Riemannian Geometry (Spring, 2022)
Final Exam

Name: No.: Department:

All Riemannian manifolds are assumed to be connected.

1. (25 marks) Let $M^n(k)$ be an n dimensional simply connected space form with constant sectional curvature $k \in \mathbb{R}$. Pick a $p \in M$ and a normal geodesic $\gamma : [0, +\infty) \to M$ with $\gamma(0) = p$. Give the answers to the following questions directly.
   (1) Find the cut point of p along γ if exists.
   (2) Find the first conjugate point of p along γ if exists.
   (3) Suppose $k > 0$. What is the index of the geodesic $\gamma|_{[0, \frac{2\pi}{\sqrt{k}}]}$?
   (4) Suppose $k < 0$, $n = 2$. For any $0 < r < +\infty$, what is the length of the curve $\{x : d(p, x) = r\}$?
   (5) Suppose $k > 0$. For any $0 < r < +\infty$, what is the volume of the ball $B_p(r)$?

2. (10 marks) Consider the upper half-plane

$$\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

We assign the following Riemannian metric:

$$g_{11} = 1, \; g_{12} = g_{21} = 0, \; g_{22} = \frac{1}{y}.$$

Please determine whether this Riemannian manifold is complete or not and explain the reason.

3. (10 marks) Let (M^n, g) be an n dimensional complete Riemannian manifold with nonnegative Ricci curvature. Let $\rho(\cdot) := d(p, \cdot)$ be the Riemannian distance function to the point p. Show that at any point $q \in M \setminus \{p, C_p\}$, where C_p is the cut locus of p, we have

$$(\Delta\rho)' \leq -\frac{(\Delta\rho)^2}{n-1},$$

where $(\Delta\rho)'(q) = \frac{d}{dt}|_{t=d(p,q)}(\Delta\rho)(\gamma(t))$ is the derivative along the radial normal geodesic γ with $\gamma(0) = p$.

4. (23 marks) Let (M^n, g) be an n-dimensional Riemannian manifold. Consider the (0,4)-tensor B defined below:

$$\begin{aligned} B(X, Y, Z, W) :=& \mathrm{Ric}(X, Z)g(Y, W) + \mathrm{Ric}(Y, W)g(X, Z) \\ & - \mathrm{Ric}(X, W)g(Y, Z) - \mathrm{Ric}(Y, Z)g(X, W) - R(X, Y, Z, W), \end{aligned}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Here we use Ric for the Ricci curvature tensor and R for the Riemannian curvature tensor.

We denote by G the following (0,4)-tensor:

$$G(X, Y, Z, W) := g(X, Z)g(Y, W) - g(X, W)g(Y, Z).$$

For any $p \in M$ and any two linearly independent tangent vectors $X, Y \in T_pM$, the *bi-Ricci curvature* of X, Y is defined by

$$\mathrm{BRic}(X, Y) := \frac{B(X, Y, X, Y)}{G(X, Y, X, Y)}.$$

(1) Show that the bi-Ricci curvature $\mathrm{BRic}(X, Y)$ only depends on the section Π spanned by X and Y, i.e., it is independent of the choices of the basis X, Y in Π.
(2) When $n = 3$, show that $\mathrm{BRic}(\Pi_p) = \frac{1}{2}S(p)$, for any $p \in M$ and any section $\Pi_p \subset T_pM$, where $S(p)$ stands for the scalar curvature at p.
(3) When $n \geq 4$, suppose that $\mathrm{BRic}(\Pi_p) = f(p)$, where Π_p is an arbitrary section of T_pM, depends only on p. We further assume that (M^n, g) is Einstein. Show that (M^n, g) has constant bi-Ricci curvature.
(4) Show that (M^n, g) has constant bi-Ricci curvature k if and only if $B = kG$.

5. (12 marks) Let (M^n, g) be a Riemannian manifold with non-positive sectional curvature.
(1) Show that, for all p, the set of conjugate points of p along any geodesic starting from p is empty.
(2) Let $\gamma : [0, \ell] \to M$ be a normal closed geodesic with $\gamma(0) = \gamma(\ell) = p \in M$ such that the parallel transport map

$$\mathcal{P}_{\gamma,0,\ell} : T_pM \to T_pM$$

has positive determinant. Assume that n is even. We define the twist α of the closed geodesic γ as below:

$$\alpha := \min_{V \in T_pM, g(V,V)=1, g(V,\gamma'(0))=0} \phi(V)$$

where $\phi(V) \in [0, \pi]$ satisfying $\cos\phi(V) = g(V, \mathcal{P}_{\gamma,0,\ell}(V))$. Show that $\alpha = 0$.

6. (20 marks) Let (M^n, g) be a Riemannian manifold. Let $\gamma : [0, \ell] \to M$ be a normal geodesic and U be a non-trivial Jacobi field along γ. Construct the following objects and show that your constructions satisfy the required properties.
(1) Find a family of geodesics whose variational field is U.
(2) Let $(\overline{M}^n, \overline{g})$ be another Riemannian manifold with the same dimension n, and $\overline{\gamma} : [0, \ell] \to \overline{M}$ be a normal geodesic. Suppose for each $t \in [0, \ell]$, we have the sectional curvatures

$$K(\Pi_{\gamma(t)}) \leq \overline{K}(\overline{\Pi}_{\overline{\gamma}(t)}),$$

holds for any sections $\Pi_{\gamma(t)}$ of $T_{\gamma(t)}M$ and $\overline{\Pi}_{\overline{\gamma}(t)}$ of $T_{\overline{\gamma}(t)\overline{M}}$. We further assume that $\overline{\gamma}$ has no conjugate points. Find a non-trivial Jacobi field $\overline{U}$ along $\overline{\gamma}$ such that the the index forms satisfy

$$I(U, U) \geq \overline{I}(\overline{U}, \overline{U}).$$