Final exam of *Advanced Algebraic Geometry* MA5130
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In this exam, K denotes for a field with a nontrivial non-Archimedean absolute value $|\cdot| : K \to \mathbb{R}$. We always assume that K is complete with respect to the valuation. Let $\bar{K}$ be an algebraic closure of K and $\hat{\bar{K}}$ be its completion. Denote again by $|\cdot|$ the unique extensions of the norm on K to $\bar{K}$ and $\hat{\bar{K}}$. In this exam, we assume $\mathrm{Char}(K) = 0$. Try to answer the questions as many as you can.

1. For an $f \in K[\zeta]$, denote $|f|$ for its Gauss norm.

   (i) Let $f = \zeta^n + c_1\zeta^{n-1} + \cdots + c_n$ be a monic polynomial, and $\alpha \in \bar{K}$ a root of f. Prove that
   $$|\alpha| \leq \sigma(f),$$
   where $\sigma(f)$ is the spectral value of f.

   (ii) Let $f, g \in K[\zeta]$ be monic polynomials of the same degree $n \geq 1$. Let α be a root of f. Prove the following inequality:
   $$|g(\alpha)| \leq |f - g| \cdot |f|^{n-1}.$$

   (iii) Let $f, g \in K[\zeta]$ be monic polynomials of the same degree $n \geq 1$. Then for each root α of f, there exists a root β of g such that
   $$|\alpha - \beta| \leq \sqrt[n]{|f - g|} \cdot |f|.$$

   (iv) For an $\alpha \in \bar{K}$ whose minimal polynomial f is of degree ≥ 2, we define $r(\alpha) = \min_\gamma |\alpha - \gamma|$, where γ runs over all roots $\neq \alpha$ of f. Prove that for any $\beta \in \bar{K}$ satisfying $|\beta - \alpha| < r(\alpha)$, it holds that
   $$\alpha \in K(\beta).$$

   (v) Consider an $\alpha \in \bar{K}$ whose minimal polynomial f is of degree $n \geq 2$. Prove that for any monic $g \in K[\zeta]$ of degree n satisfying $|f - g| < (|f|^{-1}r(\alpha))^n$, there exists a root β of g such that
   $$K(\beta) = K(\alpha).$$

(vi) Let $f \in K[\zeta]$ be an irreducible monic polynomial of degree $n \geq 2$. Prove that any monic polynomial $g \in K[\zeta]$ of degree n which is sufficiently close to f is also irreducible.

2. Consider the Tate algebra of n-variables $T_n = K\langle\zeta_1, \cdots, \zeta_n\rangle$.

   (i) Let $g \in T_n$. Prove that g is a unit if and only if it is ζ_n-distinguished of order zero.

   (ii) Let $g \in T_{n-1}[\zeta_n] \subset T_n$. Prove that g is prime in $T_{n-1}[\zeta_n]$ if and only if it is prime in T_n.

3. Let A, B be two affinoid K-algebras.

   (i) Prove that if $f \in A$ is non-nilpotent, then $|f|_{sup}$, the sup-norm of f, is nonzero.

   (ii) Let $\phi : A \to B$ be a morphism of affinoid K-algebras. Prove that ϕ maps power-bounded (resp. topologically nilpotent) elements of A to power-bounded (resp. topologically nilpotent) elements of B.

   (iii) Prove that there exist residues norms $|\cdot|_\alpha$ on A and $|\cdot|_\beta$ on B such that ϕ is contractive, that is,

$$|\phi(x)|_\beta \leq |x|_\alpha, \quad \forall x \in A.$$

4. Let $X = \mathrm{Sp}\ \hat{\bar{\mathbb{Q}}}_p\langle\zeta\rangle$ be the associated affinoid K-space to the Tate algebra of one-variable. Answer the following questions and justify your answers.

   (i) Is any Zariski open subset of X canonical open?

   (ii) Is any Zariski open subset of X an affinoid subdomain?

   (iii) Is the intersection of finitely many affinoid subdomains an affinoid subdomain?

   (iv)* Is the union of finitely many affinoid subdomains an affinoid subdomain?

(v) Is any Zariski open subset of X admissible open?

(vi) Is any canonical open subset of X admissible open?

(vii) Is any finite union of affinoid subdomains of X admissible open?

5. Let $X = \mathrm{Sp}\, A$ be an affinoid K-space. Let $X(\frac{f}{g}) \subset X$ be a rational domain, where $\frac{f}{g} = (\frac{f_1}{g}, \cdots, \frac{f_n}{g})$. Then there exists a Laurent domain $X' \subset X$ such that $X(\frac{f}{g}) \subset X' \subset X$ and $X(\frac{f}{g}) \subset X'$ is a Weierstrass domain.

6. Let X be a rigid K-space. We call X *connected* if there do not exist non-empty admissible open subsets $X_1, X_2 \subset X$ such that $X_1 \cap X_2 = \emptyset$ and (X_1, X_2) is an admissible covering of X.

   (i) Prove that an affinoid K-space X is connected if and only if A cannot be written as a non-trivial cartesian product of two K-algebras, that is, $A \ncong A_1 \times A_2$ where $A_i, i = 1, 2$ are two K-algebras.

   (ii) Prove that $\mathrm{Sp}\, \hat{\bar{\mathbb{Q}}}_p\langle\zeta\rangle$ is totally disconnected in canonical topology while connected as rigid K-space.

7. Let $X = \mathbb{P}_K^{1,rig}$ be the rigid projective line over K.

   (i) Prove that $\mathcal{O}_X(X) = K$ and hence X is connected as rigid K-space.

   (ii)* Prove that the automorphism group of X (as rigid K-space) is isomorphic to $\mathrm{PGL}_2(K)$, the projective linear group of K^2.