Online course on topology in condense matter

《A Short Course On Topology Insulator》

同调群 nakahara ch3. 加法群

$H_d(K, G)$

上同调群

$H^d(K, G)$

$\left.\begin{matrix} H_d(K,G) \\ H^d(K,G) \end{matrix}\right\}$ stokes定理

$\int_V d\omega = \int_{\partial V} \omega$

等价关系 $V \sim V + \partial\sigma$

边界无边界 $\partial^2 = 0$

H_d	$V \sim V + \partial\sigma$	$\partial^2 = 0$
H^d	$\omega \to \omega + d\eta$	$d^2 = 0$

群的分类 $\Rightarrow$ H
子群

目的：出现在许多文章中.

要求：意义.

$H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$

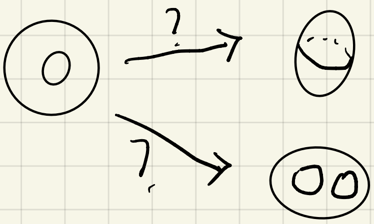
$H_1(\text{球}) = 0$

同伦群 ⇒ 广泛应用 Topo bands, Topo defect

$\pi_n(X)$

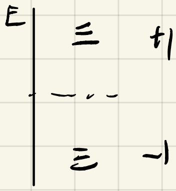
(数) (形象)

映射, 形变



※观念上的变化 $H_k = e^{i\vec{k}\cdot\vec{x}} H e^{-i\vec{k}\cdot\vec{x}}$

$H: \underset{k}{T} \to H_k$



$Q_k = U_k \begin{pmatrix} +1 & \\ & -1 \end{pmatrix} U_k^\dagger$

$Q_k^\dagger Q_k = U_k \begin{pmatrix} +1 & \\ & -1 \end{pmatrix} U_k^\dagger U_k \begin{pmatrix} +1 & \\ & -1 \end{pmatrix} U_k^\dagger$

$= U_k U_k^\dagger = \mathbb{1}$.

$Q_k \in U(n+m)$.

$\vec{V}(\vec{x})$ $V: \mathbb{R}^3 \to \mathbb{R}^d$

群 (四要素, $*$) = $\langle G, * \rangle$

同伦群 ⇒ 映射 $f: X \to Y$

群元: 映射 $f: X \to Y$.

定义 e

a^{-1}

$a*b$

$(a*b)*c = a*(b*c)$.

分类: 等价关系 $G/\sim = M$.

同伦群

$$\begin{tikzcd} & S^d \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

$U(m)$ $\mathbb{R}P^N$

$\pi_d(X) \simeq \pi_d(Y)$, 可能 $X \simeq Y$.

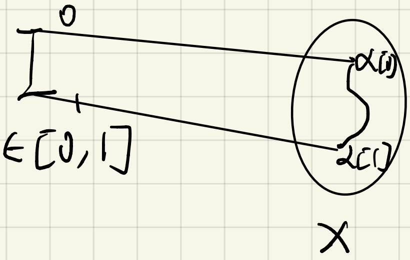
必要条件.

参考内容: nakahara CH 4

Ueda BEC CH12 ?? Homotopy group

Altland Simon CH 9 ??

path 路径 $t\in[0,1]$



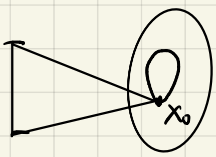
$\alpha(0)$ 可以 $\neq \alpha(1)$

$\alpha(0)=\alpha(1)$ 闭合曲线

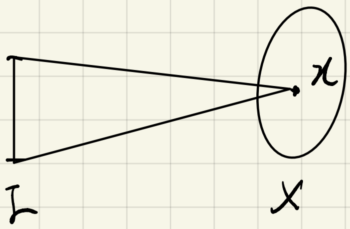
$\alpha:\ \underbrace{t\in[0,1]}_{I} \to \alpha(t)$ 是X中一条曲线

loop $\alpha(0)=\alpha(1)=x_0$ base point

x_0 基点



constant path $C_x: I\to X$ $\quad C_x(I)=x$.

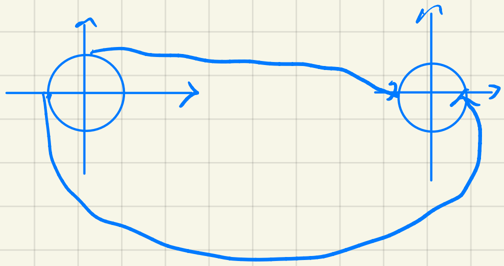


恒元 e.

constant loop

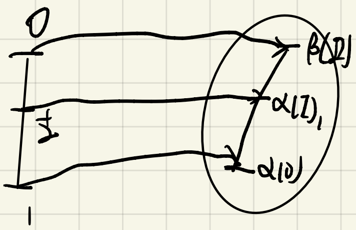
乘法. eg. $z=e^{i\theta}\in S^1 \Rightarrow z^2$

$z^{n+m}=z^n * z^m$ 乘法



$$(\alpha*\beta)=\begin{cases}\alpha(2s) & 0\le s\le \frac{1}{2}\\ \beta(2s-1) & \frac{1}{2}\le s\le 1.\end{cases}$$

$\frac{1}{2}$ 非必要

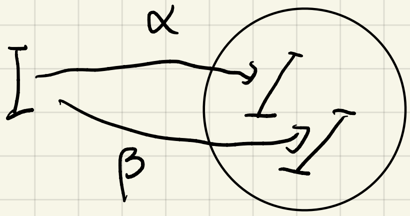


$\alpha(1)=\beta(0)$

α^{-1}: $\alpha^{-1}(s)=\alpha(1-s)$

$\alpha*\alpha^{-1}=e$? $\alpha*\alpha^{-1}=\begin{cases}\alpha(2s)\\ \alpha^{-1}(2s-1)=\alpha(2-2s)\end{cases}\neq e$.

同伦关系 Homotopy equivalence

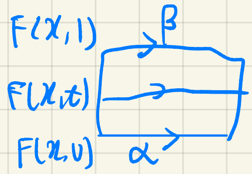


$f, g: X\to Y$

$f\sim g$: 连续形变

$F(x,t)\begin{cases}F(x,0)=f\\ F(x,1)=g\end{cases}$

e.g.



对 f 和 g: $X\to Y$

$\exists F(x,t): X\times I\to Y$

使得 $\left.\begin{array}{l}F(x,0)=f\\ F(x,1)=g\end{array}\right\}\ f\overset{F}{\sim}g$

显然 $f\sim g,\ g\sim h\Rightarrow g\sim h$.

同伦类 homotopy class

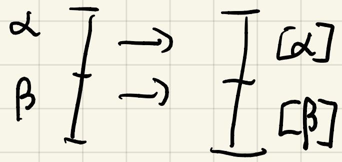
$[\alpha]=\{\beta\mid\alpha\sim\beta\}$

e.g. $[0]=\{2n\}$, $[1]=\{2n+1\}$ $\mathbb{Z}_2=\{[0],[1]\}$

助教注：参考陪集的概念 $[\alpha]=\{g\mid g\in\alpha H\}$.

发现 $[\alpha]*[\beta] = [\alpha*\beta]$

$\alpha*\beta = \alpha*\beta$



$\Downarrow$

最终 群（同伦类）

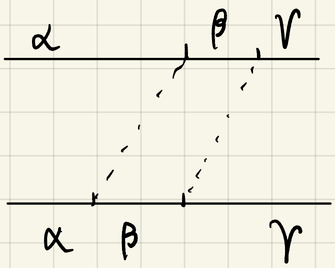
① $[C_x] = e$

② $[\alpha]*[C_x] = [\alpha]$

③ $[\alpha^{-1}]*[\alpha] = [C_x] = e$

④ $([\alpha]*[\beta])*[\gamma] = [\alpha]*([\beta]*[\gamma])$

在同伦等价的意义下相等

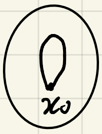


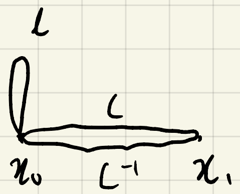
$\pi_n(X)$

原则上 $f: X \to Y$ $U(N) \to O(N)$ 过于复杂

我们关注 $f: S^d \to X$ $\pi_d(X)$

自然 $\pi_d(X) = 0 = \{[C_x]\}$ $\begin{cases} \text{平庸} \\ \text{所有映射都等价} \end{cases}$

$I \to$  C_x base point



$\pi_d(X, x_0)$ l

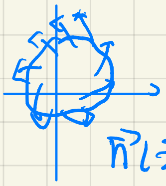
$\pi_d(X, x_1)$ $c \cdot l \cdot c^{-1}$

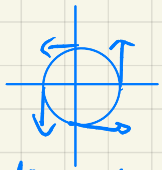
$\pi_d(X)$ ①计算(数学) 查表

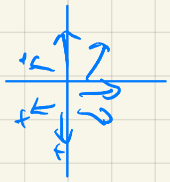
②应用

③计算(物理) $H_k \to$ Topo number

Defects XY model 液晶 0 dipole

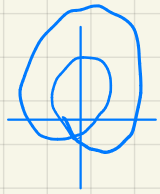






$\vec{n}(\vec{x})$

$\frac{1}{2\pi}\int d\theta$ $d\theta = \vec{n} \cdot d\vec{n}$



$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)} dz$

$\frac{1}{2\pi i}\oint \frac{dz}{z - z_0}$

Gauss $\varepsilon_0 \oint \vec{E} \cdot d\vec{s} = Ne$

Qi-Wu-Zhang

1d $\frac{1}{2\pi i}\oint \frac{dz}{z}$

2d $\frac{1}{4\pi}\oint \vec{n} \cdot (\vec{n}_x \times \vec{n}_y)\, dx\, dy$

nd $\varepsilon^{i_1 i_2 \cdots i_d} n_{i_1} (\partial_{x_1} n_{i_2}) \cdots (\partial_{x_{d-1}} n_{i_d})\, dx_1 dx_2 \cdots dx_{d-1}$