RIEMANNIAN GEOMETRY (MA0440301, SPRING, 2017)
FINAL EXAM

Name: No.: Department:

1.

(1) **(6 marks)** Let $\gamma : [0,1] \to M$ be a geodesic. State **two** equivalent definitions of the terminology "0 and 1 are conjugate values along γ".
(2) **(6 marks)** Let M be complete and let $\gamma : [0,\infty) \to M$ be a normal geodesic. State the **two** cases that may happen when $\gamma(a)$ is a cut point of $\gamma(0)$ along γ.
(2) **(6 marks)** State **two** equivalent definitions of the injectivity radius of $p \in M$. (Hint: one via exponential map, and one via cut locus.)

2.(15 marks)
Use the second variation formula to prove the **Bonnet-Myers Theorem**: Suppose (M, g) is an n-dimensional complete Riemannian manifold with Ricci curvature $\mathrm{Ric} \geq (n-1)k > 0$. Then the diameter of M is no greater than $\pi/\sqrt{k}$.

3.

(1) **(6 marks)** Let M be an n-dimensional compact Riemannian manifold with positive sectional curvature. Show that if n is even, orientability is necessary to conclude, via Synge's Theorem, that M is simply-connected. And show that if n is odd, one cannot conclude that M is simply connected.
(2) **(10 marks)** Show that the product of the real projective spaces $P^2(\mathbb{R}) \times P^2(\mathbb{R})$ has a metric with positive Ricci curvature but does not admit a metric with positive sectional curvature.

Hint: You can use the following fact: The product $M_1 \times M_2$ of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) has a natural metric $g = g_1 + g_2$: At each $(p,q) \in M_1 \times M_2$, a vector $X_{(p,q)} \in T_{(p,q)}(M_1 \times M_2)$ can be written as

$$X_{(p,q)} = (X_1, 0)_{(p,q)} + (0, X_2)_{(p,q)},$$

where $X_i \in \Gamma(TM_i), i = 1, 2$. Then the metric $g = g_1 + g_2$ is given by

$$g(X,Y) := g_1(X_1, Y_1) + g_2(X_2, Y_2),\ \forall X, Y.$$

The corresponding Riemannian curvature tensors satisfy

$$R(X,Y,Z,W)(p,q) = R(X_1,Y_1,Z_1,W_1)(p) + R(X_2,Y_2,Z_2,W_2)(q).$$

Date: 2017.06.08.

4.(15 marks) Let M be a Riemannian manifold, and $\gamma : [0, \ell] \to M$ be a normal geodesic. We say that a Jacobi filed $U(t)$ along γ is **almost parallel** if there exists a parallel vector field $V(t)$ along γ such that

$$U(t) = f(t)V(t), \text{ for some } f \in C^\infty(M).$$

Prove that if M has constant sectional curvature k, any normal Jacobi field $U(t)$ along γ with $U(0) = 0$ is almost parallel.

5. (15 marks) Let M be an n-dimensional complete simply-connected Riemannian manifold with constant sectional curvature $k < 0$. Let $\rho(x) := d(x, p)$ be the distance function to a fixed point $p \in M$. Prove that on $M \setminus \{p\}$

$$\Delta\rho = (n-1)\sqrt{-k}\frac{\cosh(\sqrt{-k}\rho)}{\sinh(\sqrt{-k}\rho)}$$

6. (10 marks) Let M be an n-dimensional complete Riemannian manifold M with $\mathrm{Ric} \geq 0$. Suppose that

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B_p(r))}{w_n r^n} = 1, \text{ for some } p \in M,$$

where w_n is the volume of the unit ball in Euclidean space. Prove that M must be isometric to Euclidean space.

7. Let M be a Riemannian manifold with sectional curvature no greater than k and $\gamma : [0, \ell] \to M$ be a normal geodesic. Let $U(t)$ be a Jacobi field along γ. Assume either (i) $k \geq 0$ or (ii) $U(t)$ is normal. Let $f_k : [0, \ell] \to \mathbb{R}$ be a function solving

$$\frac{d^2}{dt^2} f_k(t) + k f_k(t) = 0,$$
$$f(0) = |U(0)|, \frac{d}{dt} f_k(0) = \frac{d}{dt}|U|(0).$$

Suppose $f_k(t) > 0$ for $t \in (0, \ell]$, and γ contains no conjugate point of $\gamma(0)$.

(1) **(6 marks)** Show that

$$\frac{d^2}{dt^2}|U(t)| + k|U(t)| \geq 0, \text{ for } t \in [0, \ell].$$

(2) **(15 marks)** Show the following monotonicity:

$$1 \leq \frac{|U(t_1)|}{f_k(t_1)} \leq \frac{|U(t_2)|}{f_k(t_2)} \text{ for } 0 < t_1 \leq t_2 < \ell.$$

(3) **(10 marks)** When M is a complete, simply-connected Riemannian manifold of nonpositive sectional curvature, consider two geodesics

$$\gamma_1, \gamma_2 : [0, 1] \to M, \text{ such that } \gamma_1(0) = p = \gamma_2(0).$$

Use (2) to prove that

$$d(\gamma_1(t), \gamma_2(t)) \leq t d(\gamma_1(1), \gamma_2(1)).$$