Advanced Probability, MATH5007P
Autumn 2020, Final with Solutions

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1. (10 points) Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be independent σ-fields and for any $n \geq 0$ define $\mathcal{T}_n = \sigma\{\mathcal{F}_{n+1}, \mathcal{F}_{n+2}, \ldots\}$. Prove that the tail σ-field $\mathcal{T} = \bigcap_{n\geq 0} \mathcal{T}_n$ is P-trivial, that is, for any set $A \in \mathcal{T}$, $P(A) = 0$ or 1.

For any $n \geq 1$, $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n, \mathcal{T}_n$ are independent, by the grouping lemma. Then $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n, \mathcal{T}$ are independent, since $\mathcal{T} \subset \mathcal{T}_n$. Then $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{T}$ are independent, by the definition of independence. Then $\mathcal{T}_0$ and $\mathcal{T}$ are independent, again by the grouping lemma. Then $\mathcal{T}$ and $\mathcal{T}$ are independent, so for any $A \in \mathcal{T}$ we have $P(A) = P(A)P(A)$, that is, $P(A) = 0$ or 1.

2. (10 points) Let $X_1, X_2, \ldots$ be independent random variables with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. First show that $X_n \xrightarrow{P} 0$ if and only if $p_n \to 0$, then show that $X_n \to 0$ a.s. if and only if $\sum_{n \geq 1} p_n < \infty$.

First recall that $\xi_n \xrightarrow{P} \xi$ if and only if $E(|\xi_n - \xi| \wedge 1) \to 0$. So in this problem we see that $X_n \xrightarrow{P} 0$ if and only if $E(X_n \wedge 1) = p_n \to 0$.

For the a.s. convergence, note that X_n can take only two values: 0 and 1. So $X_n \to 0$ a.s. if and only if $P(X_n = 1 \text{ i.o.}) = 0$. Finally recall the 2nd Borel-Cantelli lemma, which says that $P(X_n = 1 \text{ i.o.}) = 0$ if and only if $\sum_{n \geq 1} P(X_n = 1) = \sum_{n \geq 1} p_n < \infty$.

3. (15 points) For the two-dimensional random vectors (X, Y) and (X_n, Y_n) for any $n \geq 1$, first show that $(X_n, Y_n) \xrightarrow{d} (X, Y)$ implies that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$. Next assume that X and Y are independent, and X_n and Y_n are independent for any $n \geq 1$. Then show that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ imply that $(X_n, Y_n) \xrightarrow{d} (X, Y)$.

From $(X_n, Y_n) \xrightarrow{d} (X, Y)$ we get that as $n \to \infty$,

$$Ee^{i(tX_n + sY_n)} \to Ee^{i(tX+sY)}, \quad t, s \in \mathbb{R}.$$

Taking $s = 0$ gives that

$$Ee^{itX_n} \to Ee^{itX}, \quad t \in \mathbb{R},$$

so $X_n \xrightarrow{d} X$, and similarly $Y_n \xrightarrow{d} Y$.

For the reverse direction, from $X_n \xrightarrow{d} X$ we get

$$Ee^{itX_n} \to Ee^{itX}, \quad t \in \mathbb{R},$$

and from $Y_n \xrightarrow{d} Y$ we get

$$Ee^{isY_n} \to Ee^{isY}, \quad s \in \mathbb{R}.$$

So by independence we get

$$Ee^{i(tX_n+sY_n)} = Ee^{itX_n}Ee^{isY_n} \to Ee^{itX}Ee^{isY} = Ee^{i(tX+sY)}, \quad t, s \in \mathbb{R},$$

that is, $(X_n, Y_n) \xrightarrow{d} (X, Y)$.

4. (20 points) Suppose that X_n has a normal distribution with mean $m_n \in (-\infty, \infty)$ and variance $\sigma_n^2 \in [0, \infty)$ for each $n \geq 1$, and $X_n \xrightarrow{d} X$ for some random variable X. Prove that the limit X also has a normal distribution, with mean m and variance σ^2, where $m = \lim_{n\to\infty} m_n \in (-\infty, \infty)$ and $\sigma^2 = \lim_{n\to\infty} \sigma_n^2 \in [0, \infty)$. (Note that the convergences of m_n and σ_n^2 are not assumed.)

From $X_n \xrightarrow{d} X$ we get that as $n \to \infty$,

$$\varphi_n(t) = Ee^{itX_n} = e^{im_n t - \sigma_n^2 t^2/2} \to \varphi(t) = Ee^{itX}, \quad t \in \mathbb{R}.$$

Taking $t = 1$ gives that $\lim_{n\to\infty} |e^{im_n - \sigma_n^2/2}| = \lim_{n\to\infty} e^{-\sigma_n^2/2} \in [0, 1]$.

Note that $\lim_{n\to\infty} e^{-\sigma_n^2/2} = 0$ is equivalent to $\lim_{n\to\infty} \sigma_n^2 = \infty$. In this case clearly $|\varphi_n(t)| \to 0$ when $t \neq 0$ and $|\varphi_n(t)| \to 1$ when $t = 0$, which contradicts the continuity of φ at $t = 0$. So $\sigma^2 = \lim_{n\to\infty} \sigma_n^2 \in [0, \infty)$.

Let $Y_n = m_n, n \geq 1$. Then as $n \to \infty$,

$$\phi_n(t) = Ee^{itY_n} = e^{im_n t} = \varphi_n(t) e^{\sigma_n^2 t^2/2} \to \phi(t) = \varphi(t) e^{\sigma^2 t^2/2}, \quad t \in \mathbb{R}.$$

Notice that ϕ is continuous at $t = 0$. The extended continuity theorem implies that $Y_n \xrightarrow{d} Y$ for some random variable Y. By Skorohod's representation theorem, we can construct the random variables $(Y_n')_{n\geq 1}$ and Y' on a common probability space, such that $Y_n' \stackrel{d}{=} Y_n, n \geq 1$, $Y' \stackrel{d}{=} Y$, and $Y_n' \to Y'$ a.s. Since clearly a.s. $Y_n' = m_n, n \geq 1$, we see that a.s. $m_n \to Y' \in (-\infty, \infty)$, which implies that $m = \lim_{n\to\infty} m_n \in (-\infty, \infty)$.

Finally we get that as $n \to \infty$,

$$\varphi_n(t) = Ee^{itX_n} = e^{im_n t - \sigma_n^2 t^2/2} \to e^{imt - \sigma^2 t^2/2}, \quad t \in \mathbb{R}.$$

So $\varphi(t) = Ee^{itX} = e^{imt - \sigma^2 t^2/2}, t \in \mathbb{R}$, which shows that the limit X also has a normal distribution, with mean m and variance σ^2.

5. (25 points) Suppose that $Y_n \geq 0$ for any $n \geq 1$, and, $EY_n^\alpha \to 1$ and $EY_n^\beta \to 1$ as $n \to \infty$ for some $0 < \alpha < \beta$. Show that $Y_n \xrightarrow{P} 1$. (Recall that the condition $\sup_{n\geq 1} E|Y_n|^p < \infty$ for some $p > 1$ implies that the sequence $(Y_n)_{n\geq 1}$ is uniformly integrable. Also recall that for the sequence of nonnegative random variables $(Y_n)_{n\geq 1}$, $Y_n \xrightarrow{d} Y$ for some random variable Y and the uniform integrability of $(Y_n)_{n\geq 1}$ imply that $EY_n \to EY$.)

Clearly we may and do assume that $\sup_{n\geq 1} EY_n^\alpha < \infty$ and also $\sup_{n\geq 1} EY_n^\beta < \infty$.

First we show that the sequence $(Y_n)_{n\geq 1}$ is tight, which follows from the inequality

$$P(Y_n > r) \leq \frac{EY_n^\alpha}{r^\alpha} \leq \frac{\sup_{n\geq 1} EY_n^\alpha}{r^\alpha}, \quad r > 0.$$

Next we show that the sequence $(Y_n^\alpha)_{n\geq 1}$ is uniformly integrable, which follows from the inequality

$$\sup_{n\geq 1} E(Y_n^\alpha)^{\beta/\alpha} = \sup_{n\geq 1} EY_n^\beta < \infty.$$

By the tightness of the sequence $(Y_n)_{n\geq 1}$, for any subsequence of $\mathbb{N}$ we can find a further subsequence such that along this further subsequence $Y_n \xrightarrow{d} Y$ for some random variable Y. Clearly $Y_n^\alpha \xrightarrow{d} Y^\alpha$. Then by the uniform integrability of $(Y_n^\alpha)_{n\geq 1}$ we get $EY^\alpha = 1$.

Notice that $Y_n^\beta \xrightarrow{d} Y^\beta$ along that further subsequence. By Fatou's lemma we get that along that further subsequence,

$$EY^\beta \leq \liminf_{n\to\infty} EY_n^\beta = 1.$$

However by Jensen's inequality we get

$$EY^\beta = E(Y^\alpha)^{\beta/\alpha} \geq (EY^\alpha)^{\beta/\alpha} = 1.$$

So $EY^\beta = 1$ and in this case the inequality in the last display is an equality. Clearly $f(x) = x^{\beta/\alpha}, x \geq 0$ is a strictly convex function. Recall that Jensen's inequality for a strictly convex function is actually an equality if and only if the involved random variable is not random. So we get $Y^\alpha = EY^\alpha = 1$ a.s., that is, $Y = 1$ a.s.

Since for any subsequence of $\mathbb{N}$ we can find a further subsequence such that along this further subsequence $Y_n \xrightarrow{d} 1$, we get $Y_n \xrightarrow{d} 1$ as $n \to \infty$, that is, $Y_n \xrightarrow{P} 1$ as $n \to \infty$.

6. (20 points) Let $N^1 = (N_t^1)_{t\geq 0}$ and $N^2 = (N_t^2)_{t\geq 0}$ be two independent Poisson processes. Recall that for any $t > 0$, $\Delta N_t^1 = N_t^1 - N_{t-}^1$ is the jump size of the Poisson process N^1 at time t, and similarly $\Delta N_t^2 = N_t^2 - N_{t-}^2$ is the jump size of the Poisson process N^2 at time t. Prove that

$$\sum_{t>0} \Delta N_t^1 \Delta N_t^2 = 0 \quad \text{a.s.};$$

in other words, the two processes almost surely do not jump simultaneously.

We use $T_1^1, T_2^1, \ldots$ to denote the successive jump times of N^1, and $T_1^2, T_2^2, \ldots$ the successive jump times of N^2. Then since $\Delta N_t^2 = 1$ when t is equal to one of $T_1^2, T_2^2, \ldots$, and $\Delta N_t^2 = 0$ otherwise, we get

$$\sum_{t>0} \Delta N_t^1 \Delta N_t^2 = \sum_{n\geq 1} \Delta N_{T_n^2}^1 \Delta N_{T_n^2}^2 = \sum_{n\geq 1} \Delta N_{T_n^2}^1.$$

So it suffices to show that $\Delta N_{T_n^2}^1 = 0$ a.s. for each $n \geq 1$. Then since $\Delta N_t^1 = 1$ when t is equal to one of $T_1^1, T_2^1, \ldots$, and $\Delta N_t^1 = 0$ otherwise, we only need to show that

$$P(T_m^1 = T_n^2) = 0, \quad m, n \geq 1.$$

This follows from Exercise 2.1.5 in Durrett PTE, once we get the independence between T_m^1 and T_n^2 from $T_m^1 \in \sigma(N^1)$ and $T_n^2 \in \sigma(N^2)$ (also recall that T_m^1 and T_n^2 have density functions). However in our case here it can also be argued more directly as follows:

Use $f_m^1 = f_m^1(x)_{x\in\mathbb{R}}$ to denote the density function of T_m^1, and $f_n^2 = f_n^2(x)_{x\in\mathbb{R}}$ the density function of T_n^2. By independence we see that the random vector (T_m^1, T_n^2) has the density function $(f_m^1(x)f_n^2(y))_{x,y\in\mathbb{R}}$, which implies that the distribution of (T_m^1, T_n^2) is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. The Lebesgue measure of the Borel set $\{(x, x); x \in \mathbb{R}\}$ in $\mathbb{R}^2$ is clearly 0, so $P(T_m^1 = T_n^2) = 0$.

7. (10 points) Let X and Y be two i.i.d. random variables in L^1. Show that a.s.

$$E[X|X+Y] = E[Y|X+Y] = \frac{1}{2}(X+Y).$$

First $E[X|X+Y] + E[Y|X+Y] = E[X+Y|X+Y] = X+Y$, so we only need to show that $E[X|X+Y] = E[Y|X+Y]$, that is, for any $B \in \mathcal{B}$,

$$E[X; X+Y \in B] = E[Y; X+Y \in B].$$

Since X and Y are i.i.d., we have $(X,Y) \stackrel{d}{=} (Y,X)$. Then notice that $E[X; X+Y \in B] = Ef((X,Y))$ and $E[Y; X+Y \in B] = Ef((Y,X))$ with $f((x,y)) = x\mathbf{1}\{x+y \in B\}$. So $(X,Y) \stackrel{d}{=} (Y,X)$ implies that $Ef((X,Y)) = Ef((Y,X))$, that is,

$$E[X; X+Y \in B] = E[Y; X+Y \in B].$$

Notice that the following argument is not correct:

For any $B \in \mathcal{B}$,

$$E[X; X+Y \in B] = E[Y; X+Y \in B],$$

since $X \stackrel{d}{=} Y$ implies that $E[X; A] = E[Y; A]$ for any $A \in \mathcal{A}$.

Consider X with $P(X=1) = P(X=-1) = 1/2$, and $Y = -X$, then $X \stackrel{d}{=} Y$. But $E[X; X=1] = 1/2$ and $E[Y; X=1] = -1/2$.

8. (10 points) Let the nonnegative random variables $X_1, X_2, \ldots$ in L^1 and σ-fields $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be such that $E[X_n|\mathcal{F}_n] \xrightarrow{P} 0$. Show that $X_n \xrightarrow{P} 0$.

First recall that $E[X_n|\mathcal{F}_n] \xrightarrow{P} 0$ is equivalent to $E(E[X_n|\mathcal{F}_n]\wedge 1) \to 0$.

Then apply Jensen's inequality for conditional expectations to the convex function $f(x) = -(x \wedge 1)$ to get

$$-(E[X_n|\mathcal{F}_n] \wedge 1) \leq -E[X_n \wedge 1|\mathcal{F}_n],$$

that is, $E[X_n \wedge 1|\mathcal{F}_n] \leq E[X_n|\mathcal{F}_n] \wedge 1$. This implies that

$$E(X_n \wedge 1) = E(E[X_n \wedge 1|\mathcal{F}_n]) \leq E(E[X_n|\mathcal{F}_n] \wedge 1) \to 0,$$

so $X_n \xrightarrow{P} 0$.

The use of Jensen's inequality can be avoided by using instead the following simple argument:

The inequality $X_n \wedge 1 \leq X_n$ implies that

$$E[X_n \wedge 1|\mathcal{F}_n] \leq E[X_n|\mathcal{F}_n],$$

and the inequality $X_n \wedge 1 \leq 1$ implies that

$$E[X_n \wedge 1|\mathcal{F}_n] \leq 1,$$

so

$$E[X_n \wedge 1|\mathcal{F}_n] \leq E[X_n|\mathcal{F}_n] \wedge 1.$$