \; \rho_\phi[\phi^+_{>{\rm i}}, \phi^-_{>{\rm i}}, t_0] \int d\phi_{>{\rm f}} \\ &\times \int_{\phi^+_{>{\rm i}}}^{\phi_{>{\rm f}}} \mathcal{D}\phi^+_> \int_{\phi^-_{>{\rm i}}}^{\phi_{>{\rm f}}} \mathcal{D}\phi^-_> \exp\left(i\{S[\phi^+_>] + S_{\rm int}[\phi^+_<, \phi^+_>]\right) \\ &\times \exp\left(-i\{S[\phi^-_>] + S_{\rm int}[\phi^-_<, \phi^-_>]\}\right), \end{aligned}$$

where we have used the closed time-path of Fig.1. From the influence functional we define the influence action $\delta A[\phi^+, \phi^-]$ as $F[\phi^+_<, \phi^-_<] = \exp i\delta A[\phi^+_<, \phi^-_<]$

We calculate to one loop. After defining

$$\Delta = \frac{1}{2}(\phi_<^{+2} - \phi_<^{-2}) \quad , \quad \Sigma = \frac{1}{2}(\phi_<^{+2} + \phi_<^{-2});$$

and using simple and well known identities between propagators, the real and imaginary parts of the influence action can be written as

$$\mathrm{Re}\delta A = \frac{\lambda^2}{4}\int d^4x \int d^4y \Delta(x) K_{\mathrm{q}}(x-y)\Sigma(y), \tag{102}$$

$$\mathrm{Im}\delta A = -\frac{\lambda^2}{8}\int d^4x \int d^4y \Delta(x) N_{\mathrm{q}}(x,y)\Delta(y), \tag{103}$$

where $K_{\mathrm{q}}(x-y) = \mathrm{Im}G_{++}^2(x,y)\theta(y^0 - x^0)$ is the dissipation kernel and $N_{\mathrm{q}}(x-y) = \mathrm{Re}G_{++}^2(x,y)$ is the noise (diffusion) kernel.

6.2. The Master Equation

The first step in the evaluation of the evolution equation for $\hat{\rho}$, *the master equation*, is the calculation of the density matrix propagator J_{r} from Eq. (101). We first perform a saddle point approximation

$$J_{\mathrm{r}}[\phi_{<\mathrm{f}}^+, \phi_{<\mathrm{f}}^-, t|\phi_{<\mathrm{i}}^+, \phi_{<\mathrm{i}}^-, t_0] \approx \exp iA[\phi_{\mathrm{cl}}^+, \phi_{\mathrm{cl}}^-], \tag{104}$$

where $\phi_{\mathrm{cl}}^\pm$ is the solution of the equation of motion $\frac{\delta ReA}{\delta\phi_<^+}|_{\phi_<^+=\phi_<^-} = 0$ with boundary conditions $\phi_{\mathrm{cl}}^\pm(t_0) = \phi_{<\mathrm{i}}^\pm$ and $\phi_{\mathrm{cl}}^\pm(t) = \phi_{<\mathrm{f}}^\pm$. Next, we assume that the system-field contains only one Fourier mode with $\vec{k} = \vec{k}_0$, which we identify with the Bragg peak when it arises.

To estimate t_D it is sufficient to calculate the correction to the usual unitary evolution coming from the noise kernel. For clarity we drop the suffix f on the final state fields. If $\Delta = (\phi_<^{+2} - \phi_<^{-2})/2$ for the *final* field configurations, then the master equation for $\rho_r(t) = \langle\phi_<^+|\hat{\rho}|\phi_<^-\rangle$ is

$$i\dot{\rho}_{\mathrm{r}} = \langle\phi_<^+|[\hat{H}, \hat{\rho}_{\mathrm{r}}]|\phi_<^-\rangle - i\frac{\lambda^2}{8}V\Delta^2 D(k_0,t)\rho_{\mathrm{r}} + ... \tag{105}$$

The volume factor V arises because we are considering a density matrix which is a functional of two different field configurations, $\phi_<^\pm(\vec{x}) = \phi_<^\pm \cos\vec{k}_0.\vec{x}$ spread over all space. For $k_0 < 1$ the diffusion coefficient $D(k_0,t)$ shows the inevitable exponential growth ($T_0 = O(T_c)$)

$$D(k_0,t) \sim T_c^2\Omega(k_0)\exp[2\Omega(k)t], \tag{106}$$

where $\Omega^2(k) = M^2 - k^2$, (or $1 - k^2$ in natural units), associated with the instability of the k_0 mode. Eq.106 is only valid once $2\Omega(k)t > 1$, which will have happened by $t = t_{sp}$. Once $t > t_{\rm sp}$ the diffusion coefficient stop growing, and oscillates around $D(k_0, t = t_{\rm sp})$.

On the other hand, for short times $t \ll 1$ we find that

$$D(k_0, t) \sim T_c^2 t \tag{107}$$

up to coefficients $\mathcal{O}(1)$.

6.3. The Decoherence Time

We estimate $t_{\rm D}$ for the model by considering the approximate solution to the master equation (105),

$$\rho_{\rm r}[\phi^+_<, \phi^-_<; t] \approx \rho^{\rm u}_{\rm r}[\phi^+_<, \phi^-_<; t] \ \exp\Big[- V\Gamma \int_0^t ds \ D(k_0, s)\Big], \tag{108}$$

where $\rho^{\rm u}_{\rm r}$ is the solution of the unitary part of the master equation (i.e. without environment). In terms of the fields $\bar{\phi} = (\phi^+_< + \phi^-_<)/2$, and $\delta = (\phi^+_< - \phi^-_<)/2$, $\Gamma = (\lambda^2/8)\bar{\phi}^2\delta^2$.

The system will decohere when the non-diagonal elements of the reduced density matrix are much smaller than the diagonal ones. We therefore look at the ratio

$$\left|\frac{\rho_{\rm r}[\bar{\phi}+\delta, \bar{\phi}-\delta; t]}{\rho_{\rm r}[\bar{\phi}, \bar{\phi}; t]}\right| \approx \left|\frac{\rho^{\rm u}_{\rm r}[\bar{\phi}+\delta, \bar{\phi}-\delta; t]}{\rho^{\rm u}_{\rm r}[\bar{\phi}, \bar{\phi}; t]}\right| \times \ \exp\Big[- V\Gamma \int_0^t ds \ D(k_0, s)\Big]. \tag{109}$$

In general, it is not possible to obtain an analytic expression for the ratio of density matrices that appears in Eq.(109) since we do not even know the diagonal matrix well. However, since diagonalisation will be found to occur for $t < t_{sp}$ it is sufficient to neglect the self-coupling of the system field in the diagonal matrix elements. In this case the unitary density matrix remains Gaussian at all times as

$$\left|\frac{\rho^{\rm u}_{\rm r}[\bar{\phi}+\delta, \bar{\phi}-\delta; t]}{\rho^{\rm u}_{\rm r}[\bar{\phi}, \bar{\phi}; t]}\right| = \exp[-T_{\rm c}\delta^2 p^{-1}(t)], \tag{110}$$

where $p^{-1}(t)$, essentially[47–49] $\langle\phi^2\rangle_t^{-1}$, decreases exponentially with time to a value $\mathcal{O}(\lambda)$. In the unitary part of the reduced density matrix the non-diagonal terms are not suppressed. [This should not be confused with the observation that the unitary Gaussian density matrix does show classical correlation, whereby the Wigner functional becomes localised in phase space about its classical solutions[48]. However, this has nothing to do with eliminating quantum interference between different field histories.] In order to obtain classical behaviour, the relevant part of the reduced density matrix is the term proportional to the diffusion coefficient in Eq.(109), since it is this that enforces its diagonalisation.

The decoherence time t_D can be defined as the solution to

$$1 \approx V\Gamma \int_0^{t_D} ds\ D(k_0, s). \tag{111}$$

It corresponds to the time after which we are able to distinguish between two different field amplitudes (inside a given volume V).

To estimate t_D we have to fix the values of V, δ, and $\bar{\phi}$. V is understood as the minimal volume inside which we do not accept coherent superpositions of macroscopically distinguishable states for the field. Thus, our choice is that this volume factor is $\mathcal{O}(M^{-3}) = \mathcal{O}(1)$ in dimensionless units, since the Compton wavelength is the smallest scale at which we need to look. Inside this volume, we do not discriminate between field amplitudes which differ by $\mathcal{O}(1)$, and therefore we take $\delta \sim \mathcal{O}(1)$. For $\bar{\phi}$ we set $\bar{\phi}^2 = \mathcal{O}(1/\lambda)$, its post-transition value.

From the short-times expression for the diffusion coeffient Eq.(107) is very easy to show that decoherence does not occur while $\mu t \ll 1$ due to the small value of the coupling constants. Consequently, in order to evaluate the decoherence time in our model, we have to use Eq.(106). We obtain

$$\exp[2t_D] = O(1) \tag{112}$$

from which it follows that

$$1 < t_D < t_{\rm sp}. \tag{113}$$

This result is welcome in that it suggests that our previous analysis can probably go ahead to slower quenches. Moreover, if (113) is valid there we anticipate that classical defects can appear as soon as the spinodal time is reached. However, some caution is required since perturbation theory is, generally, not valid for non-equilibrium processes. Fortunately, within the same one-loop framework it is not difficult to find circumstances in which (113) holds until t_{sp}. In particular if, in addition to the short wavelength modes, we include a large number of weakly coupled external fields to decohere the long wavelength modes (without any linear couplings), the result (113) survives. Details are given elsewhere[44].

Finally, once the long wavelength modes have decohered they satisy stochastic equations of a generalised Langevin form[21,22,50]

$$\partial^2\phi(x) - \phi + \frac{\lambda}{6}\phi^3(x) + \frac{\lambda^2}{2}\phi(x)\int d^4y\, K(x,y)\phi^2(y) = \lambda\phi(x)\eta(x), \quad (114)$$

where the Gaussian noise η satisfies $\langle\eta(x)\eta(0)\rangle \propto N(x)$ in this approximation.

Since K is retarded the integral describes Landau damping. Because of the instantaneous nature of the quench we shall not pursue this equation beyond saying that, even given that, with its weak $O(\lambda)$ multiplicative noise and weak quasi-dissipative term, it is very different from the generalised relativistic Langevin equation invoked in Refs.2,3.

7. EXPERIMENTS: CONDENSED MATTER

Now that we understand the Kibble-Zurek bounds better, we give a brief review of the main experimental evidence, as well as suggesting further experiments that could provide further support for them.

Our first observation is that, in QFT, there is no evidence for or against Kibble's bounds. Although we have every confidence in the standard model of Electroweak unification (and its breaking) in the early universe, we only have very circumstantial evidence for phase transitions at the higher energies that are of more interest to us for defect formation. In particular, to date there are no reliable sightings of any topological defect that could have been produced in the early universe. At best we have possible indirect evidence, such as the suggestion that very high energy cosmic rays arise from the intersection and release of energy from the high energy density cores of cosmic strings. [There is a parallel in condensed matter in quasiparticle production due to the interaction of vortices in ^4He]. The original hope that large-scale structure was determined by cosmic strings (vortices) seems unfounded[9].

On the other hand, for condensed matter systems there are several experiments that have been devised to check Zurek's predictions.

7.1. Superfluid ^3He

We begin with superfluid ^3He since, to date, it provides the strongest support for the validity of Zurek's bounds. ^3He is a *fermion* but, somewhat as in a BCS superconductor, these fermions form the counterpart to Cooper pairs. However, whereas the (electron) Cooper pairs in a superconductor form a ^1S state, the ^3He pairs form a ^3P state. The order parameter $A_{\alpha i}$

is a complex 3×3 matrix $A_{\alpha i}$. There are two distinct superfluid phases, depending on how the global $SO(3) \times SO(3) \times U(1)$ symmetry is broken. If the normal fluid is cooled at low pressures, it makes a transition to the $^3He - B$ phase, in which $A_{\alpha i}$ takes the form $A_{\alpha i} = R_{\alpha i}(\omega)e^{i\Phi}$, where R is a real rotation matrix, corresponding to a rotation through an arbitrary ω[15]. Although more complicated than other systems it can be easier to count vortices, since one can use NMR to detect them.

So far, experiments have been of two types. In the Helsinki experiment[11] superfluid $^3He - B$ in a rotating cryostat is bombarded by slow neutrons. Each neutron entering the chamber releases 760 keV, via the reaction $n +^3 He \rightarrow p +^3 He + 760keV$. The energy goes into the kinetic energy of the proton and triton, and is dissipated by ionisation, heating a region of the sample above its transition temperature. The heated region then cools back through the transition temperature, creating vortices. Vortices above a critical size grow and migrate to the centre of the apparatus, where they are counted by an NMR absorption measurement. The quench is very fast, with $\tau_Q/\tau_0 = O(10^3)$. Agreement with Eq.8 is good.

The second type of experiment has been performed at Grenoble and Lancaster[12]. Rather than count individual vortices, the experiment detects the total energy going into vortex formation. As before, ^3He is irradiated by neutrons. After each absorption the energy released in the form of quasiparticles is measured, and found to be less than the total 760 keV. This missing energy is assumed to have been expended on vortex production. Again, agreement with Zurek's prediction Eq.8 is good.

7.2. ^4He Experiments

In ^4He the bose superfluid has as its order parameter a complex field ϕ, whose squared modulus $|\phi|^2$ is the superfluid density. The global $U(1)$ symmetry breaks to yield simple vortices, whose winding number measures the superflow around them.

The experiments in ^4He, conducted at Lancaster, follow Zurek's original suggestion. A sample of normal fluid helium, in a container with bellows, is expanded so that it becomes superfluid at essentially constant temperature. That is, we change $1 - T/T_c$ from negative to positive by increasing T_c. As the system goes into the superfluid phase a network of vortices is formed, which are detected by measuring the attenuation of second sound within the bellows. A mechanical quench is slow, with τ_Q some tens of milliseconds, and $\tau_Q/\tau_0 = O(10^{10})$. Two experiments have been performed. In the first[51] fair agreement was found with the prediction Eq.8. However, there were potential problems with hydrodynamic effects at the bellows, and at the capillary with which the bellows were filled. A second experiment[14], designed to minimise these and other problems has failed to see any vortices whatsoever.

However, unlike the experiments with ^3He, the experimental prediction for ^4He needs more than causality to go further. The density of vortices n is assumed to obey Vinen's equation[52]

$$\frac{\partial n_{def}}{\partial t} = -\chi_2 \frac{\hbar}{m} n_{def}^2. \tag{115}$$

This behaviour, requiring $\xi_{def}(t) \propto t^{1/2}$ is the general behaviour for non-conserved order parameters. We stress that causality does not provide a value for χ_2.

In our simple TDGL model, in which this behaviour follows at early times in the absence of thermal fluctuations, $\chi_2 = 4\pi\hbar\Gamma = 4\pi\hbar/\alpha_0\tau_0$. Taking $\tau_0 \approx 8.0 \times 10^{-12}s$ and $\xi_0 \approx 5.6\mathring{A}$ the mean-field approximation for ^4He gives $\chi_2 \approx 5 \times 4\pi$. [It would have the value 4π if we motivated τ_0 from the Gross-Pitaevskii equation[1], in which $\alpha_0\tau_0 = \hbar$.] This decay law is assumed in the analysis of the Lancaster experiments, in which the empirical value of χ_2 was not taken from quenches, but turbulent flow experiments. It was suggested[14] that $\chi_2 \approx 0.005$, orders of magnitude smaller than our suggestion above. Although the TDLG theory is not very reliable for ^4He, if our estimate is at all sensible it does imply that vortices produced in a *temperature* quench decay much faster than those produced in turbulence.

As a separate observation, we note that the large value of f^2 in the prefactor of n_{zero} is, in itself, almost enough to make it impossible to see vortices in ^4He experiments, should they be present.

To compound the problem, for early time at least, thermal fluctuations are large in the Lancaster experiments. With $\tau_Q/\tau_0 = O(10^{10})$ and a Ginzburg regime so large that $(1 - T_G/T_c) = O(1)$ the inequality (66) is hugely violated. In such circumstances the density of zeroes $n_{zero} = O(l^{-2})$ after t_{sp} depends exactly on the scale l at which we look and they are not candidates for vortices. Since the whole of the quench takes place within the Ginzburg regime this is not implausible. However, the Ginzburg-Landau theory is not reliable for ^4He and, even though the thermal noise never switches off, there is probably no more than a postponement of vortex production.

7.3. Superconductors: Flux Through a Surface

The situation with *local* symmetries, in which the ϕ-field interacts with the electromagnetic fields, is different, and only beginning to be understood[53].

Whereas the 'domains' of the global symmetry breaking are characterised by approximately constant field phases, for a local (gauge) symmetry the phase can be changed at will by a gauge transformation (except at a zero of the field where it is not defined). Phase correlation lengths are not physical observables. Nonetheless, vortices are present, identified by their magnetic flux.

For the relativistic case discussed in Ref.53 it is Landau damping that controls the relaxation time, commensurate with our earlier comments on stochastic equations. Details are given elsewhere in these proceedings[54]. In this case it is not clear if causality provides a useful constraint, except in the case of extremely weak electromagnetic coupling, very fast transitions or low temperatures[2] when, for condensed matter, we have (7) for the density of Abrikosov vortices[30].

Even if the situation were clear here, there is a separate issue as to how we could confirm any predictions. Since total flux (proportional to the number of vortices *minus* the number of antivortices) is the gauge-invariant measurable quantity, rather than the density (the number of vortices *plus* the number of antivortices), we have to infer the separation $\bar{\xi}_{def}$ differently.

We assume a temperature quench of a superconductor with zero average flux. Consider a closed path in the superconductor with circumference $C \gg \bar{\xi}$. The phase difference θ_C around the path is gauge invariant and we estimate the r.m.s phase difference along the path as

$$\Delta\theta_C = O(\sqrt{C/\bar{\xi}_{def}}). \tag{116}$$

The variance of magnetic flux within the path is then

$$\Delta\Phi_C = O\left(\frac{\Phi_0}{2\pi}\sqrt{\frac{C}{\bar{\xi}_{def}}}\right), \tag{117}$$

where $\Phi_0 = h/2e$ is the basic unit of magnetic flux. Since topological charge is conserved, there is no haste to measure the trapped flux. Such an experiment has been performed[55] with a high-T_c superconductor, but there is no agreement with (117) from its simple estimates. The reason is unclear.

In this context we should also mention the production of vortices produced in the wake of Bose-Einstein condensation, for which the Zurek mechanism also applies[20], but for which the relevant experiments have yet to be performed.

7.4. Annular experiments

As an alternative way to check the predictions for $\bar{\xi}_{def}$ by measuring conserved topological charge, Zurek suggested experiments with annuli of condensed matter. If we quenched a superconducting ring of circumference C we would expect the variance in flux through it to be given also by (117), provided its width is is less than $\bar{\xi}_{def}$, and assuming that the fluctuations in the electromagnetic field are negligible.

In fact, it is more convenient to quench annular Josephson Junctions, in which two rings of superconductor are held apart by an oxide layer through which Cooper pairs can tunnel. Let the complex scalar Higgs fields for the superconductors (labeled 1 and 2) be ϕ_1 and ϕ_2 respectively. Despite our concerns about the local symmetries within the individual superconductors enabling us to gauge transform away the field phases θ_1 and θ_2, the difference between these phases determines a physical observable, the Josephson current,

$$J = J_c \sin\theta, \tag{118}$$

where $\theta = \theta_1 - \theta_2$.

The defects of this system are the 'fluxons' of the Josephson Junction and are easy to observe experimentally[56]. The variance in fluxon number at their formation is expected to be

$$\Delta N_C = \frac{1}{2\pi}\Delta\theta \approx \frac{1}{2\pi}\sqrt{\frac{C}{\bar{\xi}_Z}}, \tag{119}$$

where $\bar{\xi}_Z$ is Zurek's causal length for such a system. There are two competing mechanisms. At very early times we can treat the two superconducting annuli as independent. If this decoupling lasts long enough this would suggest[31] that $(\Delta\theta)^2 = (\Delta\theta_1)^2 + (\Delta\theta_2)^2$, for which we use the Zurek causal constraints for free propagators. On the other hand, if the coupling between the JTJs is important by the causal time $\bar{t}_Z$, then what matters is that the causal horizon can encompass a fluxon.

Although some caution is required in the interpretation of past experiments with JTJs, for which $\bar{\xi} \approx C$, which were not devised with this prediction in mind, we would have expected to see a fluxon a few percent of the time if the latter were true. This was the case. If the former mechanism is correct, we would have expected to see them more often. New experiments with more relevant parameters are being devised and have been reported elsewhere[13,57].

In a different context an experiment with an annular superconductor, separated into segments by JTJs, shows that the assumed phase separation at the onset of a transition (the 'Kibble mechanism' in the context of QFT) is correct[58]. A further experiment, to quench an annular container of similar circumference C of superfluid ^4He, was also proposed by Zurek[1], but has not been attempted. Since the phase gradient is directly proportional to the superflow velocity we expect a flow after the quench with r.m.s velocity

$$\Delta v = O\left(\frac{\hbar}{m}\sqrt{\frac{1}{C\bar{\xi}}}\right). \tag{120}$$

Although not large it is measurable, in principle. A similar experiment for BEC has been proposed[20], but has also not been performed.

8. CONCLUSIONS

The strong causal bounds (6) and (7) of Zurek and (10) and (11) of Kibble have a common origin that purports to show strong links between phase transitions in condensed matter and particle physics. In reality, this masks huge differences between the two types of system. Nonetheless, each has a certain qualitative validity, but for different reasons. What we have found in common is that explicit causality is absent from both.

In TDGL condensed matter physics we have seen how fluctuations separate initially into unstable long wavelength modes which order the field and short wavelength thermal fluctuations that make vortices (and other defects) fractal. Dimensional analysis sets the Zurek time $\bar{t}_Z$ of (6) and length scale $\bar{\xi}_Z$ of (7) as the natural scales when fluctuations can be ignored. However, the density of defects is not set by $\bar{\xi}_Z$ (as the correlation length of the field) since it freezes too soon for instabilities to have grown. Instead, the density is set by the separation of field zeroes, a very different length, in principle.

In practice, because instabilities grow exponentially, the time it takes to order the field is, qualitatively, $O(\bar{t}_Z)$, with only logarithmic dependence on the parameters of the system. Further, since the defect density is, essentially, a ratio of moments of the power spectrum of the fluctuations, there is strong cancellation of the prefactors that contain the detailed information of the system. As a result, if thermal fluctuations are negligible the defect density is that of a free field for longer than we might have thought, and Zurek's causal predictions are valid, although not for directly causal reasons. If we want to see explicit causality in the TDGL formalism we need to go to its dual worldline representation, which we have not discussed here, but which we have examined elsewhere[59].

Although thermal fluctuations and long wavelength (unstable) modes are not additive in determining defect densities, simple analysis suggests that thermal fluctuations are roughly proportional to the current temperature, with little memory of the past temperature profile. As a result, the Ginzburg regime plays no role in incubating defects.

Quantum Field Theory has some characteristics in common with condensed matter: exponential (albeit different exponential) behaviour makes

$\bar{t}_K$ the natural time-scale, up to logarithmic terms. We recover $\bar{\xi}_K$ as the relevant length-scale after instabilities have formed, rather than before. However, there is a potential complication in that we can only adopt a classical probabilistic approach to defects once there are no substantial quantum interference effects. We have shown that decoherence is established before field ordering in a simplified model and more work is being done. However, once the initial defects are established, qualitatively as suggested by Kibble, their long-term behaviour is unclear. Dissipation is determined by Landau damping and long wavelength noise is now multiplicative. Unlike the case of the TDGL equation, where we have many numerical calculations, we have very little for QFT. As before, the Ginzburg regime only looks to make defects fuzzy, rather than create them.

As for prediction and experiment, we have no concrete proposals for QFT. For condensed matter, experiments in superfluid ^3He are in agreement with Zurek-Kibble, while those for ^4He seem not to be. However, we understand this as due, in large part, to an over-optimistic assumption about their decay rate. The role of large thermal fluctuations is unclear. Although the one experiment with High-T_c superconductors sees nothing, an earlier experiment with Josephson Tunnel Junctions seems to give support to Zurek's predictions, and further experiments with JTJs are planned.

In none of these experiments is there any agreement with the predictions of thermal production in the Ginzburg regime.

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