应用　Liquid Crystal, XY model, F/B/A, QWZ model

图像：几何

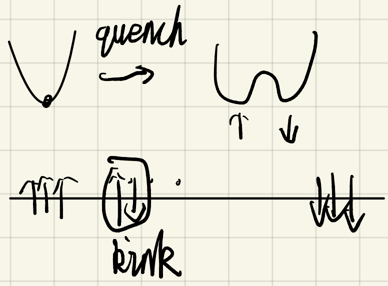
Topo 不变量

$\pi_d(X)$

典型空间 简单 $\begin{cases} S^n \\ \text{Lie group} \\ \mathbb{R}P^d,\ \mathbb{C}P^d \end{cases}$

自发对称破缺

Kibble-Zurek 机制



$\pi_d(M) = \pi_d(G/H)$

H: little group 小群

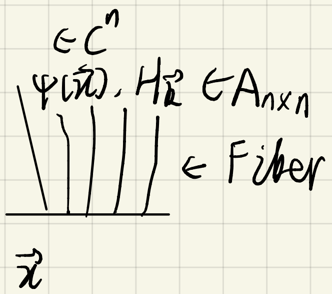
isotropical group

迷向群

Fiber Bundle 纤维丛

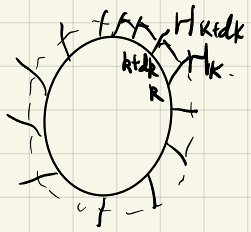
$\psi(\vec{x})$

$H_{\vec{x}}$



底空间

x, k



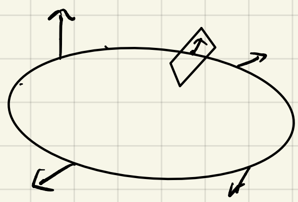
$\psi_k \in \mathbb{R}$

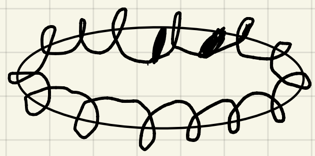
$S^1 \to \mathbb{R}$

$x \to \psi$

$S^1 \to \mathbb{R}$

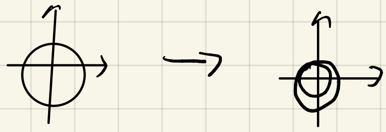
$S^1 \to S^1$





图像: 几何

$S^1 \to S^1 \Rightarrow \pi_1(S^1) = \mathbb{Z}$ $\quad z' = z^n$



$\vec{x} \in S^d \qquad \psi_{\vec{x}} \in X$

$\psi_{\vec{x}}$ 随空间 ($\vec{x}$, $\vec{k}$, ...) 变化而变化 $\Rightarrow$ 几何

空间: $\vec{x} \to M$ 空间, M 是什么?

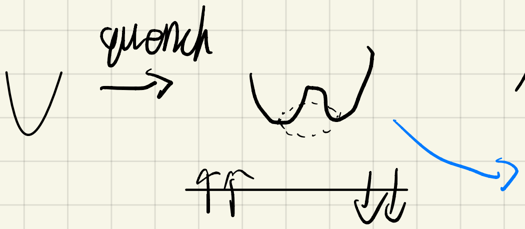
① Liquid crystal $\quad \vec{n}(\vec{x}) \quad \vec{n}: \vec{x} \to S^d, \mathbb{R}P^d$

② QM $\quad \psi_x \in \mathbb{C}, \mathbb{C}^n$, BEC, Ground state

③ Topo $\quad H_{\vec{k}} \sim U(N+M)/U(N)\times U(M)$

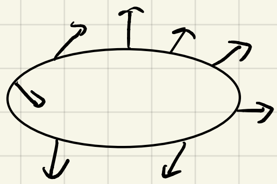
④ SSB $\quad M \cong G/H$

SSB

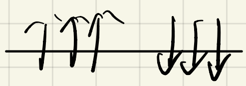


$M \cong G/H$

墨西哥帽



$S(x): \vec{x} \to \mathbb{Z}_2$



局部基态

整体不是

几个结论：

$$\pi_d(G/H) \sim \pi_d(G), \pi_d(H)$$

正合序列 exact sequence

短正合 short $0 \to A \to B \to 0$

长正合 long $0 \to A \to B \to C \to 0$

由许多短正合序列构成

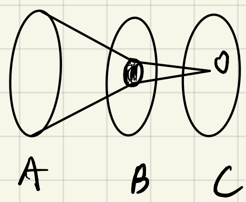
$0 \to A_1 \to B_1 \to 0 \to A_2 \to B_2 \to C_2 \to 0 \to 0 \to A_3 \cdots$

正合：

$A \to B \to C$ 是正合的

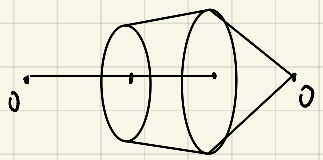
前的像 = 后的核

$\mathrm{Im}(A \to B) = \ker(B \to C)$



$0 \to A \to B \to 0 \Rightarrow A \cong B$

同构

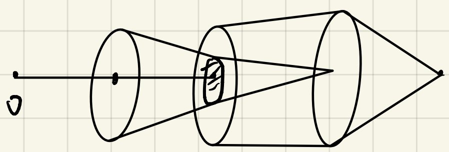


$0 \to A \to B \to C \to 0 \Rightarrow B \cong A \otimes C$

$B/A \cong C$

同态基本定理：$f: A \to B$

$\mathrm{Im} f = A/\ker f$



找 $\pi_d(G/H)$

查表

	H	G	$M \cong (G/H)$
π_3	$\pi_3(H) \to$	$\pi_3(G) \to$	$\pi_3(M)$
π_2	$\pi_2(H) \to$	$\pi_2(G) \to$	$\pi_2(M)$
π_1	- - - -		
π_0			

正合序列

$$\to \pi_d(H) \to \pi_d(G) \to \pi_d(M) \to \pi_{d-1}(H) \to \pi_{d-1}(G) \to \cdots$$

几何相（绝热过程）$\Rightarrow$ Berry phase.

绝热过程：（热力学 $dU = dQ + dW$）

$dQ = 0$ adiabatic process

经典力学 $\Rightarrow$ 绝热不变量 $I = \frac{1}{2\pi}\oint p\,dq$ 意义？

原子物理 $\frac{1}{2\pi}\oint p\,dq = n\hbar$. 量子化

Recall Bohr model

QM

QM中 $I = \frac{1}{2\pi}\oint p\,dq$

①不变量

②面积

③几何属性

$\downarrow$

$\frac{1}{2\pi}\oint \langle\psi|P|\psi\rangle dq$

$\downarrow$

$\frac{1}{2\pi}\oint \langle\psi|\frac{\partial}{\partial R}|\psi\rangle d\vec{R}$

几何相

Berry phase

$$\frac{1}{2\pi}\oint_{\partial V} p\,dq \overset{\text{stokes}}{=} \frac{1}{2\pi}\oint_{V} dp\,dq = \frac{\text{面积}}{2\pi}$$

推导几何相：
$$\begin{cases} H(\vec{x}, \vec{R}(t)) & \leftarrow \text{外部参数} \\ R(0) = R(t) \end{cases}$$

e.g. $H = \frac{p^2}{2m} + \frac{1}{2}m(\omega+\alpha R)^2 x^2$

$H = \frac{p^2}{2m(1+\alpha R)} + \frac{1}{2}m\omega^2 x^2$

$H = (\vec{B} + \alpha\vec{R})\cdot\vec{\sigma}$

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = H\psi(\vec{x},t).$$

Adiabatic process

δx

$|\varphi\rangle \to |\varphi\rangle e^{i\theta}$

$H(\vec{x}, \vec{R}(t)) \Rightarrow$ 本征态

$$H(\vec{x},\vec{R}(t))\,\varphi_n(\vec{x},\vec{R}(t)) = \lambda_n(t)\,\varphi_n(\vec{x},\vec{R}(t))$$

1. 满足 Schrodinger eq.

2. Adiabatic condition $|\varphi_n(\vec{x},t)\rangle = |\varphi_n(\vec{x},\vec{R}(t))\rangle e^{i\theta(t)}$ $\leftarrow$ 不可能跟 x 有关.

$$\Rightarrow i\frac{\partial}{\partial t}|\varphi_n(x,\vec{R}(t))\rangle e^{i\theta(t)} - \dot{\theta}(t)|\varphi_n(x,\vec{R}(t))\rangle = H|\varphi_n\rangle e^{i\theta}$$

$$\Rightarrow \langle\varphi_n|i\frac{\partial}{\partial t}|\varphi_n\rangle - \dot{\theta} = \langle\varphi_n|H|\varphi_n\rangle = \lambda_n$$

$$\Rightarrow \quad \dot{\theta} = \langle \varphi_n | i\frac{\partial}{\partial t} | \varphi_n \rangle - \lambda_n$$

$$\theta(T) - \theta(0) = \int_0^T \dot{\theta}\, dt = -\int_0^T \lambda_n dt + i\int_0^T \langle \varphi_n | \frac{\partial}{\partial t} | \varphi_n \rangle dt$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \vec{R}}\varphi_n \cdot \frac{\partial \vec{R}}{\partial t}$$

$$= -\int_0^T \lambda_n dt + i\oint \langle \varphi_n | \frac{\partial}{\partial \vec{R}} | \varphi_n \rangle \cdot d\vec{R}$$

$$= \underbrace{-\int_0^T \lambda_n dt}_{\text{动力学相位}} + \underbrace{\oint \langle \varphi_n | i\nabla_{\vec{R}} | \varphi_n \rangle \cdot d\vec{R}}_{\gamma}$$

动力学相位　　几何相位　　不变量

Berry phase

图像 1983

γ = solid angle

作业: $H = B_1 \sigma_z + B_2 \cos\theta\, \sigma_x + B_2 \sin\theta\, \sigma_y$

$$\left.\begin{array}{l} ①\ |\varphi(\theta)\rangle \\ ②\ \frac{\partial}{\partial \theta}|\varphi(\theta)\rangle \\ ③\ \int_0^{2\pi} \langle \varphi(\theta) | \frac{\partial}{\partial \theta} | \varphi(\theta) \rangle d\theta \end{array}\right\} \text{立体角}$$