RIEMANNIAN GEOMETRY (MA0440301, SPRING, 2017)
MID-TERM EXAM

Name: No.: Department:

1. (15 marks) Let (M, g) be a Riemannian manifold.

(1) Let U be a normal neighborhood of $p \in M$ with coordinates $(x^1, \dots, x^n)$. Consider the radial function on $(U \setminus \{p\}, x^1, \dots, x^n)$:

$$r = \sqrt{\sum_i (x^i)^2}.$$

Show that

$$\operatorname{grad} r = \frac{\partial}{\partial r}.$$

(Hint: Use Riemannian polar coordinates.)

(2) For any $X, Y \in \Gamma(TM)$, prove that

$$\mathrm{Hess} f(X, Y) = g(\nabla_X \mathrm{grad} f, Y),$$

where $\mathrm{Hess} f := \nabla^2 f$ is the Hessian of f.

(3) Let (M, g) be compact without boundary, and φ_1, φ_2 are two smooth functions on M such that

$$\Delta \varphi_i + \lambda_i \varphi_i = 0, \ \ \lambda_i \in \mathbb{R}.$$

Show that if $\lambda_1 \neq \lambda_2$, then

$$\int_M \varphi_1 \varphi_2 d\mathrm{vol} = 0.$$

(Hint: use Green formula.)

2. (15 marks) Let (M, g) be a Riemannian manifold. For any $p \in M$, the *injectivity radius* of p is defined as

$$i(p) := \sup\{\rho > 0 : \exp_p \text{ is a diffeomorphism on } B(0, \rho) \subset T_pM\}.$$

The injectivity radius of M is then defined as

$$i(M) := \inf_{p \in M} i(p).$$

(1) Compute the injectivity radius of the sphere $S^2(\frac{1}{k})$ of radius $\frac{1}{k}$.

(2) Prove that if M is compact, then the injectivity radius $i(M)$ is positive. (Hint: Use totally normal neighborhood.)

Date: 2017.04.15.

3. (25 marks) (Geodesic equation in Finsler geometry)
Finsler geometry is a natural generalization of Riemannian geometry. Let M be an n-dimensional smooth manifold. Let $TM := \bigcup_{x\in M} T_xM$ be the tangent bundle of M. Each element of TM has the form (x,y), where $x \in M$ and $y \in T_xM$. The natural projection $\pi : TM \to M$ is given by $\pi(x,y) = x$.

A *Finsler structure* of M is a function

$$F : TM \to [0,\infty)$$

with the following properties:

(i) *Regularity*: F is C^∞ on $TM \setminus 0$.
(ii) *Absolute homogeneity*: $F(x,\lambda y) = |\lambda| F(x,y)$ for all $\lambda \in \mathbb{R}$.
(iii) *Strong convexity*: The $n \times n$ Hessian matrix

$$(g_{ij}) := \left(\left[\frac{1}{2}F^2\right]_{y^iy^j} \right)$$

is positive-definite at every point of $TM \setminus 0$. (Explanation of y^i: For any basis $\{\frac{\partial}{\partial x^i}\}$, express y as $y^i\frac{\partial}{\partial x^i}$. The Finsler structure F is then a function of $(x^1,\dots,x^n,y^1,\dots,y^n)$, and

$$\left[\frac{1}{2}F^2\right]_{y^iy^j} := \frac{\partial^2}{\partial y^i \partial y^j}\left[\frac{1}{2}F^2\right].$$

It can be checked that the positive-definiteness is independent of the choice of $\{\frac{\partial}{\partial x^i}\}$.)

Let $\gamma : [a,b] \to M$ be a smooth curve in M. Suppose the parametrization of γ is regular, i.e., $\dot\gamma(t) \neq 0$, $\forall t \in [a,b]$. We can define the length and energy of γ to be

$$L(\gamma) := \int_a^b F(\gamma(t),\dot\gamma(t))dt,$$
$$E(\gamma) := \frac{1}{2}\int_a^b F^2(\gamma(t),\dot\gamma(t))dt,$$

respectively.

(1) Prove that $L(\gamma)$ does not depend on the choice of a regular parametrization.
(2) Prove that $L(\gamma)^2 \leq 2(b-a)E(\gamma)$, and characterize the case when "=" holds.
(3) Suppose that the image $\gamma([a,b])$ falls in a local coordinate $(U, x^1,\dots,x^n)$. Denote by

$$\gamma(t) := (x^1(t),\dots,x^n(t)).$$

Show that the Euler-Lagrange equation for $E(\gamma)$ (defined to be the geodesic equation) is

$$\ddot x^\ell + \frac{1}{4}g^{i\ell}\left(\left[F^2\right]_{x^jy^i} y^j - \left[F^2\right]_{x^i}\right) = 0,\ \forall \ell = 1,\dots,n,$$

where $\left(g^{i\ell}\right)$ is the inverse matrix of (g_{ij}).

4. (25 marks) Let (M,g) be a Riemannian manifold. Let ∇^g be the Levi-Civita connection of the metric g.

(1) Derive the following *Koszul formula* from the definition of Levi-Civita connection: For any $X,Y,Z \in \Gamma(TM)$,

$$2g(\nabla^g_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]).$$

(2) Show that for any constant $c > 0$, we have $\nabla^{cg} = \nabla^g$.

(3) Show that for any constant $c > 0$, the sectional curvature $K(cg)$, Ricci curvature tensor $\mathrm{Ric}(cg)$, and scalar curvature $S(cg)$ of (M, cg) satisfy

$$K(cg) = \frac{1}{c}K(g), \quad \mathrm{Ric}(cg) = \mathrm{Ric}(g), \quad S(cg) = \frac{1}{c}S(g).$$

Suppose we have two Riemannian manifolds (M_1, g_1) and (M_2, g_2). Then the product $M_1 \times M_2$ has a natural metric $g = g_1 + g_2$: At each $(p, q) \in M_1 \times M_2$, a vector $X_{(p,q)} \in T_{(p,q)(}M_1 \times M_2)$ can be written as

$$X_{(p,q)} = (X_1, 0)_{(p,q)} + (0, X_2)_{(p,q)},$$

where $X_i \in \Gamma(TM_i), i = 1, 2$. Then the metric $g = g_1 + g_2$ is given by

$$g(X, Y) := g_1(X_1, Y_1) + g_2(X_2, Y_2), \ \forall X, Y.$$

(4) Show that $\nabla^g_{(X_1,0)}(0, Y_2) = 0$ for any $X_1 \in \Gamma(TM_1), Y_2 \in \Gamma(TM_2)$.

(5) Compute the sectional curvature $K((V, 0)_{(p,q)}, (0, W)_{(p,q)})$, where $V \in T_pM_1$, $W \in T_qM_2$. Does $S^2 \times S^2$ have positive sectional curvature everywhere with the metric $g = g_{can} + g_{can}$? (Here g_{can} is the canonical metric on S^2.)

5. (20 marks) Let (M, g) be a complete Riemannian manifold, and let $N \subset M$ be a compact submanifold of M without boundary.

(1) Show that N with the induced metric from (M, g) is complete.

(2) Let $p_0 \in M$, $p_0 \notin N$, and let $d(p_0, N) := \inf_{q \in N} d(p_0, q)$ be the distance from p_0 to N. Show that there exists a point $q_0 \in N$ such that

$$d(p_0, q_0) = d(p_0, N).$$

Moreover, a minimizing geodesic $\gamma : [a, b] \to M$ which joins p_0 to q_0 is orthogonal to N at q_0, that is, $g(\dot\gamma(b), V) = 0$, for any $V \in T_{q_0}N \subset T_{q_0}M$.

(3) Given $p \in M$. Suppose exp_p is a diffeomorphism on $B(0, r) \subset T_pM$. Then we denote by $B_r(p) := exp_p(B(0, r))$ the normal ball with center p and radius r. Consider the particular submanifold $N := exp_p(\partial B(0, r))$. For any $p_0 \notin B_r(p)$, prove that there exists $q_0 \in N$ such that

$$d(p, p_0) = r + d(q_0, p_0).$$