1 Introduction

In classical electrodynamics, electromagnetic (EM) field is a 4-vector $A_\mu$ in Minkowski spacetime. When we make a transition from the classical to quantum electrodynamics we must make an assumption about properties of the quantum of EM field, namely, whether photon is an elementary particle or a particle composed of other particles. Besides, under any quantization procedures, the result has to be physical and compatible with experiment, that photon is a quanta of free, or transverse part of EM field, even if we quantize the whole field in the first hand.

Throughout this paper, we will only talk about relativistic quantization of photon. In relativistic QED, photon must be a boson because fermion field does not transform as a vector. Normally, we quantize free EM field by Conjugate Quantization, that $A_\mu$ and its conjugate momentum $\pi^\nu$ have to obey commutation relation:

$$[A_\mu(t, x), \pi^\nu(t, y)] = ig_{\mu\nu}\delta^3(x - y).$$

This relation takes on relativistic requirement, since both $A_\mu$ and $\pi^\nu$ are 4-vectors. Then we do mode expansion and give out creation and annihilation operators for $A_\mu$:

$$A_\mu = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\alpha=0}^2 [a_\alpha(p)e^{ipx}e^{-ipx} + h.c.]$$

where $p$ is 4-vector of energy and momentum, $px = -E_p t + p \cdot x$. And $\int d\vec{p}$ denotes $\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}}$, containing a normalization factor and an invariance coefficient.

Normal choice of polarization vectors is $\varepsilon^1$ as timelike, $\varepsilon^4$ as longitudinal while other two independently take over transverse components. This gives $\varepsilon^\mu_\alpha \varepsilon^\nu_{\alpha}^* = g_{\mu\nu} g^{rs}$, and in order to consist with Eq[1], the commutation relation of $a_\alpha(p)$ and $a^{\dagger}_{\beta}(q)$ has to be $[a_\alpha(p), a^{\dagger}_{\beta}(q)] = g_{rs}\delta^3(p - q)$. Till now, we have 4 quanta, with 2 more freedoms unphysical. Besides, normally we choose Minkowski matric as diag(-1,1,1,1), which will lead to the problem of negative norm, that inner product $\langle 0 | a^{\dagger}_\mu a^\mu_\nu | 0 \rangle = -\delta^4(p - q)$.

In literature, Gupta-Bleuer condition is added to reduce these problems, by asserting that only states obeying quantum lorenz gauge $\partial^\mu A_\mu |\Phi\rangle$ are physical, and these states are transverse states $a^{\dagger}_1|0\rangle$, $a^{\dagger}_3|0\rangle$ and $(a^{\dagger}_0 - a^{\dagger}_2)|0\rangle$, with the third state’s norm always zero. Thus there’s only two components left.

Here we introduce a more fundamental quantization method for photon, by proving from EM interaction Hamiltonian that photon is a composite particle composed by massive and charged...
fermion-antifermion pairs. Step from this, we can quantize photon from fermions’ anticommutation relation, but not from Conjugate Quantization. Directly from this procedure, only two components are left, which obey boson’s commutators. Moreover, it can be proven that this procedure is not only relativistic but also compatible with Conjugate Quantization.

2 Derivation of transition operators

As we know, EM field’s Hamiltonian density is: $H = H_0 + H_{int} = F^\mu_0 \partial_\mu A_\mu + \frac{1}{4} F^\mu_\nu F_{\mu\nu} - \frac{1}{c} j_\mu A^\mu$, and in Quantum motion, we can choose interaction picture to reduce Hamiltonian into (c=1):

$$H_{int} = - \int d^3x j^\mu A_\mu$$

Here $j^\mu$ is the source of this EM field, which reflects charged particle’s flow. When we quantize this system, charged particle is naturally a Dirac particle, whose wave function obeys Dirac equation $(i\gamma^\mu \partial_\mu - m)\Psi = 0$, and charge flow $j^\mu = e\nabla_\mu \Psi$. Here we assume Dirac field $\Psi$ is already quantized:

$$\Psi = \int d\tilde{p}[\hat{b}_\mu(p)u^\lambda(p)e^{-ipx} + \hat{d}^\mu_s(p)v^\lambda(p)e^{ipx}]$$

where $\hat{b}_\mu$ stands for positive energy fermion and $\hat{d}^\mu_s$ stands for negative energy antifermion. Note that in literature, we always use $\hat{d}^\mu_s$ to replace the second term with positive energy antifermion.

Then we don’t rush to quantize $A_\mu$, but to keep it classical, and take mode expansion containing only c-numbers:

$$A_\mu = \int d\tilde{k} \sum_{\lambda=0}^3 [a_\lambda(k)\varepsilon_\mu^\lambda(k)e^{-ikx} + c.c.]$$

so far $a_\lambda(k)$ and $a^*_\lambda(k)$ are coefficients in front of plane EM waves, which obeys Maxwell equation by $E_k = c|k|$.

Total expansion for $H_{int} = - \int d^3x j^\mu A_\mu = -e \int d^3x \bar{\Psi}\gamma^\mu \Psi A_\mu$ gives eight different terms (c=1):
\[ H_1 = \int d\vec{p}d\vec{q}d\vec{k} a_s(\vec{q}, t)b_r(\vec{p}, t)a_\lambda^*(\vec{k}, t)h_1 \quad (6) \]
\[ H_2 = \int d\vec{p}d\vec{q}b_r^\dagger(\vec{q}, t)d_s^\dagger(\vec{p}, t)a_\lambda(\vec{k}, t)h_2 \quad (7) \]
\[ H_3 = \int d\vec{p}d\vec{q}b_r^\dagger(\vec{q}, t)b_r(\vec{p}, t)a_\lambda^*(\vec{k}, t)h_3 \quad (8) \]
\[ H_4 = \int d\vec{p}d\vec{q}d_s(\vec{q}, t)b_r(\vec{p}, t)a_\lambda(\vec{k}, t)h_4 \quad (9) \]
\[ H_5 = \int d\vec{p}d\vec{q}d_s(\vec{q}, t)d_s^\dagger(\vec{p}, t)a_\lambda(\vec{k}, t)h_5 \quad (10) \]
\[ H_6 = \int d\vec{p}d\vec{q}d_s(\vec{q}, t)d_s^\dagger(\vec{p}, t)a_\lambda(\vec{k}, t)h_6 \quad (11) \]
\[ H_7 = \int d\vec{p}d\vec{q}b_r^\dagger(\vec{q}, t)d_r^\dagger(\vec{p}, t)a_\lambda^*(\vec{k}, t)h_7 \quad (12) \]
\[ H_8 = \int d\vec{p}d\vec{q}b_s(\vec{q}, t)b_r(\vec{p}, t)a_\lambda(\vec{k}, t)h_8 \quad (13) \]

Its easy to compute, for instance, \( h_2 = \varphi^\dagger(\vec{p})\gamma^\mu\nu^\tau(\vec{q})\phi^\lambda(\vec{k}) \). Here \( t \) in operators and c-numbers means that the time evolution \( e^{-iEt} \) is reduced into them.

Now we quantize \( a_\lambda \) by assuming that \( H_2 \) is a transition operator which annihilates a photon state \( |\varphi_{ph}(\vec{k}, t)\rangle \) into vacuum with operator \( \hat{a}_\lambda(\vec{k}, t) \) and then creates a fermion and an antifermion by \( b_r^\dagger(\vec{q}, t)d_r^\dagger(\vec{p}, t) \), reading:

\[ H_2|\varphi_{ph}(\vec{k}, t)\rangle = \int d\vec{p}d\vec{q} h_2 \hat{b}_s(\vec{q}, t) \otimes \hat{a}_\lambda(\vec{k}, t) \]

(14)

From two points: 1) \( H_2 \)'s physical meaning is just inner interaction within photon state, or from its components: fermion, antifermion pairs. 2) \( H_{int} \) is the only term in evolution \( T\{\exp[\int_0^t ds H_{int}(s)]\} \) that can give out relationship that only contains one photon, we propose that in \( H_2 \)'s case \( |\varphi_{ph}(\vec{k}, t)\rangle \) is completely equal to combination of \( h_2|\hat{b}_s(\vec{q}, t) \otimes \hat{a}_\lambda(\vec{p}, t)\rangle \). From Eq[14] we get:

\[ |\varphi_{ph}(\vec{k}, t)\rangle = \int d\vec{p}d\vec{q} h_2|\hat{b}_s(\vec{q}, t) \otimes \hat{a}_\lambda(\vec{p}, t)\rangle \]

(15)

Now that before quantizing photon, vacuum state and fermion operators are well defined, that \( |\hat{b}_s(\vec{q}, t) \otimes \hat{a}_\lambda(\vec{p}, t)\rangle = b_r^\dagger(\vec{q}, t)d_r^\dagger(\vec{p}, t)|0\rangle \), we define \( |\varphi_{ph}\rangle = \hat{a}_\lambda^\dagger(\vec{k}, t)|0\rangle \), then from Eq[15]:

\[ \hat{a}_\lambda^\dagger(\vec{k}) = \int d\vec{p}d\vec{q} b_r^\dagger(\vec{q})d_r^\dagger(\vec{p})\gamma^\mu\nu^\tau(\vec{q})\phi^\lambda(\vec{k})\delta(\vec{p} + \vec{q} - \vec{k}) \]

(16)

where we remove time evolution terms and replace it with \( \delta(p^0 + q^0 - k^0) \).

In sum, we can define creation operator \( \hat{a}_\lambda^\dagger \), and its hermitian conjugation \( \hat{a}_\lambda \) by superposition of fermion and antifermion pairs, with the presumption that \( \hat{a}_\lambda \) can annihilate a single photon state. This logic should be understandable, though it's totally different from conjugation quantization, where we can directly and independently derive commutators of bosons.
3 Vacuum fluctuation and conservation law

3.1 Conservation law

A detail that should be noticed, is the energy-momentum conservation in these process. We choose $H_2$’s transition and the notation of $p + q = k$. Then the energy conservation is:

$$\sqrt{p^2 + m^2} + \sqrt{(k - p)^2 + m^2} = |k|$$  \hspace{1cm} (17)

which is simplified to $k \cdot p = |k||p|\cos \theta = |k|\sqrt{|p|^2 + m^2}$. A physical process can only happen when $|p|\cos \theta = \sqrt{|p|^2 + m^2}$, for normal massive particles it’s violated, if we don’t postulate further that there’s energy loss in these processes.

However, the strategy here is, we introduce a kind of particle (or quanta of quantum field) called Tachyon, whose velocity is always greater than the speed of light. Since energy of particles should be real in the first hand, which reads:

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$  \hspace{1cm} (18)

with $\sqrt{1 - \frac{v^2}{c^2}}$ an imaginary number, thus Tachyon’s mass should also be imaginary $m = \imath \tilde{m}$, then it’s plausible for $|p|\cos \theta = \sqrt{|p|^2 + \tilde{m}^2}$ to be true. Also, we should make it clear physically, that two $v > c$ particles can actually form a photon, since we require momentum conservation, but not velocity conservation. So in our case, the fermions and anti-fermions should be Tachyons and anti-Tachyons, which obey Dirac equation with imaginary mass, and get positive and negative energy from the solution.

3.2 Basic scattering process

Before computing the commutators for photon, it should be noticed that in Eq[6 13], there are 4 different combination of creation and annihilation operators $H_{2,4,6,8}$ and their hermitian operators $H_{1,3,5,7}$, and each transition matrix can be interpreted as a scattering process. Then it’s clear that $\hat{a}_1^\dagger$ contains more combinations than terms in $H_2$. However, these eight different terms are actually describing the same process. This can be explained using the concept Vacuum fluctuation:

Two particles with opposite energy $+E$ and $-E$, opposite charges $q$ and $-q$, opposite momentum $p$ and $-p$ but the same chirality come from or annihilate into vacuum.

This is easily understandable, because two particles with properties above cancel their energy, charges, momentum and spin, leaving literally vacuum in spacetime. And in terms of Dirac spinors, we can also check that the spinors of these two particles differ only in a minus sign. This concept can be used to explain why $H_{2,4,6,8}$ or its h.c. $H_{1,3,5,7}$ describe the same process. As shown in following feynmann diagrams, we treat $H_{1,3,5,7}$ process as basic scattering:

Two particles with opposite energy $+E$ and $-E$, opposite charges $q$ and $-q$, opposite momentum $p$ and $-p$ but the same chirality come from or annihilate into vacuum.

To make their momentum notations correspond to each other, as in $H_1$ should be $p + q = k$, while in $H_3$ it’s $p = q + k$, the $q$ here has been changed in sign. Then it should be clear that, there’s only one basic process: $H_1$, where a fermion and an antifermion form a photon. $H_{3,5,7}$ are the same with $H_1$ only by adding that incoming fermion or antifermion comes from vacuum, and gives out an outgoing opposite particle. This vacuum won’t influence their scattering amplitudes.

Since there’s only one basic process, there should be only one kind of combination for $\hat{a}_1^\dagger$, which can be chose directly as the form of Eq[16]. It can be proven that other combinations...
Figure 1: Basic scattering

derived from $H_{int}$ are exactly the same with Eq[16], when we write vacuum fluctuation in form of opposite particles’ creation and annihilation operators’ relationship.

We plot the basic processes with Feynmann diagram and explain how they are formulated with vacuum fluctuation, here F means fermion and AF means antifermion:

4 Commutation relation

Before computing the integral:

$$\hat{a}_\lambda^\dagger (\mathbf{k}) = \int d\mathbf{p} d\mathbf{q} \hat{b}_\lambda^\dagger (\mathbf{q}) \hat{b}_\mu^\dagger (\mathbf{p}) \bar{\psi} (\mathbf{p}) \gamma^\mu \psi (\mathbf{q}) \epsilon_\mu^\lambda (\mathbf{k}) \delta^4 (p + q - k)$$  \hspace{1cm} (19)

It should be noticed that when fermions and antifermions have unignorable mass, then these three momentums will not be on the same line, and the spinors will also be vary complicated. However, if we assume their mass is ignorable compared to their energy, then according to energy-momentum conservation: $p + q = k$ and $|p| + |q| = |k|$, they should move in the same direction, with the restriction that negative energy antifermion moves oppositely from photon and fermion.

From now on, we choose the notation as shown in following picture for simplicity, where
fermion has momentum \( p + k \), while antifermion has momentum to opposite direction \(-p\), they combine to form photon with momentum \( k\):

Choosing the only direction as \( z\)-axis, we can simplify the computation to be completely analytical. Spinors \( u^s \) and \( v^r \) have two independent components each, in the case of \( u^s(z) \) and \( v^r(-z) \), they can be normalized to be:

\[
\begin{align*}
u^1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\end{align*}
\] (20)

here \( u^1 \) and \( v^2 \) is left handed, their spins are parallel to their momentums, while \( u^2 \) and \( v^1 \) is right handed, with spins antiparallel to momentums.

As shown in the picture, in order to construct photon as a spin-1 particle, fermion and antifermion should have opposite chirality, which means only \( u^1 \gamma^\mu v^1 = (0, 1, -i, 0) \) and \( \pi^2 \gamma^\mu v^2 = (0, -1, -i, 0) \) are left in the integral. In group theory, spinors transform under SL(2,C) group, only some combinations of them can form Lorentz vectors. Indeed, \( u^1 \gamma^\mu v^1 \) and \( \pi^1 \gamma^\mu v^2 \) are the way spinors form vectors, which mathematically explains what composite photon theory is.

It’s easiest to choose transverse part of polarization vectors \( \epsilon^1 = (0, 1, -i, 0) \) and \( \epsilon^2 = (0, -1, -i, 0) \) as the states created by \( \hat{a}^\dagger_1,\hat{a}^\dagger_2 \), since they are already 4-vectors. Then \( \pi^s(z)\gamma^\mu v^r(-z)\epsilon^1,2(z) \) can be normalized to be 1, leading to the result:

\[
\hat{a}^\dagger_r(k) = \frac{1}{N||} \sum_{k||p} \hat{b}^\dagger_r(p+k)\hat{d}^\dagger_l(-p) \quad r = 1, 2
\] (21)

where the \( N|| \) represents a normalization factor of all possible combination of \( k||p \) in the space.

Using the anticommutation relation of fermion and antifermion: \( \{\hat{b}_r(p), \hat{b}^\dagger_s(q)\} = \{\hat{d}_r(p), \hat{d}^\dagger_s(q)\} = \delta_{rs}\delta^3(p-q) \), photon’s commutation relation is:

\[
[\hat{a}_r(p), \hat{a}^\dagger_s(q)] = \delta_{rs}\delta^3(p-q) - \delta_{rs}\frac{1}{N||} \sum_{k||p} \sum_{k'||q} [\delta^3(k-k')\hat{b}^\dagger_r(q+k)\hat{b}^\dagger_r(p+k) + \delta^3(p+k-q-k')\hat{b}^\dagger_r(-k')\hat{b}^\dagger_r(-k)]
\] (22)

Physically, the total number of \( k \) parallel to \( p \) should be infinity, so the commutation relation reduces to:

\[
[\hat{a}_r(p), \hat{a}^\dagger_s(q)] = \delta_{rs}\delta^3(p-q)
\] (23)

Figure 2: Massless limit
with \( r, s = 1, 2 \), which is exactly bosonic commutator. With this commutator, the creation and annihilation operators take on full structure, we can thus establish the whole Fock space for photon.

Meanwhile, as is seen above, in \( \bar{\psi} \gamma^\nu \epsilon^\lambda_{\nu} \), actually no timelike and longitudinal components are left, so no matter how we choose polarization vectors \( \epsilon^r_\mu(p) \), there should be exactly \( \hat{a}^\dagger_{0, 3} = 0 \), which means there’s absolutely no Fock space for timelike and longitudinal components. In this way, we have proven that only two transverse components of EM field are left.

Furthermore, when mass cannot be ignored, clearly \( p \) doesn’t parallel to \( k \), for fixed direction \( p \), vectors \( \bar{\psi} (p+k) \gamma^\nu \epsilon^\lambda_{\nu}(p) \) should have properties, such as cylindrical antisymmetry around direction \( p \). Then components that isn’t parallel to it may be integral to zero, so the commutator is still as above. But this condition hasn’t be computed by us. Note that the reason why the mass cannot be asserted to be exactly zero, is that physically \( j^\mu \) in Hamiltonian should be charged particle’s flow.

5 Discussion on physical meaning

5.1 Physical picture

Here we fix the basic process to be a Tachyon with +q and an anti-Tachyon with -q forming a photon. Physically, two particles bring their passive EM field with them, combine and form a photon with no charge, because their charges annihilate with each other. Remaining around this photon is divergent (free) EM field, which is the transverse part of \( A_\mu \). Thus photon only has two transverse components. This procedure just reflects EM interaction, indeed, the interaction Hamiltonian can derive these scattering processes.

5.2 Lorentz invariance and conjugate quantization

Now that we know photon has only two components, free EM field \( A_\mu \) should just be expanded upon \( r, s = 1, 2 \). Since polarization vectors \( \epsilon^r(p) \), \( r = 1, 2 \) are 4-vectors, \( A_\mu \) should still be a 4-vector. This is guaranteed by the fact that two spinors are mapped to be a vector through \( \bar{\psi} \gamma^\mu \psi \).

Then we come back to classical case, and focus only on how to quantize transverse 4-vectors \( A_\mu \). We can still construct EM’s Lagrangian. And similar mounts can be derived, like \( A_\mu \)'s conjugate momentum \( \pi^\mu \), the only difference is there’re only two components in expansion. For 4-vectors, the commutation relation should still be Eq[1], and we can check, with only two components left in \( A^\mu \):

\[
\int dp dq \sum_{r=1}^{2} \sum_{s=1}^{2} [\hat{a}_r(p), \hat{a}^\dagger_s(q)] \epsilon^r_\mu(p) \epsilon^s_\nu(q) \propto \eta_{\mu\nu}
\]  

(24)

This should be solved to get the commutators, as before. For \( \epsilon^1 = (0, 1, -i, 0) \), \( \epsilon^2 = (0, -1, -i, 0) \) as we chose in quantization, it’s easy to check \( \epsilon^r_\mu(p) \epsilon^s_\nu(p) = g_{\mu\nu} g^{rs} \), then the solution is exactly \( [\hat{a}_r(p), \hat{a}^\dagger_s(q)] = \delta_{rs} \delta^3(p-q) \). Note what’s lorentz invariant: \( A_\mu \) is always a lorentz vector, and the relation Eq[1] is invariant. But \( \epsilon^r_\mu(p) \) have only two components \( r = 1, 2 \), so the result \( [\hat{a}_r(p), \hat{a}^\dagger_s(q)] = g_{r\nu} \delta^3(p-q) \) comes to be \( [\hat{a}_r(p), \hat{a}^\dagger_s(q)] = \delta_{rs} \delta^3(p-q) \). Nothing is wrong here, because creation operators are not necessarily vectors, they are just insuring that Eq[1] is lorentz invariant.

The logic is, after we know photon is quantized from transverse (free) EM field, we come back to classical condition with transverse 4-vector \( A_\mu \), whose Lorentz invariance is guaranteed.
by the mapping from fermion spinors. Then quantize it again using conjugate quantization, it will give the same result. Thus, the method is still consistent with conjugate quantization.