2022 年随机过程期中试卷

姓 名 ________________ 学 号 ________________ 学 院 ________________

题号	一	二	三	四	五	六	七	总分
得分								

一、【10 分】

Let $\{X_n\}_{n\geq 0}$ be a square integrable martingale w.r.t. the family of σ-fields $\mathcal{F}_n, n \geq 0$. Prove

(a). If $m \neq n$, then $E[(X_{n+1} - X_n)(X_{m+1} - X_m)] = 0$.

(b). $E[(X_{n+1} - X_n)^2] = E[X_{n+1}^2] - E[X_n^2]$.

二、【15 分】

Let $\{X_n\}_{n\geq 0}$, $\{Y_n\}_{n\geq 0}$ be two martingales w.r.t. the family of σ-fields $\mathcal{F}_n, n \geq 0$. Let T be a stopping time. Define $Z_n = X_n I_{\{n\leq T\}} + Y_n I_{\{n>T\}}, n \geq 0$. Assume $X_T = Y_T$ on the event $\{\omega; T(\omega) < \infty\}$. Prove that $\{Z_n\}_{n\geq 0}$ is a martingale.

三、【15 分】

Let $\{X_n\}_{n\geq 0}$ be a martingales with $E[X_0^2] < \infty$ and let $\xi_n = X_n - X_{n-1}$ for $n \geq 1$. Use the martingale convergence theorem to prove that $X_n \to X_\infty$ in L^2 if and only if $\sum_{m=1}^{\infty} E[\xi_m^2] < \infty$.

四、【20 分】

Let $X_n, n \geq 1$ be independent random variables with $P(X_n = 1) = p$ and $P(X_n = -1) = q = 1-p$. Let $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$ be the σ-field generated by $X_1, X_2, ..., X_n$ and $\mathcal{F}_0 = \{\Omega, \emptyset\}$. Consider the random walk $S_n = \sum_{i=1}^{n} X_i, n \geq 1$ with $S_0 = 0$. Suppose $0 < p < 1$. It is known that $\{Z_n = (\frac{1-p}{p})^{S_n}, n \geq 0\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 0$. For an integer x, define the stopping time $T_x = min\{n; S_n = x\}$, the first time at which the random walk hits the position x. For integers $a < 0 < b$, let $T = T_a \wedge T_b = min(T_a, T_b)$.

(i) Determine the value of $E[Z_T]$ and give your reason.

(ii) Use the result in (i) to show that

$$P(T_a < T_b) = \frac{\phi(b) - \phi(0)}{\phi(b) - \phi(a)},$$

where $\phi(x) = (\frac{1-p}{p})^x$.

(iii) Suppose $\frac{1}{2} < p < 1$. If $a < 0 < b$, determine the probability that the random walk hits a,i.e., $P(T_a < \infty)$.

五、【10 分】

Let $(X_n, n \geq 0)$ be a Markov chain with transition matrix $(p(x,y))$. Let $f(y)$ be a bounded function such that

$$f(x) = \sum_y p(x,y)f(y), x \in S.$$

Prove that under P_μ, $\{f(X_n)\}_{n\geq 0}$ is a martingale with respect to $\{\mathcal{F}_n = \sigma(X_k, k \leq n)\}_{n\geq 0}$, where μ is the distribution of X_0.

六、【15 分】

Let $(X_n, n \geq 0)$ be a Markov chain. Let $T_y^0 = 0$, and $T_y^k = \inf\{m > T_y^{k-1}; X_m = y\}$ for $k \geq 1$. Recall $p^n(x,y) = P_x(X_n = y)$. Show that

(i). $P_x(T_x^k < \infty) = P_x(T_x^1 < \infty)^k$.

(ii). $p^n(x,y) = \sum_{m=1}^n P_x(T_y^1 = m)p^{n-m}(y,y)$.

七、【15 分】

Let $(X_n, n \geq 0)$ be a Markov chain with transition matrix $(p(x,y))$. Assume $(X_n, n \geq 0)$ is irreducible and positive recurrent.

(i) Let $y \neq x$. Define $n = \inf\{m \geq 1; p^m(y,x) > 0\}$. Explain why one can pick $y_1, ..y_{n-1} \neq y$ so that $p(y,y_1)p(y_1,y_2) \cdot p(y_{n-1},x) > 0$.

(ii) Prove $E_x[T_y] < \infty$.

(Hint: Use $E_y[T_y] \geq E_y[T_y; X_1 = y_1, X_2 = y_2, ..., X_n = x]$.)