中国科学技术大学

2019–2020学年第一学期考试试卷

考试科目：同调代数　　　　　　　　　　　　得分＿＿＿＿＿＿

学生所在系：[redacted]　　　　姓名 [redacted]　　　　学号 [redacted]

1. Let $\mathcal{C}$, $\mathcal{D}$ be additive categories.

(1) Let $X, Y \in \mathcal{C}$. Then the following are equivalent:

(i) $(X \oplus Y, \iota_1 : X \longrightarrow X \oplus Y, \iota_2 : Y \longrightarrow X \oplus Y)$ is a coproduct.

(ii) There exist morphisms $p_1 : X \oplus Y \longrightarrow X$ and $p_2 : X \oplus Y \longrightarrow Y$ such that

$$p_1\iota_1 = 1_X,\ p_2\iota_2 = 1_Y,\ \iota_1 p_1 + \iota_2 p_2 = 1_{X\oplus Y}.$$

(2) Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a (not necessarily additive) functor which admits a left (or right) adjoint. Then F is additive.

2. Let $\mathcal{C}$ be a category with zero object. Prove that:

(1) Coequalizers and cokernels are direct limits.

(2) Assume $\mathcal{C}$ has finite coproduct, then $\mathcal{C}$ admits coequalizers iff $\mathcal{C}$ admits pushouts.

3. Denote the category of left R-modules by ${}_R\mathbf{Mod}$. Prove that:

(1) Let $0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$ be a family of short exact sequences in ${}_R\mathbf{Mod}$, then $0 \longrightarrow \Pi_i A_i \longrightarrow \Pi_i B_i \longrightarrow \Pi_i C_i \longrightarrow 0$ is exact.

(2) Let I be a direct set, and $\{A_i, \alpha_j^i\}, \{B_i, \beta_j^i\}, \{C_i, \gamma_j^i\}$ be direct systems in ${}_R\mathbf{Mod}$ over I. Assume that $r : \{A_i, \alpha_j^i\} \longrightarrow \{B_i, \beta_j^i\}$ and $s : \{B_i, \beta_j^i\} \longrightarrow \{C_i, \gamma_j^i\}$ are morphism of direct systems and such that $0 \longrightarrow A_i \xrightarrow{r_i} B_i \xrightarrow{s_i} C_i \longrightarrow 0$ is exact for each $i \in I$. Then there is an exact sequence

$$0 \longrightarrow \varinjlim A_i \longrightarrow \varinjlim B_i \longrightarrow \varinjlim C_i \longrightarrow 0.$$

(3) Keep assumptions as in (2). Prove or disprove the exactness of

$$0 \longrightarrow \varprojlim A_i \longrightarrow \varprojlim B_i \longrightarrow \varprojlim C_i.$$

(4) Give a counter-example to show that (2) is NOT true if I is NOT direct.

4. Compute $\mathrm{H}_i(\mathbf{Z}_m \times \mathbf{Z}_n, \mathbf{C}^\times)$ and $\mathrm{H}^i(\mathbf{Z}_m \times \mathbf{Z}_n, \mathbf{C}^\times)$ for $i = 0, 1, 2$, where $\mathbf{C}^\times$ is the multiplicative group of the complex number field.

5. Let R be a P.I.D., B a left R-module, and $(\mathbf{C}, d)$ be a complex of free left R-module. Then there exist exact sequences

$$0 \longrightarrow \mathrm{Ext}_R^1(\mathrm{H}_{n-1}(\mathbf{C}), B) \xrightarrow{\lambda_n} \mathrm{H}^n(\mathrm{Hom}_R(\mathbf{C}, B)) \xrightarrow{\mu_n} \mathrm{Hom}_R(\mathrm{H}_n(\mathbf{C}), B) \longrightarrow 0, n \in \mathbf{Z}.$$

6. Let $f: R \longrightarrow R^*$ be a ring homomorphism.

(1) The induced functor $f^* : {}_{R^*}\mathbf{Mod} \longrightarrow {}_R\mathbf{Mod}$ is exact and admits left adjoint and right adjoint.

(2) If R^* is a projective R-module, then

$$\mathrm{Ext}^n_{R^*}(A^*, \mathrm{Hom}_R(R^*, B)) \xrightarrow{\cong} \mathrm{Ext}^n_R(f^* A^*, B).$$

(3) If R^* is a flat R-module, then

$$\mathrm{Ext}^n_{R^*}(R^* \otimes_R A, B^*) \xrightarrow{\cong} \mathrm{Ext}^n_R(A, f^* B^*).$$

7. Let $\mathbb{K}$ be a field and A be a $\mathbb{K}$-algebra. The n-th Hochschild cohomology group of A with coefficient in A is defined as $\mathrm{HH}^n(A, A) := \mathrm{Ext}^n_{A^e}(A, A)$, where $A^e := A \otimes_{\mathbb{K}} A^{op}$.

(1) Give all indecomposable projective modules of A^e.

(2) Compute $\mathrm{HH}^0(A, A) = Z(A)$ and $\mathrm{HH}^1(A, A)$.

(3) Let A be a matrix subalgebra of $\mathrm{Mat}_3(\mathbb{K})$ whose elements with the form $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$

Compute $\mathrm{HH}^n(A, A)$, $n = 0, 1, 2, \cdots$.

8. If every finitely generated submodule of a left R-module M is flat. Show that M is flat.