1. 题: $dx_i = a_{ij} dx_j'$, 证明 $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = J\, dx_1' \wedge dx_2' \wedge \cdots \wedge dx_n'$

其中 $J = \det(a)$.

证明: $\because dx_i = a_{ij} dx_j'$

$\therefore dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = (a_{1j_1} dx_{j_1}') \wedge (a_{2j_2} dx_{j_2}') \wedge \cdots \wedge (a_{nj_n} dx_{j_n}')$

$= a_{1j_1} a_{2j_2} \cdots a_{nj_n}\, dx_{j_1}' \wedge dx_{j_2}' \wedge \cdots \wedge dx_{j_n}'$

$\because dx_i \wedge dx_i = 0$, $dx_i \wedge dx_j = - dx_j \wedge dx_i \quad (i \neq j)$

即 $dx_{j_1}' \wedge dx_{j_2}' \wedge \cdots dx_{j_n}' = \varepsilon_{j_1 j_2 \cdots j_n} dx_1' \wedge dx_2' \wedge \cdots \wedge dx_n'$

$\therefore$ 原式 $= \varepsilon_{j_1 j_2 \cdots j_n} a_{1j_1} a_{2j_2} \cdots a_{nj_n}\, dx_1' \wedge dx_2' \wedge \cdots \wedge dx_n'$

$= \det(a)\, dx_1' \wedge dx_2' \wedge \cdots \wedge dx_n'$

证毕

2. 通过 $\frac{dz}{z} = \lim\limits_{\varepsilon \to 0} \frac{x - iy}{x^2 + y^2 + \varepsilon^2} dz$ 和 stokes 定理, 计算 $\frac{1}{2\pi i} \oint \frac{dz}{z}$

解: $\frac{dz}{z} = \lim\limits_{\varepsilon \to 0} \frac{x - iy}{x^2+y^2+\varepsilon^2} d(x + iy) = \lim\limits_{\varepsilon \to 0} \frac{(x - iy) dx + (ix + y) dy}{x^2 + y^2 + \varepsilon^2} = \lim\limits_{\varepsilon \to 0} Q dx + P dy$

$\therefore \frac{1}{2\pi i} \oint \frac{dz}{z} = \frac{1}{2\pi i} \iint_S d \frac{dz}{z} = \frac{1}{2\pi i} \iint_S \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dx \wedge dy$

$= \frac{1}{2\pi i} \iint_S \lim\limits_{\varepsilon \to 0} \frac{2i \cdot \varepsilon^2}{(x^2 + y^2 + \varepsilon^2)^2} dx dy$

不妨取极坐标. $dx dy = 2\pi r\, dr d\theta$

$\frac{1}{2\pi i} \oint \frac{dz}{z} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \lim\limits_{\varepsilon \to 0} \int_0^{r(\theta)} \frac{2 r \varepsilon^2}{(r^2 + \varepsilon^2)^2} dr$

$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \lim\limits_{\varepsilon \to 0} \left. \frac{-\varepsilon^2}{r^2 + \varepsilon^2} \right|_{r^2 = 0}^{r^2 = r^2(\theta)}$

$= \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1.$

3. 证明: $\frac{1}{2\pi i}\oint \mathrm{Tr}(A^{-1}dA) = \frac{1}{2\pi i}\oint \sum_j \frac{d\lambda_j}{\lambda_j}$

证明: 设 A 可以对角化.

$$A = SDS^{-1} = S\begin{pmatrix} a_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}S^{-1}$$

$$A^{-1} = SD^{-1}S^{-1} = S\begin{pmatrix} \lambda_1^{-1} & & & \\ & \lambda_2^{-1} & & \\ & & \ddots & \\ & & & \lambda_n^{-1} \end{pmatrix}S^{-1}$$

$$dA = d(SDS^{-1}) = dS\,DS^{-1} + S\,dD\,S^{-1} + SD\,dS^{-1}$$

$$\therefore A^{-1}dA = SD^{-1}S^{-1}d(SDS^{-1})$$

$$= SD^{-1}S^{-1}dSDS^{-1} + SD^{-1}dDS^{-1} + SdS^{-1}$$

$\because \mathrm{Tr}[(AB)C] = \mathrm{Tr}[(CA)B]$.

$$\therefore \mathrm{Tr}[A^{-1}dA] = \mathrm{Tr}(SD^{-1}S^{-1}dSDS^{-1} + SD^{-1}dDS^{-1} + SdS^{-1})$$

$$= \mathrm{Tr}[S^{-1}SD^{-1}dD] + \mathrm{Tr}(DS^{-1}SD^{-1}S^{-1}dS + SdS^{-1}).$$

$$= \mathrm{Tr}[D^{-1}dD] + \underline{\mathrm{Tr}(S^{-1}dS + SdS^{-1})}$$

$\mathrm{Tr}(dSS^{-1}) = 0.$

$$= \sum_j \frac{d\lambda_j}{\lambda_j}$$

$$\therefore \oint \mathrm{Tr}[A^{-1}dA] = \oint \sum_j \frac{d\lambda_j}{\lambda_j}$$