Entropy and Area

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The ground-state density matrix for a massless free field is traced over the degrees of freedom residing inside an imaginary sphere; the resulting entropy is shown to be proportional to the area (and not the volume) of the sphere. Possible connections with the physics of black holes are discussed.

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A free, massless, scalar, quantum field (which could just as well represent, say, the acoustic modes of a crystal, or any other three-dimensional system with dispersion relation $\omega = c|\mathbf{k}|$) is in its nondegenerate ground state, $|0\rangle$. We form the ground-state density matrix, $\rho_0 = |0\rangle\langle 0|$, and trace over the field degrees of freedom located inside an imaginary sphere of radius R. The resulting density matrix, $\rho_{\rm out}$, depends only on the degrees of freedom outside the sphere. We now compute the associated entropy, $S = -{\rm Tr}\rho_{\rm out}\ln\rho_{\rm out}$. How does S depend on R?

Entropy is usually an extensive quantity, so we might expect that $S \sim R^3$. However, this is not likely to be correct, as can be seen from the following argument. Consider tracing over the outside degrees of freedom instead, to produce a density matrix $\rho_{\rm in}$ which depends only on the inside degrees of freedom. If we now compute $S' = -{\rm Tr}\rho_{\rm in}\ln\rho_{\rm in}$, we would expect that S' scales like the volume *outside* the sphere. However, it is straightforward to show that $\rho_{\rm in}$ and $\rho_{\rm out}$ have the same eigenvalues (with extra zeros for the larger, if they have different rank), so that in fact $S' = S$ [1]. This indicates that S should depend only on properties which are shared by the two regions (inside and outside the sphere). The one feature they have in common is their shared boundary, so it is reasonable to expect that S depends only on the area of this boundary, $A = 4\pi R^2$. S is dimensionless, so to get a nontrivial dependence of S on A requires another dimensionful parameter. We have two at hand: the ultraviolet cutoff M and the infrared cutoff μ, both of which are necessary to give a precise definition of the theory. (For a crystal, M would be the inverse atomic spacing, and μ the inverse linear size, in units with $\hbar = c = 1$.) However, if the ground-state correlations between the inside and outside degrees of freedom fall off fast enough with distance from the boundary, S should be independent of μ. We therefore expect that S is some function of M^2A.

In fact, as will be shown below, $S = \kappa M^2 A$, where κ is a numerical constant which depends only on the precise definition of M that we adopt.

This result bears a striking similarity to the formula for the intrinsic entropy of a black hole, $S_{\rm BH} = \frac{1}{4}M_{\rm Pl}^2 A$, where $M_{\rm Pl}$ is the Planck mass and A is the surface area of the horizon of the black hole [2]. The links in the chain of reasoning establishing this formula are remarkably diverse, involving, in turn, classical geometry, thermodynamic analogies, and quantum field theory in curved space. The result is thus rather mysterious. In particular, we would like to know whether or not $S_{\rm BH}$ has anything to do with the number of quantum states accessible to the black hole.

As a black hole evaporates and shrinks, it produces Hawking radiation whose entropy, $S_{\rm HR}$, can be computed by standard methods of statistical mechanics. One finds, after the black hole has shrunk to negligible size, that $S_{\rm HR}$ is a number of order 1 (depending on the masses and spins of the elementary particles) times the original black hole entropy [3]. This calculation of $S_{\rm HR}$ is done by counting quantum states, and the fact that $S_{\rm BH} \simeq S_{\rm HR}$ lends support to the idea that $S_{\rm BH}$ should also be related to a counting of quantum states. It is then tempting to think of the horizon as a kind of membrane [4], with approximately 1 degree of freedom per Planck area. However, in classical general relativity, the horizon does not appear to be a special place to a nearby observer, so it is hard to see why it should behave as an object with local dynamics. The new result quoted above indicates that $S \sim A$ is a much more general formula than has heretofore been realized. It shows that the amount of missing information represented by $S_{\rm BH}$ is about right, in the sense that we would get the same answer in the vacuum of flat space if we did not permit ourselves access to the interior of a sphere with surface area A, and set the ultraviolet cutoff to be of order $M_{\rm Pl}$ (perfectly reasonable for comparison with a quantum theory that includes gravity). Furthermore, getting $S \sim A$ clearly does not require the boundary of the inaccessible region to be dynamical, since in our case it is entirely imaginary.

To establish that $S = \kappa M^2 A$ for the problem at hand, let us begin with the simplest possible version of it: two coupled harmonic oscillators, with Hamiltonian

$$H = \tfrac{1}{2}[p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2]\,. \qquad (1)$$

The normalized ground-state wave function is

$$\psi_0(x_1,x_2) = \pi^{-1/2}(\omega_+\omega_-)^{1/4}\exp[-(\omega_+ x_+^2 + \omega_- x_-^2)/2]\,, \qquad (2)$$

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where $x_\pm = (x_1 \pm x_2)/\sqrt{2}$, $\omega_+ = k_0^{1/2}$, and $\omega_- = (k_0 + 2k_1)^{1/2}$. We now form the ground-state density matrix, and trace over the first ("inside") oscillator, resulting in a density matrix for the second ("outside") oscillator alone:

$$\rho_{\rm out}(x_2, x_2') = \int_{-\infty}^{+\infty} dx_1\, \psi_0(x_1,x_2)\psi_0^*(x_1,x_2') = \pi^{-1/2}(\gamma-\beta)^{1/2}\exp[-\gamma(x_2^2+x_2'^2)/2+\beta x_2 x_2'] , \tag{3}$$

where $\beta = \frac{1}{4}(\omega_+ - \omega_-)^2/(\omega_+ + \omega_-)$ and $\gamma - \beta = 2\omega_+ \times \omega_-/(\omega_+ + \omega_-)$. We would like to find the eigenvalues p_n of $\rho_{\rm out}(x,x')$:

$$\int_{-\infty}^{+\infty} dx'\rho_{\rm out}(x,x') f_n(x') = p_n f_n(x) , \tag{4}$$

because in terms of them the entropy is simply $S = -\sum_n p_n \ln p_n$. The solution of Eq. (4) is found most easily by guessing, and is

$$\begin{aligned} p_n &= (1-\xi)\xi^n , \\ f_n(x) &= H_n(\alpha^{1/2}x)\exp(-\alpha x^2/2) , \end{aligned} \tag{5}$$

where H_n is a Hermite polynomial, $\alpha = (\gamma^2 - \beta^2)^{1/2} = (\omega_+\omega_-)^{1/2}$, $\xi = \beta/(\gamma+\alpha)$, and n runs from zero to infinity. Equation (5) implies that $\rho_{\rm out}$ is equivalent to a thermal density matrix for a single harmonic oscillator specified by frequency α and temperature $T = \alpha/\ln(1/\xi)$. The entropy is

$$S(\xi) = -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln\xi , \tag{6}$$

where ξ is ultimately a function only of the ratio k_1/k_0.

We can easily expand this analysis to a system of N coupled harmonic oscillators with Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j , \tag{7}$$

where K is a real symmetric matrix with positive eigenvalues. The normalized ground-state wave function is

$$\psi_0(x_1,\ldots,x_N) = \pi^{-N/4}(\det\Omega)^{1/4}\exp[-x\cdot\Omega\cdot x/2] , \tag{8}$$

where Ω is the square root of K: If $K = U^T K_D U$, where K_D is diagonal and U is orthogonal, then $\Omega = U^T K_D^{1/2} U$. We now trace over the first n ("inside") oscillators to get

$$\rho_{\rm out}(x_{n+1},\ldots,x_N; x_{n+1}',\ldots,x_N') = \int \prod_{i=1}^{n} dx_i\, \psi_0(x_1,\ldots,x_n,x_{n+1},\ldots,x_N)\psi_0^*(x_1,\ldots,x_n,x_{n+1}',\ldots,x_N') . \tag{9}$$

To carry out these integrals explicitly, we write

$$\Omega = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} , \tag{10}$$

where A is $n\times n$ and C is $(N-n)\times(N-n)$. We find

$$\rho_{\rm out}(x,x') \sim \exp[-(x\cdot\gamma\cdot x + x'\cdot\gamma\cdot x')/2 + x\cdot\beta\cdot x'] , \tag{11}$$

where x now has $N-n$ components, $\beta = \frac{1}{2}B^T A^{-1} B$, and $\gamma = C - \beta$. In general β and γ will not commute, which implies that Eq. (11) is *not* equivalent to a thermal density matrix for a systems of oscillators.

We need not keep track of the normalization of $\rho_{\rm out}$, since we know that its eigenvalues must sum to 1. To find them, we note that the appropriate generalization of Eq. (4) implies that $(\det G)\rho_{\rm out}(Gx,Gx')$ has the same eigenvalues as $\rho_{\rm out}(x,x')$, where G is any nonsingular matrix. Let $\gamma = V^T\gamma_D V$, where γ_D is diagonal and V is orthogonal; then let $x = V^T\gamma_D^{-1/2}y$. (The eigenvalues of γ are guaranteed to be positive, so this transformation is well defined.) We then have

$$\rho_{\rm out}(y,y') \sim \exp[-(y\cdot y + y'\cdot y')/2 + y\cdot\beta'\cdot y'] , \tag{12}$$

where $\beta' = \gamma_D^{-1/2}V\beta V^T\gamma_D^{-1/2}$. If we now set $y = Wz$, where W is orthogonal and $W^T\beta' W$ is diagonal, we get

$$\rho_{\rm out}(z,z') \sim \prod_{i=n+1}^{N} \exp[-(z_i^2 + z_i'^2)/2 + \beta_i' z_i z_i'] , \tag{13}$$

where β_i' is an eigenvalue of β'. Each term in this product is identical to the $\rho_{\rm out}$ of Eq. (3), with $\gamma \to 1$ and $\beta \to \beta_i'$. Therefore, the entropy associated with the $\rho_{\rm out}$ of Eq. (13) is just $S = \sum_i S(\xi_i)$, where $S(\xi)$ is given by Eq. (6), and $\xi_i = \beta_i'/[1 + (1-\beta_i'^2)^{1/2}]$.

We now wish to apply this general result to a quantum field with Hamiltonian

$$H = \frac{1}{2}\int d^3x [\pi^2(\mathbf{x}) + |\nabla\varphi(\mathbf{x})|^2] . \tag{14}$$

To regulate this theory, we first introduce the partial wave components

$$\begin{aligned} \varphi_{lm}(x) &= x\int d\Omega\, Z_{lm}(\theta,\phi)\varphi(\mathbf{x}) , \\ \pi_{lm}(x) &= x\int d\Omega\, Z_{lm}(\theta,\phi)\pi(\mathbf{x}) , \end{aligned} \tag{15}$$

where $x = |\mathbf{x}|$ and the Z_{lm} are real spherical harmonics: $Z_{l0} = Y_{l0}$, $Z_{lm} = \sqrt{2}\,\mathrm{Re}Y_{lm}$ for $m>0$, and $Z_{lm} = \sqrt{2}\,\mathrm{Im}Y_{lm}$ for $m<0$; the Z_{lm} are orthonormal and complete. The operators defined in Eq. (15) are Hermitian, and obey the canonical commutation relations

$$[\varphi_{lm}(x), \pi_{l'm'}(x')] = i\delta_{ll'}\delta_{mm'}\delta(x-x') . \tag{16}$$

In terms of them, we can write $H = \sum_{lm} H_{lm}$, where

$$H_{lm} = \frac{1}{2}\int_0^\infty dx \left\{ \pi_{lm}^2(x) + x^2\left[\frac{\partial}{\partial x}\left(\frac{\varphi_{lm}(x)}{x}\right)\right]^2 + \frac{l(l+1)}{x^2}\varphi_{lm}^2(x) \right\} . \tag{17}$$

So far we have made no approximations or regularizations.

Now, as an ultraviolet regulator, we replace the continuous radial coordinate x by a lattice of discrete points with spacing a; the ultraviolet cutoff M is thus a^{-1}. As an infrared regulator, we put the system in a spherical box of radius $L=(N+1)a$, where N is a large integer, and demand that $\varphi_{lm}(x)$ vanish for $x \geq L$; the infrared cutoff μ is thus L^{-1}. Altogether, this yields

$$H_{lm}=\frac{1}{2a}\sum_{j=1}^{N}\left[\pi_{lm,j}^2+(j+\tfrac{1}{2})^2\left(\frac{\varphi_{lm,j}}{j}-\frac{\varphi_{lm,j+1}}{j+1}\right)^2+\frac{l(l+1)}{j^2}\varphi_{lm,j}^2\right], \tag{18}$$

where $\varphi_{lm,N+1}=0$;$\varphi_{lm,j}$ and $\pi_{lm,j}$ are dimensionless, Hermitian, and obey the canonical commutation relations

$$[\varphi_{lm,j},\pi_{l'm',j'}]=i\delta_{ll'}\delta_{mm'}\delta_{jj'}. \tag{19}$$

Thus, H_{lm} has the general form of Eq. (7), and for a fixed value of N we can compute (numerically) the entropy $S_{lm}(n,N)$ produced by tracing the ground state of H_{lm} over the first n sites. The ground state of H is a direct product of the ground states of each H_{lm}, and so the total entropy is found by summing over l and m: $S(n,N)=\sum_{lm}S_{lm}(n,N)$. As can be seen from Eq. (18), H_{lm} is actually independent of m, and therefore so is $S_{lm}(n,N)=S_l(n,N)$. Summing over m just yields a factor of $2l+1$, and so we have $S(n,N)=\sum_l(2l+1)\times S_l(n,N)$. From Eq. (18) we also see that the l-dependent term dominates if $l\gg N$, and in this case we can compute $S_l(n,N)$ perturbatively. The result is that, for $l\gg N$, $S_l(n,N)$ is independent of N, and is given by

$$S_l(n,N)=\xi_l(n)[-\ln\xi_l(n)+1], \tag{20}$$

where

$$\xi_l(n)=\frac{n(n+1)(2n+1)^2}{64l^2(l+1)^2}+O(l^{-6}). \tag{21}$$

Equations (20) and (21) demonstrate that the sum over l will converge, and also provide a useful check on the numerical results.

Let us define $R=(n+\frac{1}{2})a$, a radius midway between the outermost point which was traced over, and the innermost point which was not. The computed values of $S(n,N)$ are shown for $N=60$ and $1\leq n\leq 30$ as a function of R^2 in Fig. 1. As can be seen, the points are beautifully fitted by a straight line:

$$S=0.30M^2R^2, \tag{22}$$

where $M=a^{-1}$. Furthermore, $S(n,N)$ turns out to be independent of N (and hence the value of the infrared cutoff). Specifically, for fixed n, with $n\leq\frac{1}{2}N$, the values of $S(n,N)$ turn out to be identical (in the worst case, to within 0.5%) for $N=20$, 40, and 60. The restriction to $n\leq\frac{1}{2}N$ is necessary, since the linear behavior in Fig. 1 cannot continue all the way to $n=N$: At this point we will have traced over *all* the degrees of freedom, and must find $S=0$. S must therefore start falling as R begins to approach the wall of the box at radius $L=(N+1)a$.

One can still question whether the results depend on the specific form of the ultraviolet cutoff which is used. To check this point, the values of S with the infrared and ultraviolet cutoffs provided by a $6\times6\times6$ cubic lattice (with antiperiodic boundary conditions) are shown in Fig. 2. Again we find $S\simeq0.30M^2R^2$, where M is the inverse lattice spacing, and R is midway between the outermost point, which was traced, and the innermost point, which was not. The data are considerably noisier, however, due to the much smaller ratio of ultraviolet to infrared cutoffs: $M/\mu=6$, compared to $M/\mu=61$ for the results shown in Fig. 1. Computations on $N\times N\times N$ lattices with much larger values of N would be needed to confirm that S is still independent of N (as it is with the radial lattice cutoff), but there is no reason to expect otherwise.

Of course, similar calculations can be done for one- and two-dimensional systems as well. For $d=2$, our introductory arguments would lead us to expect that $S=\kappa MR$, since the relevant "area" is the circumference of the dividing circle of radius R. This is confirmed by the numerical results, which will be presented in detail elsewhere [5]. For $d=1$, our arguments must break down: They would lead to the conclusion that S is independent of R, and this is clearly impossible. In fact, the numerical results indicate that $S=\kappa_1\ln(MR)+\kappa_2\ln(\mu R)$ in one dimension; for the first time, we see a dependence on the infrared cutoff μ [6]. For $d\geq4$, regularization by a radial lattice turns out to be insufficient; the sum over partial waves does not converge. Regulari-

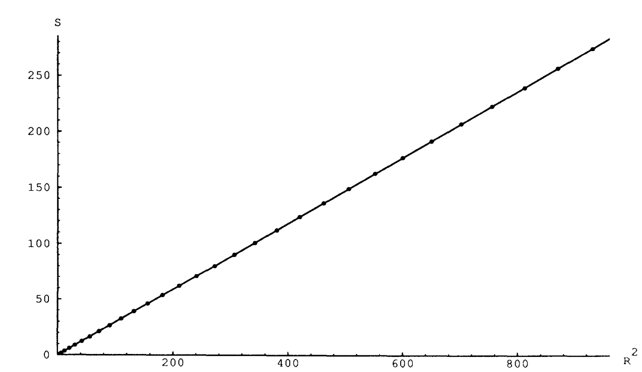

FIG. 1. The entropy S resulting from tracing the ground state of a massless scalar field over the degrees of freedom inside a sphere of radius R. The points shown correspond to regularization by a radial lattice with $N=60$ sites; the line is the best linear fit.

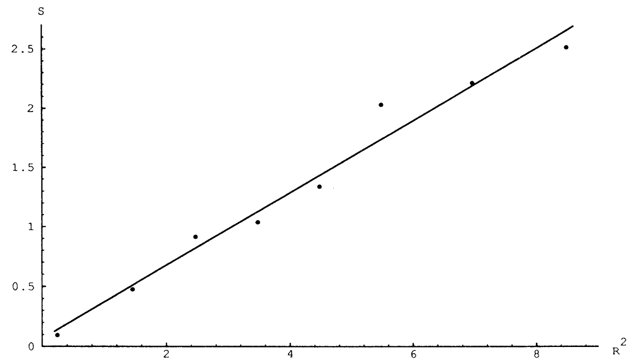

FIG. 2. Same as Fig. 1, except that the points shown correspond to regularization by a $6\times6\times6$ cubic lattice.

zation by a cubic lattice would certainly produce a finite S, but the calculation would require considerably more computer time.

To summarize, a straightforward counting of quantum states in a simple, well-defined context has produced an entropy proportional to the surface area of the inaccessible region, inaccessible in the sense that we ignore the information contained there. Equation (22) is strikingly similar to the formula for the entropy of a black hole, $S_{\rm BH}=\frac{1}{4}M_{\rm Pl}^2A$, and so may provide some clues as to its deeper meaning.

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Note added.—After this paper was completed and circulated as a preprint, I learned of related work by Bombelli *et al.* [7]. Also motivated by the black hole analogy, these authors find an equivalent result for the entropy of a coupled system of oscillators. They also argue that, for a quantum field, the entropy should be proportional to the area of the boundary; the argument they give is different from those presented here, and is valid only if the field has a mass m which is large enough to make the Compton wavelength $1/m$ much less than R. I thank Erik Matinez for bringing this paper to my attention.

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[1] Let $|0\rangle=\sum_{ia}\psi_{ia}|i\rangle_{\rm in}|a\rangle_{\rm out}$, so that $(\rho_{\rm in})_{ij}=(\psi\psi^\dagger)_{ij}$ and $(\rho_{\rm out})_{ab}=(\psi^T\psi^*)_{ab}$. Now it is clear that ${\rm Tr}\rho_{\rm in}^k={\rm Tr}\rho_{\rm out}^k$ for any positive integer k. This can only be true if $\rho_{\rm in}$ and $\rho_{\rm out}$ have the same eigenvalues, up to extra zeros.

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