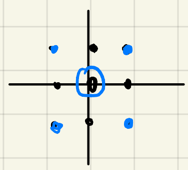
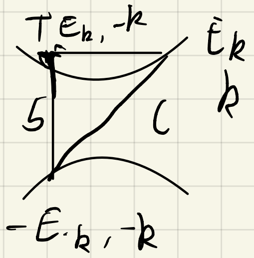
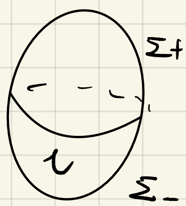
总结: 1. T, C, S 对称性





2. Stokes 定理



$$\int \vec{B}\cdot d\vec{r} = \left(\int_{\Sigma_+} + \int_{\Sigma_-}\right)\vec{B}\cdot d\vec{l}$$

$$= \int_l (\vec{A}_+ - \vec{A}_-)\, d\vec{l} = \int_l d\theta = 2\pi N$$

$$\vec{B} = \nabla\times\vec{A}$$

$$ch_{2n+2}(F) = d\, Q_{2n+1}(A, F)$$

几何 $\begin{cases} d\theta \\ d\Omega \end{cases}$

奇数维 AIII $\Rightarrow$ $\{H, S\}=0$

$HS+SH=0$ 或 $SHS^{-1}=H$

$S^2=1$

设 $S=\sigma_z$ $H=\begin{pmatrix} a & b \\ b^\dagger & c \end{pmatrix}$

$SH+HS$

$=\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} a & b \\ b^\dagger & c \end{pmatrix}+\begin{pmatrix} a & b \\ b^\dagger & c \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$=\begin{pmatrix} a & b \\ -b^\dagger & -c \end{pmatrix}+\begin{pmatrix} a & -b \\ b^\dagger & -c \end{pmatrix}=0$

$\Rightarrow \begin{cases} a=0 \\ c=0 \end{cases} \Rightarrow H=\begin{pmatrix} 0 & b \\ b^\dagger & 0 \end{pmatrix}$

$\because S^2=1 \Rightarrow \lambda=\pm 1$

$\therefore S=U\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}U^\dagger$

$\therefore \exists U, \quad UHU^\dagger=\begin{pmatrix} 0 & q \\ q^\dagger & 0 \end{pmatrix}$

$\because H^\dagger H=1 \Rightarrow q\in U(n)$

$\pi_d(U(n))$ 1d winding number

$\frac{1}{2\pi i}\oint \frac{dz}{z} \Rightarrow \frac{1}{2\pi i}\oint \mathrm{Tr}(q^{-1}dq)$

$(qq^\dagger=1) = \frac{1}{2\pi i}\oint \mathrm{Tr}(q^\dagger dq)$

$\begin{cases} 1d & \frac{dz}{z}=d\theta \\ 2d & \vec{n}\cdot(\partial_x\vec{n}\times\partial_y\vec{n})=d\Omega \\ nd & \varepsilon^{i_0 i_1\cdots i_n} n_{i_0}\partial_1 n_{i_1}\cdots\partial_n n_{i_n} \end{cases}$

有人使用 Green function

$\mathrm{Tr}(G^{-1}dG)$

symbol $A^2 = A\wedge A$

$A^3 = A\wedge A\wedge A$

$(q^\dagger dq)^2 = q^\dagger dq \wedge q^\dagger dq$

$$\mathrm{Tr}[(q^\dagger dq)^2] = \mathrm{Tr}(q^\dagger dq\wedge q^\dagger dq)$$
$$= \mathrm{Tr}(q^\dagger d_x q\wedge q^\dagger d_y q) - \mathrm{Tr}(q^\dagger d_y q\wedge q^\dagger d_x q)$$
$$= 0$$

$$\nu_{2n+1}(q) = \int \omega_{2n+1}(q)$$

$$\omega_{2n+1}(q) = \frac{(-1)^n n!}{(2n+1)!}\left(\frac{i}{2\pi}\right)^{n+1} \mathrm{tr}[(q^\dagger dq)^{2n+1}]$$
$$= \frac{(-1)^n n!}{(2n+1)!}\left(\frac{i}{2\pi}\right)^{n+1} \varepsilon^{\alpha_1\alpha_2\cdots\alpha_{2n+1}} \mathrm{Tr}(q^{-1}\partial_{\alpha_1} q\, q^{-1}\partial_{\alpha_2} q \cdots)$$

偶数维 $d=2n+2$

$$\underset{\in\Lambda^{2n+2}}{ch_{2n+2}} = d\,\underset{\Lambda^{2n+1}}{Q_{2n+1}}$$

可以独立

可以不独立.

奇数维

$$\nu_{2n+1} = \int \omega_{2n+1}(q)$$

Chein - Simons invarient

$$CS_{2n+1}[A,F] = \int_{BZ^{2n+1}} Q_{2n+1}(A,F)$$

$$CS_{2n+1}[A',F'] = \int_{BZ} Q_{2n+1}(A',F')$$

$A = \langle\psi_i|d|\psi_j\rangle \Rightarrow$ non-abelian geometry phase

$|\psi_\alpha\rangle = g_{\alpha i}|\psi_i\rangle$

$$\begin{cases} A' = g^{-1}\wedge g + g^{-1}dg \\ F' = g^{-1}Fg. \end{cases}$$

证明: $Q_{2n+1}(A', F') - Q_{2n+1}(A, F) = Q_{2n+1}(g^{-1}dg, 0) + d\alpha_{2n-2}$

eq. 24 NJP, D, 065010 (2010)

$Q_{2n+1}(g^{-1}dg, 0) = W_{2n+1}(g)$ eq. 25

$W_{2n+1} = e^{i2\pi CS_{2n+1}}$ well-defined

$\pi_d(G/H)$ 计算

G和H U, O, SP.

$U(N+M)/U(N)\times U(M)$ Grassmanian space

$\cong N$ — +1

$\cong M$ — -1

Hall effect

d	1	2	3	4		
	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

文献: Lundell, Concise Tables of James Number And some Homotopy of ... 1990 会议论文

P_2 表格

引用 Bott 1959

长正合序列

$\Rightarrow$ $0 \to A \to B \to 0$, $A \cong B$

$0 \to A \to B \to C \to 0$, $B = A \otimes C$

$\vdots$

$\pi_d(H) \to \pi_d(G) \to \pi_d(G/H) \to$

$\to \pi_{d-1}(H) \to \pi_{d-1}(G) \to \pi_{d-1}(G/H) \to$

$\vdots$

$\to \pi_0(H) \to \cdots$

$\pi_d(U(n)) = \pi_d(U)$

$U = \lim_{n\to\infty} U(n)$

$$\begin{cases} d=0 & 0 \\ 1 & \mathbb{Z} \\ & 0 \\ & \mathbb{Z} \end{cases}$$

	$U(N)\times U(M)$	$U(N+M)$	$U(N+M)/U(N)\times U(M)$
3	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}$	
2	0	0	
1	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}$	
0	0	0	

stiefel manifold

$U(N+M)/U(N)$

$Sp(N+M)/Sp(N)$

$O(N+M)/O(N)$

	$U(N)$	$U(N+M)$	$U(N+M)/U(N)$
5	$\mathbb{Z}$	$\mathbb{Z}$	0
4	0	0	0
3	$\mathbb{Z}$	$\mathbb{Z}$	0
2	0	0	0
1	$\mathbb{Z}$	$\mathbb{Z}$	0
0	0	0	0

	$U(M)$	$U(N+M)/U(N)$	$U(N+M)/U(N)\times U(M)$
3	$\mathbb{Z}$	0	0
2	0	0	$\mathbb{Z}$
1	$\mathbb{Z}$	0	0
0	0	0	

类似 $Sp(N+M)/Sp(N)\times Sp(M)$

$Sp(N)$	$Sp(N+M)/Sp(M)$	$Sp(N+M)/Sp(M)\times Sp(N)$
	0	$\cancel{0}$
$\cancel{0}$	0	$\cancel{0}$
$\cancel{0}$	0	0
0	0	

$$\pi_d[Sp(N+M)/Sp(N)\times Sp(M)] = \pi_{d-1}[Sp(N)]$$

作业：利用 Lundell Tabel I

stiefel manifold 推导 NJP. D. 065010 (2010)

Table 3

1) 具体计算 $H_k \rightarrow$ Topo # 困难 $\langle \psi | d | \psi \rangle$

$$\frac{d}{dx}\psi(x) \neq \frac{\psi(x+\delta x)\, e^{\text{迹}} - \psi(x)}{\delta(x)}$$

2) 具体材料

TI Time reversal

涉及 Topo # 计算

3) 边界效应

Hall effect

chiral current