$\pi_d(X)$ $S^d \to X$ 的映射有多少种不同伦等价的类

① 数学定义 → 群

② 简单表 $\pi_d(S^n)$ $\pi_d(U(n))$ $\begin{cases} n \text{小} \Rightarrow \text{unstable} \\ n \text{足够大} \Rightarrow \pi_d(U) \text{ stable} \end{cases}$

$\pi_d(O(n))$ $\pi_d(SO(n))$

$\pi_d(Sp(n))$ $\pi_d(\mathbb{R}P^n)$ $\pi_d(\mathbb{C}P^n)$

$U(n)$ $H = U^\dagger \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix} U$ $\mu$ 空 $+1$ 满 $-1$ $\Leftrightarrow \tilde{H} = U^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U$

$\Leftrightarrow \tilde{H}^\dagger \tilde{H} = \mathbb{1}$

$\Leftrightarrow \tilde{H} \in U(n+m)$

实际上 $U(n+m)/(U(n) \times U(m))$

无表可查
要自己计算

$\pi_d(G/H) \sim \pi_d(G)$ 和 $\pi_d(H)$

③ 应用 Homotopy group 引入到物理中 1976 Touleure and Kleman

liquid crystal 1979 Rev

1981 RMP

a. liquid crystal

$\vec{n} \sim -\vec{n}$ ($\because F(\vec{n}) = F(-\vec{n})$)

$\vec{n}$: dipole

$\Big\}$ $\vec{n} \sim \frac{S^d}{\mathbb{Z}_2} = \mathbb{R}P^d$ $\pi_d(\mathbb{R}P^d)$

$\pi_1(\mathbb{R}P^d) = \begin{cases} \mathbb{Z} & d=1 \\ \mathbb{Z}_2 & d \geq 2 \end{cases}$

b. Fermion / Boson Anyon

相对坐标 $\vec{r} \sim -\vec{r}$

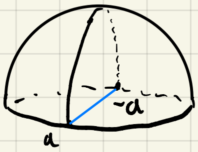
$\vec{r} \in \mathbb{R}^+ \times S^{d-1}/\mathbb{Z}_2 = \mathbb{R}^+ \times \mathbb{R}P^{d-1}$

$$\pi_1(\mathbb{R}^+\times\mathbb{RP}^{d-1}) = \pi_1(\mathbb{R}^+)\times\pi_1(\mathbb{RP}^{d-1}) = \begin{cases}\mathbb{Z} & d=2 \quad \text{Anyon}\\ \mathbb{Z}_2 & d>2 \quad \text{Femion / Boson}\end{cases}$$

$\mathbb{R}^+\times\mathbb{RP}^1 \Rightarrow \mathbb{Z}$ $\qquad\qquad$ $S^2/\mathbb{Z}_2 \Rightarrow \mathbb{Z}_2$

1

2





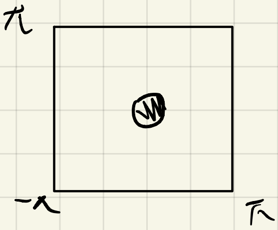
$a\sim -a$

c. $\begin{cases}\text{Topo band}\\ \text{QWZ model}\end{cases}$

为什么计算Topo band 时，$k\in \text{Torus}\simeq S^d$.

原因：BZ 边界等价为一个点 $\Rightarrow$ $\text{Torus} \overset{近似}{\sim} S^d$

$$\underbrace{[-\pi,\pi]\times[-\pi,\pi]}_{k\in T^2} \rightarrow H_k\in S^2$$



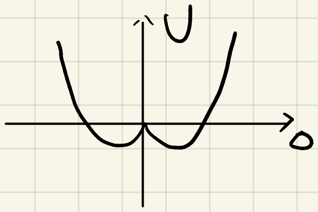


低部分有贡献大

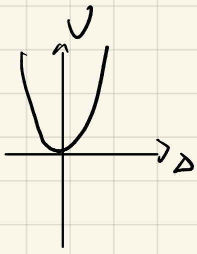
# Spontaneous Symmetry Breaking (SSB) 自发对称性破缺

$U = \alpha(T-T_c)|\Delta|^2 + \beta|\Delta|^4 \qquad \alpha>0 \quad \beta>0$

$\Delta$ order parameter 序参量




$T<T_c$




$T>T_c$

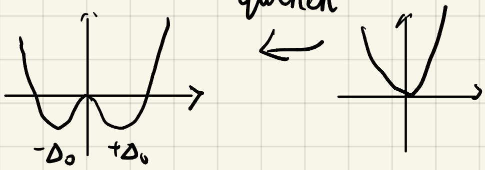
两个结论: ① order parameter space $M = G/H$ $\quad \dfrac{\pi_n(M)}{\text{Topo defect}}$

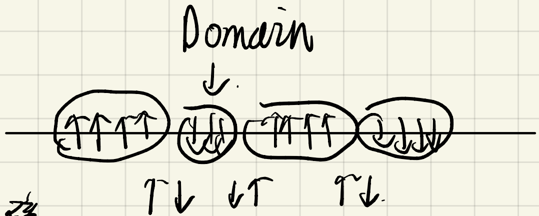
② Quench 过程 产生 Topo defect

i. $\Delta$为实数 1d system

$U = \alpha(T-T_c)\Delta^2 + \beta\Delta^4$

quench




e.g. Magnetic Domain
磁畴

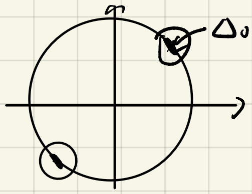
畴壁

1976 Kibble string and topo defect in cosmology

1985 Zurek Cosmological experiments in superfluid helium?

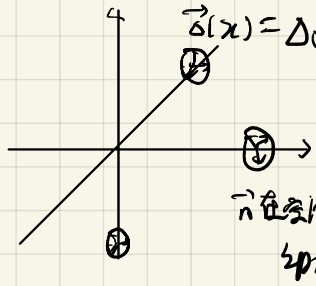
KZ 机制 $\quad n = \frac{N}{L} \sim \frac{1}{\tau_Q^{\nu}}$

ii 复数 $\Delta$ or $\vec{\Delta}$



$\Delta_0 e^{i\varphi(\vec{x})} \in \mathbb{C}$

$S^1 \to S^1$



$\vec{\Delta}(x) = \Delta_0 \hat{n}(\vec{x})$

$\vec{n}$在空间上有一些结构.
spin texture
Topo defect在空间的分布.

# Ueda - CH12 Topological excitation

isotropy group 迷向群
Little group 小群
} 子群.

Lie group $G = \{g \mid F(\psi) = F(g\psi)\}$

isotropy group $H = \{h \mid \psi = h\psi\}$.

order parameter manifold $M = G/H$

e.g. BECs

global gauge transformation

$$G = U(1)_\phi \times SO(3)_S$$

arbitrary rotation

spin-1 BEC
1 0 −1
} 相互作用与内态无关.

$\psi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\qquad U(1)\psi = \psi e^{i\theta}$

$SO(3) = e^{iS_z\alpha} e^{iS_y\beta} e^{iS_z\gamma}$ 欧拉角

$h\psi = \psi$

$= e^{i\gamma} U(0,0,\gamma)\psi$

$$\therefore M = \frac{U(1)_\phi \times SO(3)_S}{U(1)_{\phi+S_\gamma}} = SO(3)_{\phi,S}$$

立体角 $N_2 = \frac{1}{4\pi}\int_{\Sigma} d\theta d\phi \sin\theta \left|\frac{\partial(\alpha,\beta)}{\partial(\theta,\phi)}\right|$

$= \frac{1}{4\pi}\int_{\Sigma} \vec{n}\cdot d\vec{S}$