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School of Mathematical Sciences, USTC

2019—2020 Spring Semester, Final EXAMINATION

Class NAME Hamonic Analysis Class NUMBER MA04310.01

DATE 2020.06.19 EXAM Patern OPEN

Student NAME__________ Student NUMBER__________

TEST NUMBER	I	II	III	IV	V	TOTAL
Scores						

I: Riemann-Lebesgue Lemma. Let f be a function on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For $f \in L^1(\mathbb{T})$, and an integer k, define its k-th Fourier coefficient by

$$c_k(f) = \int_{\mathbb{T}} f(x) e^{-2\pi i k x} dx$$

.

1. Show that if $g \in L^1(\mathbb{T})$, then
$$c_k(g) \to 0$$
as $|k| \to \infty$.

2. If in addition $g \in C^N(\mathbb{T})$ for some positive integer N, then
$$c_k(g) = \mathbf{o}\left(\frac{1}{|k|^N}\right).$$

3. Identify $\mathbb{T}$ with the interval $(0, 1]$, and let $g(x) = |x| - \frac{1}{2}$ for $x \in (0, 1]$. Give the exact decay of $c_k(g)$ as $|k| \to \infty$.

4. Show that the restriction of the meromorphic function $\frac{1}{1-z/2} + \frac{1}{1-1/(2z)}$ onto the circle has its Fourier coefficients decay exponentially.

II: Approximation of the identity.

1. Give the definition of an approximation of the identity.

2. Write out the expression of the Dirichlet kernel on the real line $\mathbb{R}$. Show that we can not form an approximation of the identity from the Dirichlet kernel.

3. Express the Fejer kernel F in terms of the Dirichlet kernel. Demonstrate that we can form an approximation of the identity from the Fejer kernel, denoted by $\{F_R(x)\}_{R>0}$.

4. Show that for $f \in L^p(\mathbb{R})$ for some $p \in [1, \infty)$, we have

$$\lim_{R\to\infty} \|F_R * f - f\|_{L^p(\mathbb{R})} = 0.$$

III: Calderon-Zygmund theorem

Let K be a tempered distribution in $\mathbb{R}^n$ which coincides with a locally integrable function on $\mathbb{R}^n$ away from the origin, and is such that

$$\left|\hat{K}(\xi)\right| \leq A, \tag{1}$$

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \leq B, \quad y \in \mathbb{R}^n. \tag{2}$$

1. Show that

$$\|K * f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \tag{3}$$

and

$$\left|\left\{x \in \mathbb{R}^n : |K * f|(x) > \lambda\right\}\right| \leq \frac{c}{\lambda}\|f\|_{L^1(\mathbb{R}^n)} \tag{4}$$

2. Recall that for $j \in \{1, 2, \ldots, n\}$, the Riesz transform is defined by

$$R_j f(x) = c_n \texttt{p.v.} \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x-y)dy \tag{5}$$

for some positive constant $c_n > 0$. Using Caderon-Zygmund theorem, indicate (do not need to prove) that R_j is a bounded operator acting on $L^p(\mathbb{R}^n), 1 < p < \infty$.

IV: Hormander Multiplier theorem Let's work on the space $\mathbb{R}^n$ for some integer $n \geq 2$.

1. Give the definition of a multiplier on $L^p(\mathbb{R}^n)$.

2. Give the definition of $L^2(\mathbb{R}^n)$-based inhomogeneous Sobolev space $H^s(\mathbb{R}^n)$ of regularity s. And show that if $m \in H^s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$, then m is a multiplier on $L^p(\mathbb{R}^n), 1 \leq p \leq \infty$.

3. State Hormander multiplier theorem (NO need to prove it).

4. Using Hormander multiplier theorem, show that both of the functions

   (a) $m(\xi) = |\xi|^{it}, \xi \in \mathbb{R}^n, t \in \mathbb{R}\backslash\{0\}$;

   (b) m: a function that is homogeneous of degree 0 and is of class $C^k(\mathbb{S}^{n-1})$ for some integer $k > \frac{n}{2}$;

   define multipliers on $L^p(\mathbb{R}^n), 1 < p < \infty$.

V: Principle of Stationary phase when $n = 1$. Consider

$$I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda\phi(x)}\psi(x)dx \tag{6}$$

where ψ is a smooth function of compact support and $x = 0$ is the only critical point of ϕ in the support of ψ, with $\phi''(0) \neq 0$. Show that for every integer $N > 0$,

$$I(\lambda) = \frac{e^{i\lambda\phi(0)}}{\lambda^{1/2}}\left(a_0 + a_1\lambda^{-1} + \cdots + a_N\lambda^{-N}\right) + \mathbf{O}(\lambda^{-N-1}) \tag{7}$$

as $\lambda \to \infty$, where a_k are constants determined by values of ϕ and its derivatives at 0.