理论力学A. 2022年11月30日.

参考书目 Arnold. 经典力学中的数学方法.

数学 ⇒ | 空间.
| 映射.

① 映射 (函数)

$f: X \longmapsto Y$

表示 $x \in X$. $f(x) = y \in Y$

物理: $\mathcal{L}(x, \dot{x}, t) = \frac{m}{2}\dot{x}^2 - U(x,t) = T - U$

$(x, \dot{x}, t) \in \Gamma$, 相空间.

$\mathcal{L}$ 和 f

数学: 映射.

$\mathcal{L}: \Gamma \to \underline{\underline{\mathbb{R}}}$ 实数空间.

空间 $E = \mathbb{R}^3$, 3维, 欧氏

$\mathbb{R}^3 \times \mathbb{R}$, (3+1)维. 闵氏.

* 矢量.
* 距离. $ds^2 = dx^2 + dy^2 + dz^2$
* 运算.

总结: (1) $L: \Gamma \longmapsto \mathbb{R}$.

(2) $E = T + V = \frac{1}{2}mv^2 + U(x)$

$\Rightarrow E(x, v) \Rightarrow \Gamma \to \mathbb{R}$.

(3) 不同的空间 Γ. x 与 $\dot{x}$ 是独立的;

广义坐标的空间.

(4) $\mathcal{L}: TM \longmapsto \mathbb{R}$

例如：单摆. $L(\theta, \dot{\theta})$

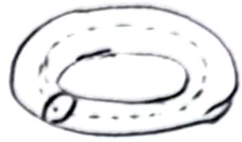
$\dot{\theta} \in \mathbb{R}$, $\theta \in S$

$\Gamma = S \times \mathbb{R}$

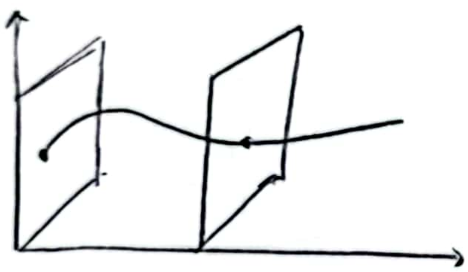
双摆 $L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)$

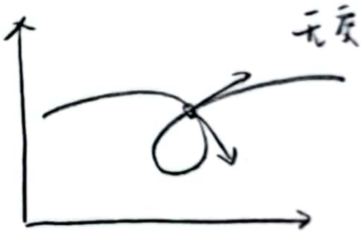
$\Gamma = S^1 \times S^1 \times \mathbb{R}^2$

$= T^2 \times \mathbb{R}^2$, T 为 ~~Torrence~~ Torus.



时间：世界线. world line.



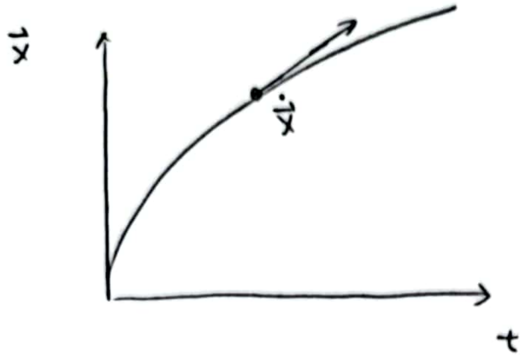


$\vec{x}(t) : \mathbb{R} \to \mathbb{R}^d$

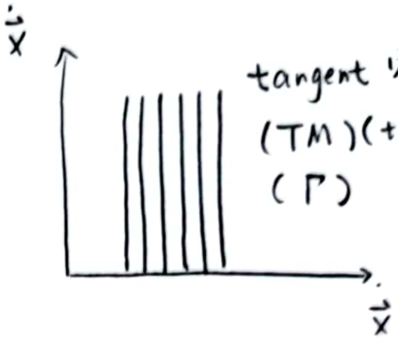
$\dot{\vec{x}}(t) : \mathbb{R} \to \mathbb{R}^d$

切空间. $\vec{x}$ 和 $\dot{\vec{x}}$ 什么关系?

纤维丛.



$$\dot{\vec{x}} = \frac{\vec{x}(t+dt)-\vec{x}(t)}{dt}$$

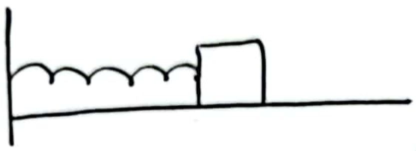


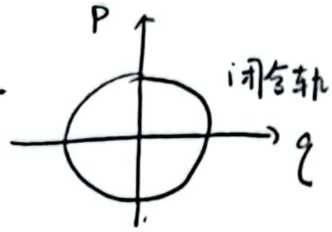
tangent 空间

(TM)(切丛)

(Γ)

相空间.





闭合轨迹

纤维丛.

$$|Ae^{i\theta}| = |A|$$

$e^{i\theta}$ 只增加相位，不改变模.(等价类).

TA建议: 清华的近世代数，林 第一章，习题6.

② 勒让德变换. Legendre transformation.

$$L = T - V$$

$$L = L(q, \dot{q}, t)$$

$$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial t} dt$$

$$dL = \frac{\partial L}{\partial q} dq + p\, d\dot{q} + \frac{\partial L}{\partial t} dt$$

$$= \frac{\partial L}{\partial q} dq + d(p\dot{q}) + \frac{\partial L}{\partial t} dt - \dot{q}\, dp$$

$$d(p\dot{q} - L) = \dot{q}\, dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt$$

☆ $L(q, \dot{q})$ 和 t 无关. $\frac{\partial L}{\partial t} = 0$.

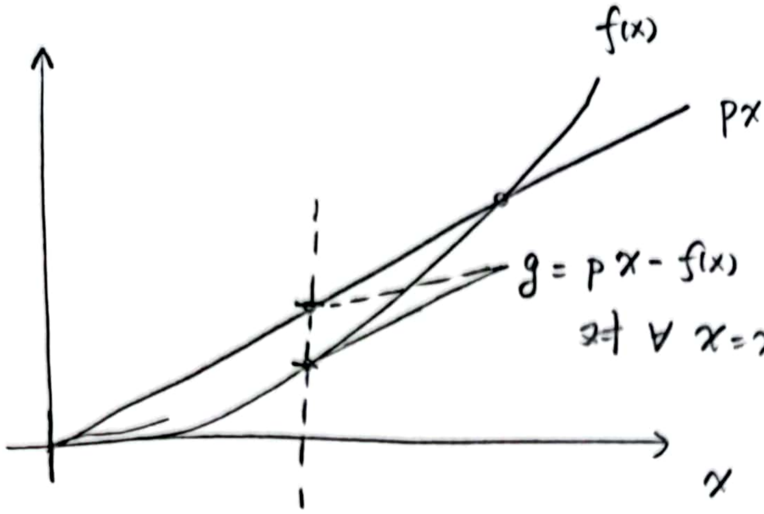
☆ 令 $p\dot{q} - L = H$. 能量 Hamiltonian.

$$\begin{cases} L = L(q, \dot{q}) \\ H = H(q, p) \end{cases}$$

P. 广义动量.
$\dot{q}$. 广义速度

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$, 两个一阶方程.



对 $\forall\ x = x(p)$ 都最大. 求 $p = f'(x)$:

1. $g = px - f(x)$ 找一个 p. 令 g 最大.

$$\frac{\partial g}{\partial x} = p - f'(x) = 0.$$

$$g(p) = px - f(x) = \frac{\partial f}{\partial x} x - f(x).$$

$$H \overset{(q,p)}{=} \frac{\partial L}{\partial \dot{q}} \dot{q} - L(q, \dot{q})$$

$$= p\dot{q} - L$$

2. 柯氏不等式.

3. $px - f(x) = g(p).$

$$f(x) \xrightarrow{LT} g(p) \xrightarrow{LT} f(x)$$

③. H的物理意义.

$$L = \frac{1}{2} m \dot{q}^2 - U(q)$$

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \Leftrightarrow \quad \dot{q} = \frac{p}{m}$$

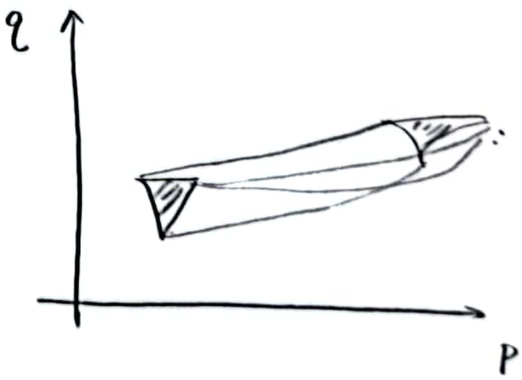
$$\begin{aligned} H &= p\dot{q} - L \\ &= p\frac{p}{m} - \frac{1}{2}m\left(\frac{p}{m}\right)^2 + U(q) \\ &= \frac{p^2}{2m} + U(q) \end{aligned}$$

④. 后续内容

(1) 最小作用量原理. (变分法).

(2) Hamiltonian 代数. (泊松括号).

(3) Liuviell 方程. (流守恒).



形状变化

但总体积 (或面积) 不变.

作业: Arnold书

$P_{10} \sim P_{11}$, 用Legendre transformation

把 (1)~(4) 的 L 写成 H.