2018年秋季学期 微分流形期中考试
2018 Fall Mid-term Exam: Differential Manifolds

整理人: 章俊彦 zhangjy9610@gmail.com

主讲教师：王作勤(Zuoqin Wang)

Problem 1 (20 points, 4 points each)

Let M be a smooth manifold. Write down the definitions of the following conceptions.

(1) What does a smooth function i $f: M \to \mathbb{R}$ mean?

(2) What does a smooth map i $f: M \to M$ mean?

(3) When we say N is a smooth submanifold of M, what do we mean?

(4) When we say X is a smooth vector field on M, what do we mean?

(5) A smooth k-dimensional distribution $\mathcal{V}$ on M is...

(6) A smooth action of a Lie group G on M $\tau: G \to \mathrm{Diff}(M)$ is...

Problem 2 (20 points, 2 points each)

TRUE or FALSE.

( ) If a smooth manifold M is connected, it must be path-connected;

( ) If (ϕ, U, V) is a smooth chart on a smooth manifold M, then $\phi: U \to V$ is a diffeomorphism;

( ) If N is a smooth submanifold of M and S is a smooth submanifold of N, then S is a smooth submanifold of M;

( ) The set of critical values of any smooth map is of measure zero;

( ) If M_1, M_2 are smooth submanifold of M, then so is $N_1 \cap M_2$;

( ) $S^2 \times S^2$ cannot be embedded into $\mathbb{R}^5$;

( ) Any smooth vector field on $\mathbb{R}P^n$ is complete;

( ) If $f: M \to N$ is a submersion, then $\mathcal{V}_p := \ker(df_p)$ defines an involutive distribution on M;

( ) Each smooth manifold admits at most one Lie group structure;

( ) If a Lie group G acts on M smoothly, then any orbit $G \cdot m$ is an immersed submanifold of M;

( ) Let $X, Y \in \Gamma^\infty(TM)$ be complete vector fields on M. If the Lie derivative of X along Y is zero, then the Lie derivative of Y along X is also zero.

Problem 3 (20 points, 4 points each)

Write down the solutions. No detail is needed.

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $f(x, y) = (x^3 - y, xy^2)$, then $df_{(x,y)}$=_______________;

(2) All the critical points of $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x^3 - y, xy^2)$ is _______________;

(3) Let $X = e^x \partial_x$ is a vector field on $\mathbb{R}$. Find the equation of the maximal integral curve γ of X with $\gamma(0) = 0$:_______________;

(4) Let $X = y\partial_z - x^2y^2\partial_x,\ Y = z\partial_y$ be vector fields on $\mathbb{R}^3$, then $[X, Y]$=_______________;

(5) The lie algebra $\mathfrak{h}$ of the Heisenberg group

$$H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

is _______________;

Problem 4 (15 points)

Let M be a smooth manifold and $f \in C^\infty(M)$. Suppose $\mathcal{L}_X f = 0$ holds for alll $X \in \Gamma^\infty(TM)$. Can we conclude that f is a constant function? Prove your conclusion.

Problem 5 (15 points)

Let $S^n = \{(x^1, \cdots, x^{n+1}) : \sum_{i=1}^{n+1}(x^i)^2 = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$. Define a function f on S^n by

$$f(x^1, \cdots, x^{n+1}) = \sum_{i=1}^{n+1} a_i (x^i)^2,$$

where $a_1, \cdots a_{n+1} \in \mathbb{R}$.

(1) Prove f is a smooth function on S^n;

(2) Find all the critical values of f;

(3) Suppose $a_1 = \cdots = a_k - a < b = a_{k+1} = \cdots = a_{n+1}$. Take any $c \in (a, b)$. What is the manifold $f^{-1}(c)$? Descirbe it in a geometric way: What manifold that we are familiar with is it diffeomorphic to?

Problem 6 (20 points)

For a smooth map $f : M \to M$, a point $p \in M$ satisfying $f(p) = p$ is said to be a fixed point of f. We say f is a Lefschetz map if for each $p \in \text{Fix}(f)$, 1 is not an eigenvalue of $df_p : T_pM \to T_pM$. For such an f, we define its local Lefschetz number $L_p(f)$ at $p \in \text{Fix}(p)$ to be $\text{sgn}(\det(df_p - Id))$.

Answer the follwiong questions:

(1) Let $r_\theta : S^2 \to S^2$ be the map "rotate S^2 by angle $\theta \neq 2k\pi$", defined by

$$r_\theta(x^1, x^2, x^3) = (x^1 \cos\theta - x^2 \sin\theta, x^1 \sin\theta + x^2 \cos\theta, x^3).$$

Prove that r_θ is a Lefschetz map.

(2) Let V be a vector space and $L : V \to V$ is a linear map. Let $\Delta = \{(v, v) : v \in C\}$ be the diagonal in $V \times V$, and $\Gamma_L := \{(v, Lv) : v \in V\}$ be the graph of L. Prove that Γ_L intersects Δ transversally iff 1 is not an eignvalue of L.

(3) If M is compact and f is a Lefschetz map. Show that $\text{Fix}(f)$ is finite.

(4) Define the Lefschetz numebr of f by

$$L(f) = \sum_{p \in \text{Fix}(p)} L_p(f).$$

Compute $L(r_\theta)$.

Problem 7 (15 points)

Let M be a connected smooth manifold. Prove that for any $p, q \in M$, there is a $\phi \in \mathrm{Diff}(M)$ so that $\phi(p) = q$.

Problem 8 (15 points)

Let M, N be smooth manifolds.

(1) Write down the definition of <u>submersion</u>;

(2) Let $\phi : M \to N$ be a smooth map. Write down the definition of "<u>$X \in \Gamma^\infty(TM)$ is ϕ-related to $Y \in \Gamma^\infty(TN)$</u>".

(3) Suppose $\phi : M \to N$ is a submersion. Prove that for any $Y \in \Gamma^\infty(TN)$, there exists $X \in \Gamma^\infty(TM)$ which is ϕ-related to Y.

Problem 9 (20 points)

In what follows we cite a proof of **Cartan's closed subgroup Theorem**. Find the mistakes (maybe more than one) in this proof and your brief explanation.

Theorem Any closed subgroup H of a Lie group G is a Lir subgroup.

Proof: W.L.O.G we assume H is connected. Then

$$\mathfrak{h} := \{X \in \mathfrak{g} | \exp_G(tX) \in H \forall t \in \mathbb{R}\}$$

is a vector subspace of $\mathfrak{g}$. $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ since (the formula below is correct)

$$\exp_G(tX)\exp_G(tY)\exp_G(-tX)\exp_G(-tY) = \exp_G(t^2[X,Y] + O(t^3)).$$

So there is a unique connected Lie subgroup $H' \leq G$ with Lie algebra $\mathfrak{h}$. Let $\mathfrak{s}$ be a complememntary subspace of $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$. Let U, V be twp sufficiently small convex neighbourhood of 0 in $\mathfrak{h}$ and in $\mathfrak{s}$ respectively, such that $\exp_G|_{U\times V}$ is a diffeomorphsim to its image.

We claim that $H \cap \exp_G(U \times V) = \exp_g U$. In fact, suppose $\exp_G(X+Y) \in H$, $X \in U$ and $Y \in V$. Then

$$\exp_G(Y) = \lim_{n\to\infty}\left(\exp_G\frac{X+Y}{n}\exp_G\frac{-X}{n}\right)^n$$

is in H since H is closed. Hence $Y \in \mathfrak{h} \cap \mathfrak{s} = \{0\}$. Consequently, $\exp_G U$ is an open set in H. On the other hand, $\exp_G U$ is an open set in H' since $\exp_G|_{\mathfrak{h}} = \exp_{H'}$. Therefore $H = \cup_n(\exp_G U)^n = H'$ as both are connected.