PHYSICAL REVIEW D VOLUME 14, NUMBER 2 15 JULY 1976

Dirac's monopole without strings: Classical Lagrangian theory

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(Received 16 March 1976)

The non-quantum-mechanical interaction of a Dirac magnetic monopole and a point charge through the electromagnetic field is studied. A classical action integral which is multiple-valued is found. Stability of this action integral against variations of the world lines of the point charge and the monopole, and against variations of the electromagnetic potentials, yields the correct Lorentz equations of motion of the particles and the Maxwell equations for the field. No strings are introduced in the formalism.

I. ELECTROMAGNETIC FIELD IN INTERACTION WITH CHARGED PARTICLES AND MAGNETIC MONOPOLES

In this paper and an earlier one[9] on monopole harmonics, we study some properties of the Dirac magnetic monopole without the introduction of the concept of strings. The two papers are, however, logically and technically independent, and may be read separately.

Consider first the familiar case of one positron with world line x^μ. Then the electric current density j^μ is

$$j^\mu(\xi)=4\pi e\int ds\,\frac{dx^\mu}{ds}\,\delta^4(\xi-x(s)),\qquad(1)$$

where ξ designates a space-time point. This current density leads to an electromagnetic field described by the tensor $f^{\mu\nu}=-f^{\nu\mu}$. Let $\bar{f}^{\mu\nu}$ be its dual such that

$$E_1=-f^{01}=\bar{f}^{23},\quad H_1=-f^{23}=-\bar{f}^{01},\quad\text{etc.}\qquad(2)$$

[cf. (6) below], where the metric used is

$$dt^2+dx^2+dy^2+dz^2=-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2\,.$$

Along a world line we define

$$ds=(dt^2-dx^2-dy^2-dz^2)^{1/2}=dt(1-v^2)^{1/2}\,.$$

The usual Maxwell equations for the electromagnetic field are

$$f^{\mu\nu}{}_{,\nu}(\xi)=-4\pi e\int ds\,\frac{dx^\mu}{ds}\,\delta^4(\xi-x(s)),\qquad(3)$$

and

$$f_{\mu\nu,\lambda}(\xi)+f_{\nu\lambda,\mu}(\xi)+f_{\lambda\mu,\nu}(\xi)=0$$

or (4)

$$\bar{f}^{\mu\nu}{}_{,\nu}(\xi)=0\,.$$

Simultaneously, through the Lorentz force, the electromagnetic field also acts on the positron to determine its motion[1]:

$$\frac{dp^\mu}{ds}=-ef^{\mu\nu}\frac{dx_\nu}{ds}\qquad\left(p^\mu=m\frac{dx^\mu}{ds}\right).\qquad(5)$$

The general definition of the dual field strength $\bar{f}^{\mu\nu}$ is

$$\bar{f}^{\mu\nu}=-\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}f_{\alpha\beta}\,,\qquad(6)$$

where

$$\epsilon^{0123}=-1,\quad\epsilon^{\mu\nu\alpha\beta}=\text{antisymmetrical tensor.}\qquad(7)$$

Next consider the more general problem of the electromagnetic field in interaction with one positron with world line x^μ and one Dirac magnetic monopole[2] with world line X^μ. (Generalization to a finite number of positrons and a finite number of Dirac magnetic monopoles is straightforward.) The coupled equations of motion must be

$$f^{\mu\nu}{}_{,\nu}(\xi)=-4\pi e\int ds\,\frac{dx^\mu}{ds}\,\delta^4(\xi-x)\,,\qquad\text{(Me)}$$

$$\bar{f}^{\mu\nu}{}_{,\nu}(\xi)=-4\pi g\int ds\,\frac{dX^\mu}{ds}\,\delta^4(\xi-X)\,,\qquad\text{(Mg)}$$

$$\frac{dp^\mu}{ds}=-ef^{\mu\nu}\frac{dx_\nu}{ds}\,,\qquad\text{(Le)}$$

and

$$\frac{dP^\mu}{ds}=-g\bar{f}^{\mu\nu}\frac{dX_\nu}{ds}\qquad\left(P^\mu=M\frac{dX^\mu}{ds}\right),\qquad\text{(Lg)}$$

where p^μ and P^μ are the four-momenta of the positron and the monopole, respectively, and g is the magnetic charge of the monopole, positive if the monopole is a north (i.e., north-seeking) pole. We shall refer to these equations as the Maxwell [(Me), (Mg)] and the Lorentz [(Le), (Lg)] equations.

It has been well known since the beginning of this century that classical electrodynamics (without magnetic monopoles), as described by (3), (4), and (5), can be formulated compactly in an action principle. It is the purpose of the present paper to generalize this formulation so that the Maxwell equations (Me) and (Mg) and the Lorentz equations (Le) and (Lg) for the interaction between electromagnetism, charged particles, and magnetic monopoles are formulated compactly in an action principle.

After Dirac introduced[2] the concept of magnetic monopoles, he came back in 1948 to the question[3] of the classical action principle. The aim of the present work is exactly to rediscuss this question of Dirac's 1948 paper. However, we shall avoid the introduction of the concept of strings attached to monopoles, which was necessary in Dirac's formulation.

Dirac originally introduced the string because his vector potential for a static magnetic monopole is singular along a semi-infinite line in three-space. Subsequently, the string has caused a number of problems, including the following two:

(i) The Dirac veto. The wave functions for all charged particles vanish on the string. This problem has been discussed most explicitly by Wentzel.[4]

(ii) Dynamic variables for the string. In studying the moving magnetic monopole, Dirac[3] used an infinite number of dynamic variables for the string. The resultant formalism becomes extremely complicated.

In this paper we circumvent all these complexities by introducing ideas[5] borrowed from the mathematics of fiber bundles.

Before going into the details, let us emphasize that throughout the present paper, except for remarks in Sec. VI, *we do not introduce Planck's constant* $\hbar$, so that the monopole strength need not satisfy Dirac's quantization condition. That is, it is possible that

$$\frac{2eg}{\hbar c} \neq \text{integer}. \tag{8}$$

Because $\hbar$ is not introduced, the concept that electromagnetism is a nonintegrable phase factor[5] is not used in the present paper.

II. EQUATION (Mg) AS KINEMATICS

The electromagnetic potential around a magnetic monopole cannot be chosen without singularities. This fact was proved in Ref. 5 for a monopole at rest where, in order to circumvent the singularity problem, the space-time outside of the monopole was divided into two overlapping regions R_a and R_b and singularity-free electromagnetic potentials $(A_\mu)_a$ and $(A_\mu)_b$ were found in R_a and R_b, respectively. We shall now use this same idea[6] to describe the electromagnetic potential outside of the world line X^μ of a magnetic monopole of strength g. The choice of R_a and R_b is very flexible. For definiteness we shall choose to define R_a and R_b in one Lorentz frame: For each t, R_a and R_b are respectively the regions defined by

$$\begin{aligned} R_a:&\ 0 \leq \theta < \tfrac{1}{2}\pi + \delta, \quad \text{all } \phi, \quad r > 0 \\ R_b:&\ \tfrac{1}{2}\pi - \delta < \theta \leq \pi, \quad \text{all } \phi, \quad r > 0 \end{aligned} \tag{9}$$

where r, θ, and ϕ are spherical coordinates with the monopole position at that t taken as the origin. δ is a smooth function of t satisfying $0 < \delta \leq \frac{1}{2}\pi$.

Given an electromagnetic field satisfying (Mg) we can find vector potentials $(A_\mu)_a$ and $(A_\mu)_b$ in regions R_a and R_b, respectively, so that for $i = a, b$,

$$f_{\mu\nu} = (A_{\mu,\nu})_i - (A_{\nu,\mu})_i \tag{10}$$

in R_i. In the overlap $R_{ab} = R_a \cap R_b$, the two vector potentials are related by the gauge transformation

$$(A_\mu)_a - (A_\mu)_b = \alpha_\mu \text{ in } R_{ab}, \tag{11}$$

where by (10) a_μ must satisfy

$$\alpha_{\mu,\nu} - \alpha_{\nu,\mu} = 0 \text{ in } R_{ab}. \tag{12}$$

R_{ab} is a four-dimensional region where loops around the monopole cannot be shrunk within R_{ab}. Equation (12) asserts that

$$\oint \alpha_\mu(\xi)\, d\xi^\mu = K, \tag{13}$$

where K is independent of any distortions of the loop within R_{ab}, and the integral is defined in the direction of increasing azimuthal angle ϕ (cf. Fig. 1).

To determine K we consider, at a fixed t, a spherical surface S around the monopole. The upper hemisphere S_a, where $\theta \leq \frac{1}{2}\pi$, is entirely in R_a. The lower hemisphere S_b is entirely in R_b. Hence

$$\begin{aligned} &\text{outward magnetic flux through } S_a = \oint (A_\mu)_a\, d\xi^\mu, \\ &\text{outward magnetic flux through } S_b = -\oint (A_\mu)_b\, d\xi^\mu. \end{aligned} \tag{14}$$

Thus

$$\text{outward magnetic flux through } S = \oint \alpha_\mu\, d\xi^\mu \text{ around equator}. \tag{15}$$

In (14) we have used the sign convention (2). The total outward flux is, of course, $4\pi g$. Thus

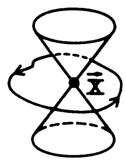

FIG. 1. Regions R_a and R_b at a given time t. $\vec{X}$ is the position of the monopole. R_a is the region above the lower cone. R_b is the region under the upper cone. The loop is in the overlap $R_{ab} = R_a \cap R_b$.

$$\oint \alpha_\mu(\xi) d\xi^\mu = 4\pi g \tag{16}$$

around *any* loop in R_{ab} which circles the monopole world line X^μ in the direction of increasing azimuthal angle ϕ.

Thus *any electromagnetic field* $f_{\mu\nu}$ *satisfying (Mg) is describable by* $(A_\mu)_a$ *and* $(A_\mu)_b$ *satisfying (10), (11), (12), and (16). Conversely, any* $(A_\mu)_a$ *and* $(A_\mu)_b$ *satisfying (10), (11), (12), and (16) gives an electromagnetic field* $f_{\mu\nu}$ *satisfying (Mg).* To prove this last statement, we observe that at a point ξ^μ not on the world line of the monopole, (10) implies the homogeneous Maxwell equation (4). Thus $\tilde{f}^{\mu\nu}{}_{,\nu}(\xi)$ is only nonvanishing on the world line of the monopole. That is,

$$\tilde{f}^{\mu\nu}{}_{,\nu}(\xi) = \int a^\mu(s) ds\, \delta^4(\xi - X(s)), \tag{17}$$

where $a^\mu(s)$ is a four-vector defined on the world line. *In any specific Lorentz frame* consider the $\mu = 0$ component of (17). Using convention (2), this component reduces to

$$-\nabla \cdot H|_\xi = a^0 \delta^3(\vec{\xi} - \vec{X}(s)) \left(\frac{dX^0}{ds}\right)^{-1}, \tag{18}$$

where on the right-hand side s is taken to be that point on the world line where $X^0(s) = \xi^0$. At this fixed ξ^0 we now integrate (18) over the interior of a sphere around the monopole at $\vec{X}(s)$. The right-hand side yields

$$a^0 \left(\frac{dX^0}{ds}\right)^{-1}.$$

The left-hand side becomes a surface integral which is (−1) times the total magnetic flux outward from the surface, which one can evaluate in the same way as in an earlier discussion that led to (15). If (16) is satisfied, this is $-4\pi g$. Thus

$$-4\pi g = a^0 \left(\frac{dX^0}{ds}\right)^{-1}, \quad \text{or } a^0 = -4\pi g \frac{dX^0}{ds}. \tag{19}$$

Since (19) holds in any Lorentz frame, we have

$$a^\mu = -4\pi g \frac{dX^\mu}{ds}.$$

Substitution of this into (17) leads to (Mg).

We have thus shown that (Mg), which describes the generation of electromagnetism by a monopole, is *equivalent* to the condition that the electromagnetic field be described by $(A_\mu)_a$ and $(A_\mu)_b$ satisfying (10), (11), (12), and (16). We shall now consider $(A_\mu)_a$ and $(A_\mu)_b$ as independent variables subject to conditions (11), (12), and (16). The field strength $f_{\mu\nu}$ described by such electromagnetic potentials *automatically* satisfies (Mg). In other words, in this approach (Mg) becomes a kinematic equation.

III. THE ACTION INTEGRAL

For the electrodynamics of positrons and electromagnetic fields described by (3), (4), and (5), the action integral is

$$\mathcal{A}(x,A) = -m\left[\int ds\right]_{\text{positron}} - (16\pi)^{-1} \int d^4\xi\, f_{\mu\nu}(\xi) f^{\mu\nu}(\xi) + e\int_{-\infty}^{\infty} A_\mu(x) \frac{dx^\mu}{ds} ds, \tag{20}$$

where the first and the last integrals are defined along the world line of the positron and ds is real and >0. Equation (4) is a kinematic condition. Equations (3) and (5) are dynamical equations which result from the condition of stability of $\mathcal{A}$ against variations respectively of $A_\mu(\xi)$ and of the world line $x^\mu(s)$.

For the electromagnetism of positrons and monopoles described by (Me), (Mg), (Le), and (Lg) we seek to find an action integral $\mathcal{A}(x,X,A)$, where x, X represent the world lines of the positron and the monopole, and A is the electromagnetic potential defined in two regions R_a and R_b and satisfies (10), (11), (12), and (16). As proved in the last section, (Mg) is a kinematic equation. We expect (Me), (Le), and (Lg) to result from the stability condition of $\mathcal{A}(x,X,A)$ against variations of A, x, and X.

Equation (20) suggests

$$\mathcal{A}(x,X,A) = -m\left[\int ds\right]_{\text{positron}} - M\left[\int ds\right]_{\text{monopole}} - (16\pi)^{-1}\int d^4\xi\, f_{\mu\nu}(\xi) f^{\mu\nu}(\xi) + \mathcal{A}_1, \tag{21}$$

where

$$\mathcal{A}_1 = e \,⨍\, A_\mu(x) dx^\mu \tag{22}$$

along the world line of the electron. We use an unusual symbol for this last integral because its definition requires careful examination. In particular, one has to define which A_μ $[(A_\mu)_a$ or $(A_\mu)_b]$ to use as the integrand. We proceed as follows:

(a) If the positron world line is entirely in one region, R_a or R_b, then the $\oint$ is defined to be the usual integral with $(A_\mu)_a$ or $(A_\mu)_b$ as the integrand. If the positron world line crosses from R_b through R_{ab} into R_a [Fig. 2(a)], the definition of $\mathcal{A}_1$ is

$$\mathcal{A}_1 = e \oint A_\mu(x)\,dx^\mu = e\int_Q^\infty (A_\mu)_a dx^\mu + e\beta(Q) + e\int_{-\infty}^Q (A_\mu)_b dx^\mu\,, \tag{23}$$

where β is defined in R_{ab} by

$$\beta_{,\mu} = \alpha_\mu\,. \tag{24}$$

Given $(A_\mu)_a$ and $(A_\mu)_b$ such a β exists, because of Eq. (12), as a multiple-valued function plus an arbitrary constant independent of space-time. In view of (16), the multiple values are different from each other by $4\pi g$ times an integer.

The necessity of the term $e\beta(Q)$ in (23) is demonstrated by the fact that (23) is independent of changing the point Q to another point Q' along the positron world line in R_{ab} (Fig. 2). To see this we use (11) and (24). This demonstration is entirely similar to, but not quite the same as, a corresponding one in Ref. 5.

(b) We can rewrite the three terms on the right-hand side of (23) in a convenient notation as follows:

$$\mathcal{A}_1 = \mathcal{A}_1(\infty_a, Q_a) + \mathcal{L}(Q_a, Q_b) + \mathcal{A}_1(Q_b, -\infty_b)\,. \tag{25}$$

We write ∞_a for ∞ to emphasize that at $t=\infty$ the world line of the positron (in this case) is in R_a. A point Q in R_{ab} is treated as two points Q_a and Q_b, which can be schematically thought of as on different "floors," as illustrated in Fig. 2 of Ref. 5. The definition of $\mathcal{L}$ is

$$\mathcal{L}(Q_a, Q_b) \equiv e\beta(Q) \equiv -\mathcal{L}(Q_b, Q_a)\,. \tag{26}$$

For a world line of the positron that goes in and out of R_a and R_b as along the path (b) of Fig. 2, we have many equivalent definitions of $\mathcal{A}_1$, such as

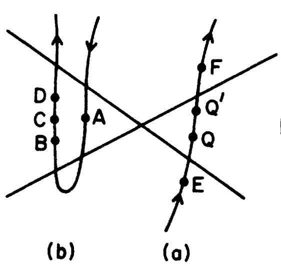

FIG. 2. Electron world lines (a) and (b) that go through overlap R_{ab}.

$$\begin{aligned}\mathcal{A}_1 &= \mathcal{A}_1(\infty_a, B_a) + \mathcal{L}(B_a, B_b) + \mathcal{A}_1(B_b, A_b) + \mathcal{L}(A_b, A_a) + \mathcal{A}_1(A_a, -\infty_a)\\ &= \mathcal{A}_1(\infty_a, C_a) + \mathcal{L}(C_a, C_b) + \mathcal{A}_1(C_b, A_b) + \mathcal{L}(A_b, A_a) + \mathcal{A}_1(A_a, -\infty_a)\,.\end{aligned} \tag{27}$$

Notice the following identity:

$$\mathcal{A}_1(\infty_a, B_a) = \mathcal{A}_1(\infty_a, D_a) + \mathcal{L}(D_a, D_b) + \mathcal{A}_1(D_b, C_b) + \mathcal{L}(C_b, C_a) + \mathcal{A}_1(C_a, B_a)\,. \tag{28}$$

(c) How does an additive constant γ (which is independent of space-time) to β affect the definition of $\mathcal{A}_1$ given above? To answer this question, we notice that for a path that begins and ends in the same region, R_a or R_b, the number of $\mathcal{L}$ terms in the definition of $\mathcal{A}_1$, such as in (27), is even and the change $\beta \to \beta+\gamma$ does not change the value of $\mathcal{A}_1$. For a path that begins and ends in different regions, the number of $\mathcal{L}$ terms is odd and the change $\beta \to \beta+\gamma$ produces a change in $\mathcal{A}_1$ by $\pm e\gamma$. In the variational principle, where one keeps the world line fixed at $t=\pm\infty$, this additive constant of $\pm e\gamma$ does not produce any changes in the final result.

(d) Since β is multiple-valued, which of its values should be chosen in the definition of $\mathcal{A}_1$ such as (23) and (26)? The answer is: Any choice is undesirable and *we just consider* $\mathcal{A}_1$ *as definable only modulo* $4\pi eg$. To see this let us consider the world line (Fig. 3) and choose a specific value of $\beta(Q)$ among its multiple values to evaluate $\mathcal{A}_1$ in (23). For simplicity we consider the case that the monopole is fixed in space at the origin so that $\vec{X}=0$ for all times. Now continuously distort the portions of the world line of the positron between E and F so that it loops around the origin (i.e., the monopole) and return to the original position, in the direction of increasing azimuthal angle. The point Q describes a loop as shown in Fig. 3. If we use the azimuthal coordinate ϕ of Q to label the action integral $\mathcal{A}(\phi)$ for this one-parameter family of positron world lines, then a comparison between the two cases $\theta=0$ and $\theta=2\pi$ shows that β at Q is the only quantity that is different. More pre-

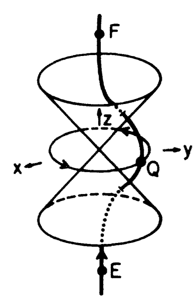

FIG. 3. Distortion of world line EF of the electron.

cisely

$$\mathcal{A}(2\pi)-\mathcal{A}(0)=\mathcal{A}_1(2\pi)-\mathcal{A}_1(0)$$
$$=e\oint \alpha_\mu\, dx^\mu$$
$$=4\pi eg\,. \tag{29}$$

Thus if we want the value of $\mathcal{A}$ to be continuous with respect to the distortion of world lines, we must define its value only up to modulo $4\pi eg$.

IV. CONSTRAINTS ON x,X,A AND STABILITY OF $\mathcal{A}$ AGAINST δx^μ AND δA_μ

We have defined above the action integral $\mathcal{A}(x,X,A)$ in terms of the dynamical variables x, X, and A_μ. To be more precise, $\vec{x}(t)$ and $\vec{X}(t)$ designate the world lines of the positron and the monopole. They represent $2\infty^3$ *independent dynamical variables* subject to the constraints discussed in the next paragraph.

The world lines x and X are constrained to be timelike. Furthermore, they must not cross. That is, $\vec{x}(t)-\vec{X}(t)\neq 0$ for all t. Without this condition it would not be possible to properly define the action integral $\mathcal{A}$. For example, the one-parameter family of positron world lines labeled by ϕ in the preceding section gives rise to an action integral $\mathcal{A}(\phi)$ which varies with ϕ, with a total variation from $\phi=0$ to $\phi=2\pi$ given by (29). If we now distort these world lines so as to make the loop described by Q shrink toward the monopole position, we can only guarantee the continuity of $\mathcal{A}$ against such distortions if the condition is imposed that the positron and monopole world lines do not cross. The necessity of this condition had been discussed before by Rosenbaum.[7]

The dynamical variables $A_\mu(\xi)$ consist of $(A_\mu)_a$ and $(A_\mu)_b$ subject to the conditions (11), (12), and (16). As proved in Sec. II, these conditions lead to (Mg) as a kinematic equation.

We have to demonstrate now that the stability of $\mathcal{A}$ against variations of these dynamical variables, keeping the conditions discussed above satisfied, yields the remaining Maxwell and Lorentz equations.

It is easy to show that the stability of $\mathcal{A}$ against variations $\delta\vec{x}(t)$ gives rise to (Le), exactly as in the usual case without monopoles. For variations of A_μ, first consider variations $(\delta A_\mu)_a$ and $(\delta A_\mu)_b$ in the subregions of R_a and R_b outside of the overlap R_{ab}. Stability of $\mathcal{A}$ leads to equation (Me) in these subregions. Next consider variations $(\delta A_\mu)_a$ and $(\delta A_\mu)_b$ in R_{ab}, but with

$$\delta((A_\mu)_a-(A_\mu)_b)=0\,. \tag{30}$$

Stability of $\mathcal{A}$ against such variations leads to equation (Me) in R_{ab}. Lastly we have to consider variations $(\delta A_\mu)_a$ and $(\delta A_\mu)_b$ in R_{ab} that violate (30). It is sufficient to consider the case $(\delta A_\mu)_b=0$. Then conditions (11) and (12) require

$$(\delta A_\mu)_a=\delta\alpha_\mu=\delta\beta_{,\mu}\,, \tag{31}$$

where we have used (24). Thus (21) and (23) give

$$\delta\mathcal{A}=\delta\mathcal{A}_1=-e\delta\beta(Q)+e\delta\beta(Q)=0\,. \tag{32}$$

Clearly this conclusion also holds for more twisted world lines such as (b) of Fig. 2.

To summarize, we have demonstrated that stability of $\mathcal{A}$ against δx^μ [i.e., $\delta\vec{x}(t)$] and δA_μ, subject to the proper conditions, yields (Me) and (Le), respectively. We emphasize that the multivaluedness of $\mathcal{A}$ (which is defined modulo $4\pi eg$) does not influence the validity of this statement.

Variation of the world line of the monopole is more cumbersome to study since it necessitates a change of regions R_a and R_b. We shall circumvent this complication by investigating the dual action integral.

V. DUAL ACTION INTEGRAL

The dual field, designated by an overbar, has already been used in Sec. I. Under the dual operation, ξ^μ does not change, but

$$\bar{e}=g,\quad \bar{g}=-e,\quad \bar{f}_{\mu\nu}=-\tfrac{1}{2}\epsilon_{\mu\nu\sigma\rho}f^{\sigma\rho}\,, \tag{33}$$

$$\bar{E}_k=H_k,\ \text{and}\ \bar{H}_k=-E_k\,. \tag{34}$$

Also

$$\bar{\bar{e}}=-e,\quad \bar{\bar{g}}=-g,\ \text{and}\ \bar{\bar{f}}_{\mu\nu}=-f_{\mu\nu}\,. \tag{35}$$

Furthermore,

$$\bar{f}_{\mu\nu}\bar{f}^{\mu\nu}=-f_{\mu\nu}f^{\mu\nu}\,. \tag{36}$$

In particular, this dual operation can be applied to the action $\mathcal{A}$ of (21) to give

$$\bar{\mathcal{A}}(x,X,\bar{A})=-m\left[\int ds\right]_{\text{positron}}-M\left[\int ds\right]_{\text{monopole}}$$
$$-(16\pi)^{-1}\int d^4\xi\,\bar{f}_{\mu\nu}(\xi)\bar{f}^{\mu\nu}(\xi)$$
$$+g\int \bar{A}_\mu(X)dX^\mu\,. \tag{37}$$

In this formula we have introduced the dual potential $\bar{A}_\mu$ defined by

$$\bar{A}_{\mu,\nu}-\bar{A}_{\nu,\mu}=\bar{f}_{\mu\nu}\,. \tag{38}$$

Because of the presence of the positron, $\bar{A}_\mu$ is defined separately in two regions $\bar{R}_a$ and $\bar{R}_b$. At any given time t, $\bar{R}_a$ and $\bar{R}_b$ are defined exactly as R_a and R_b were defined in (9), except that $\bar{r}, \bar{\theta}, \bar{\phi}$ now refer to spherical coordinates with the *positron* position at that time as the origin. The overlap size $2\bar{\delta}$ need not be related to 2δ. The definition of the integral $\oint$ in (37) is the same as the corresponding one in (21), except that the regions are now $\bar{R}_a$ and $\bar{R}_b$. Clearly,

$$\bar{\bar{\mathfrak{A}}} = \mathfrak{A} , \tag{39}$$

and $\bar{\mathfrak{A}}$ is defined modulo $4\pi g e$.

Notice that the $\int d^4\xi$ terms in (37) and (21) are equal in magnitude but opposite in sign because of (35).

The discussions of Secs. II, III, and IV can of course be duplicated with appropriate changes of all quantities into their respective duals. For example, (Me) is now kinematic, while (Mg) is dynamic resulting from the condition of stability of $\bar{\mathfrak{A}}(x,X,\bar{A})$ against $\delta\bar{A}$.

Consider for *fixed* nonintersecting world lines of the positron and the monopole x^μ and X^μ (both of which are timelike) the quantity

$$\bar{\mathfrak{A}}_0(x,X) = \text{extremum of } \bar{\mathfrak{A}}(x,X,\bar{A}) \text{ with respect to } \delta\bar{A} . \tag{40}$$

The field strength $f_{\mu\nu}$ generated from the extremizing $\bar{A}_\mu$ satisfies both Maxwell equations (Mg) and (Me). Since it satisfies (Mg), according to Sec. II, there exists an A_μ satisfying (10), (11), (12), and (16). At such an A_μ $\mathfrak{A}(x,X,A)$ attains an extremum with respect to δA since (Me) is satisfied. We define

$$\mathfrak{A}_0(x,X) = \text{extremum of } \mathfrak{A}(x,X,A) \text{ with respect to } \delta A . \tag{41}$$

Notice that the extrema (40) and (41) are attained with potentials $\bar{A}$ and A that give the *same* field strengths $f_{\mu\nu}$. We shall later prove the following:

Lemma. $\bar{\mathfrak{A}}_0(x,X) - \mathfrak{A}_0(x,X) = \text{integrals at infinity.}$ (42)

Since integrals at infinity play no role in the action principle, it follows from this lemma that the stability of $\mathfrak{A}(x,X,A)$ against all variations gives all four Maxwell and Lorentz equations. In particular (Lg) follows from the stability of $\mathfrak{A}$ against δX because (42) implies $\delta\mathfrak{A}_0 = \delta\bar{\mathfrak{A}}_0$, and $\delta\bar{\mathfrak{A}}_0 = 0$ against δX implies (Lg).

Proof of lemma. (a) First consider the special case when there is a positron but no magnetic monopole. In this special case

$$\mathfrak{A}_0 - \bar{\mathfrak{A}}_0 = e\int_{-\infty}^{\infty} A_\mu\, dx^\mu - (8\pi)^{-1}\int d^4\xi\, f_{\mu\nu}(\xi) f^{\mu\nu}(\xi) . \tag{43}$$

Since the Maxwell equations are satisfied, (3) can be used to rewrite the first term on the right-hand side of (43):

$$e\int_{-\infty}^{\infty} A_\mu\, dx^\mu = e\int d^4\xi \int_{-\infty}^{\infty} ds\, \frac{dx^\mu}{ds} A_\mu(\xi)\delta^4(\xi - x) = -(4\pi)^{-1}\int d^4\xi\, A_\mu(\xi) f^{\mu\nu}{}_{,\nu}(\xi) . \tag{44}$$

Substitution into (43) gives

$$\mathfrak{A}_0 - \bar{\mathfrak{A}}_0 = -(4\pi)^{-1}\int d^4\xi [A_\mu(\xi) f^{\mu\nu}(\xi)]_{,\nu} , \tag{45}$$

which is equal to a surface integral at infinity. The lemma is thus proved in this special case.

(b) Next consider the case where the world lines are such that for all t, $Z(t) > z(t)$. In this case we can construct a three-dimensional surface S which at each fixed t is the plane

$$\xi^3 = \tfrac{1}{2}[Z(t) + z(t)] , \tag{46}$$

as illustrated in Fig. 4. S separates space-time into two regions, an upper region G containing the world line of the monopole and a lower region E containing the world line of the positron. Furthermore, E is completely in R_b and G is completely in $\bar{R}_a$. We write

$$\mathfrak{A}_0 - \bar{\mathfrak{A}}_0 = e\oint A_\mu\, dx^\mu - g\oint \bar{A}_\mu\, dX^\mu - (8\pi)^{-1}\int d^4\xi\, f_{\mu\nu} f^{\mu\nu} = B_E - B_G , \tag{47}$$

where

$$B_E = e\oint A_\mu\, dx^\mu - (8\pi)^{-1}\int_E d^4\xi\, f_{\mu\nu} f^{\mu\nu} , \tag{48}$$

$$B_G = g\oint A_\mu\, dX^\mu - (8\pi)^{-1}\int_G d^4\xi\, \bar{f}_{\mu\nu}\bar{f}^{\mu\nu} . \tag{49}$$

Since the positron world line is entirely in E which is in R_b, we replace the A_μ in (48) by $(A_\mu)_b$. We can then process the right-hand side of (48) in exactly the same way that we processed (43)–(45), with the space-time region E replacing the whole space-time. This leads to

$$B_E = -(4\pi)^{-1}\int_S (A_\mu)_b f^{\mu\nu}\, d\sigma_\nu + \text{terms at infinity} , \tag{50}$$

where $d\sigma_\nu$ is the three-dimensional surface area on S. Similarly,

$$B_G = (4\pi)^{-1}\int_S (\bar{A}_\mu)_{\bar{a}}\, \bar{f}^{\mu\nu}\, d\sigma_\nu + \text{terms at infinity} . \tag{51}$$

The sign difference is due to the fact that G is above S while E is below. Expressing $\bar{f}^{\mu\nu}$ in (51)

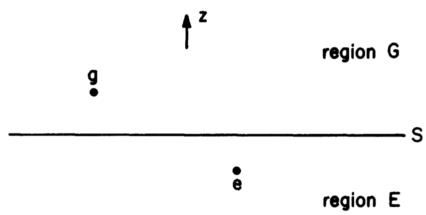

FIG. 4. Division of space-time into regions G and E. The figure shows the division at one instant of time.

in terms of $f_{\alpha\beta}$ and integrating by parts yield

$$B_G = -(8\pi)^{-1}\epsilon^{\mu\nu\alpha\beta}\int (\bar{A}_\mu)_{\bar{a}} f_{\alpha\beta}\, d\sigma_\nu + \text{terms at infinity}$$

$$= (8\pi)^{-1}\epsilon^{\mu\nu\alpha\beta}\int \bar{f}_{\mu\beta}(A_{\alpha_b})\, d\sigma_\nu + \text{terms at infinity}$$

$$= (4\pi)^{-1}\int f^{\nu\alpha}(A_\alpha)_b\, d\sigma_\nu + \text{terms at infinity}$$

$$= B_E + \text{terms at infinity}. \qquad (52)$$

The lemma follows.

(c) Next consider the general case. Since the world lines of the positron and the monopole are assumed not to intersect, there always exists a three-dimensional surface S that separates all space-time into an "upper" region G containing the monopole world line and a "lower" region E containing the positron world line. But now E (and S) in general contain both points in R_a and points in R_b. Equations (47), (48), and (49) remain valid. But (50) should be replaced by

$$B_E = -(4\pi)^{-1}\fint_S A_\mu f^{\mu\nu}\, d\sigma_\nu + \text{terms at infinity}, \qquad (53)$$

where we have introduced a new three-dimensional integral $\fint_S$, which is a generalization of the corresponding one-dimensional integral $\fint$. The precise definition of this new integral and its properties are given in the Appendix. Equations (51) and (52) then remain valid with the $\int$'s replaced by $\fint$'s. The lemma follows.

VI. REMARKS

1. We have assumed throughout that $m>0$, $M>0$. That is, the positron and the monopole both have nonvanishing masses.

2. Clearly the definitions of R_a and R_b are quite flexible. Given the monopole world line X^μ, continuous distortions of the boundaries of regions R_a and R_b starting from (9) are allowed, provided R_a, R_b together always cover the whole of space-time outside of the monopole world line. Amalgamation of boundaries is not allowed in this process.

3. Under a gauge transformation $A_\mu \to A'_\mu$, provided $f_{\mu\nu}$ is unchanged and (16) remains valid, $\mathfrak{a}(x,X,A)$ changes by terms evaluated at infinity.

4. The condition $A^\mu{}_{,\mu}=0$ is not used at all in this paper.

5. A key point of this paper is the reduction of the Maxwell equation (Mg) to kinematics in Sec. II. While the method used for this reduction is very much in the spirit[5] of the Chern-Weil theorem in fiber-bundle theory, there is considerable difference also. The present considerations focus on the "exponent" of the phase factor of Ref. 5. The phase factor itself cannot be given meaning without the introduction of the Planck constant h.

6. If one formulates, after Feynman,[8] the quantization procedure by path integration, one would be dealing with integrals of $\exp(i\mathfrak{a}/\hbar)$. Since $\mathfrak{a}$ is only definable modulo $4\pi eg$, this process is meaningful only if $4\pi eg/\hbar = 2\pi(\text{integer})$. That is Dirac's quantization rule

$$2eg/\hbar = \text{integer}$$

must be satisfied. We are currently working on this problem.

7. When there are two or more neighborhoods (e.g., R_a and R_b), the barred integrals such as $\fint A_\mu dx^\mu$ are the natural ones while the corresponding ordinary integrals such as $\int A_\mu dx^\mu$ are not even definable. Thus the bar is really superfluous. We retain the bar in this paper only to draw attention to the special nature of the integral when it spans more than one region.

APPENDIX

We assume throughout this appendix that (Me) and (Mg) are satisfied, but not necessarily (Le) and (Lg).

Before defining the three-dimensional integral $\fint$ and proving (53) and (52) we shall need two preliminaries.

1. We shall rewrite the integral in (50) as follows:

$$\int_S A_\mu f^{\mu\nu}\, d\sigma_\nu = -\tfrac{1}{6}\int A_\mu f^{\mu\nu}\epsilon_{\nu\xi\eta\zeta}\, dx^{\xi\eta\zeta}$$

$$= -\tfrac{1}{2}\int A_\xi \bar{f}_{\eta\zeta}\, dx^{\xi\eta\zeta}, \qquad (A1)$$

where we use the usual notation

$$dx^{\xi\eta\zeta} = -dx^{\eta\xi\zeta} = -dx^{\xi\zeta\eta}.$$

We have assumed that S is in only one region R_a or R_b.

It is easy to prove that

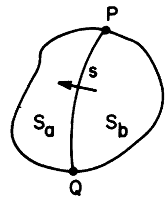

FIG. 5. Three-dimensional surface S divided into S_a and S_b. S_a is in R_a and S_b in R_b. s is the two-dimensional surface between S_a and S_b. PQ is the boundary of s and is a one-dimensional loop.

$$\oint_S A_\lambda \bar{f}_{\mu\nu} dx^{\lambda\mu\nu} = -\int_R f_{\mu\nu} f^{\mu\nu} d^4x \tag{A2}$$

if S is the boundary of a space-time region R entirely in R_a or R_b, which contains no parts of the world line of either the positron or the monopole.

2. The one-dimensional integral $\fint$ defined in Sec. III has the property that if a loop L is the border of a two-dimensional region Y, then

$$\oint_L A_\mu dx^\mu = -\tfrac{1}{2}\int_Y f_{\mu\nu} dx^\mu dx^\nu . \tag{A3}$$

This formula is obvious if Y is all in R_a or all in R_b. If Y is partly in R_a and partly in R_b, (A3) is correct because of the terms $\mathcal{L}$ in the definition of $\fint$, as in (25) or (27). The negative sign in (A3) derives from the usual convention of a right-handed loop for the sense of L.

We proceed now as follows:

3. Consider a three-dimensional surface S that is divided into two parts S_a and S_b with the boundary s between them, as illustrated in Fig. 5. We define

$$\fint_S A_\lambda \bar{f}_{\mu\nu} dx^{\lambda\mu\nu} = \int_{S_a} (A_\lambda)_a \bar{f}_{\mu\nu} dx^{\lambda\mu\nu} + \int (A_\lambda)_b \bar{f}_{\mu\nu} dx^{\lambda\mu\nu} + \int_S \beta \bar{f}_{\mu\nu} dx^{\mu\nu} , \tag{A4}$$

where s is taken to have the normal to it in the direction from S_b to S_a as indicated by the arrow in Fig. 5. The surface s is, of course, in R_{ab}. If we fix its boundary PQ, which is a one-dimensional loop, and move s, within R_{ab}, to s', it is easy to show that according to (A4), the integral $\fint$ is unchanged, provided S does not intersect either the positron or the monopole world line.

Thus the three-dimensional integral on the left-hand side of (A4) is dependent only on S and on the loop PQ which divides the boundary of S. Changing PQ on the boundary of S *does* lead to a change in the integral, just as in the one-dimensional integral case where

$$\mathcal{A}_1(B_a, D_b) \neq \mathcal{A}_1(B_b, D_b) .$$

Now if S itself is a boundary of a four-dimensional region R, then S has no boundary of its own. Thus there is no PQ and the integral defined by (A4) is well defined. We have assumed that the world line of either the positron or the monopole does not enter the region enclosed by S. In fact, in this case,

$$\oint_S A_\lambda \bar{f}_{\mu\nu} dx^{\lambda\mu\nu} = -\int_R f_{\mu\nu} f^{\mu\nu} d^4x , \tag{A5}$$

which is the generalization of (A2), and is analogous to (A3).

4. Equation (53) is now easily proved if we take R to be the lower region E which contains the electron world line. One has instead of (A5)

$$-\int_E f_{\mu\nu} f^{\mu\nu} d^4x = \fint_S A_\lambda \bar{f}_{\mu\nu} dx^{\lambda\mu\nu} - 8\pi e \fint A_\mu dx^\mu + \text{terms at infinity.} \tag{A6}$$

The first integral on the right-hand side is equal to, by the reasoning that led to (A1),

$$-2\fint_S A_\mu f^{\mu\nu} d\sigma_\nu .$$

Equation (53) follows immediately.

5. It remains to prove the generalization of (52), when S, the boundary of E, is not entirely in R_b, and not entirely in $\bar{R}_a$. We shall only demonstrate the proof in the case that S is entirely in $\bar{R}_a$, but not entirely in R_b. Divide S into regions S_a and S_b as in Fig. 5. Then (53) states that, omitting terms at infinity,

$$8\pi B_E = \fint_S A_\lambda \bar{f}_{\mu\nu} dx^{\lambda\mu\nu} = \int_{S_a} A_\lambda \bar{f}_{\mu\nu} dx^{\lambda\mu\nu} + \int_{S_b} A_\lambda \bar{f}_{\mu\nu} dx^{\lambda\mu\nu} + \int_s \beta \bar{f}_{\mu\nu} dx^{\mu\nu}$$

$$-8\pi B_G = \int_S \bar{A}_\mu f_{\lambda\nu} dx^{\lambda\mu\nu} ,$$

where we have used $\bar{\bar{f}} = -f$. Thus

$$8\pi(B_E - B_G) = 2\int_{S_a} (A_\lambda \overline{A}_\mu)_{,\nu} dx^{\lambda\mu\nu} + 2\int_{S_b} (A_\lambda \overline{A}_\mu)_{,\nu} dx^{\lambda\mu\nu} + \int_s \beta \overline{f}_{\mu\nu} dx^{\mu\nu}$$

$$= -2\int_s (A_\lambda)_a \overline{A}_\mu dx^{\lambda\mu} + 2\int_s (A_\lambda)_b \overline{A}_\mu dx^{\lambda\mu} + \int_s \beta \overline{f}_{\lambda\mu} dx^{\lambda\mu}$$

$$= \int_s (-2\beta_{,\lambda} \overline{A}_\mu + \beta \overline{f}_{\lambda\mu}) dx^{\lambda\mu} = -2\int_s (\beta \overline{A}_\mu)_{,\lambda} dx^{\lambda\mu} .$$

Since the boundary of s is at infinity we have

$B_E - B_G$ = terms at infinity.

*Work supported in part by the U. S. ERDA under Contract No. E(11-1)-3227.

†Work supported in part by the National Science Foundation under Grant MPS74-13208 A01.

[1]We use the notation $x^0 = -x_0 = t$, $x^i = x_i$ $(i = 1, 2, 3)$. Comma means derivative. It is well known that, because of the point nature of the positron, such a coupled system of equations does not possess any finite solution. We are here not concerned with such questions of infinities.

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[3]P. A. M. Dirac, Phys. Rev. 74, 817 (1948). See also J. Schwinger, *Particles, Sources and Fields* (Addison-Wesley, Reading, Mass., 1970 and 1973), Vols. 1 and 2.

[4]Gregor Wentzel, Prog. Theor. Phys. Suppl. 37–38, 163 (1966).

[5]Tai Tsun Wu and Chen Ning Yang, Phys. Rev. D 12, 3845 (1975).

[6]The idea that one divides space into two regions is familiar in the usual problem of choosing a singularity-free coordinate system on the surface of a globe. It is well known that a single coordinate system (such as the latitudes and longitudes) must have some singularities (such as the north and south poles). To avoid singularities, one can divide the globe into two overlapping regions R_a and R_b. For example, one can choose R_a to contain more than the northern hemisphere but not the south pole, and R_b to contain more than the southern hemisphere but not the north pole. In each region a singularity-free coordinate system can be chosen. In the overlap, the two systems of coordinates are transformable into each other.

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