理论力学A. 2022.10.26;

1. 回顾上一节课的内容:

微振动: $\begin{cases} A: & m\ddot{x}+m\omega_0^2 x \\ B: & 驱动.\ f\cos(\omega t) \longrightarrow \begin{cases} f\cos\omega t. & 受迫. \\ 1+h\cos\omega t. & 参变. \end{cases} \\ C: & 耗散.\ 2\lambda\dot{x} \\ D: & 非线性.\ \alpha x^2+\beta x^3+\cdots \end{cases}$ E: 多分量.

实际问题,主要是模型搭建. 可以由上述几种组合;
丰富的动力学行为.

Laudau书中的方法:(缺点:可执行性不强)(小量大量不清楚).

$$\ddot{x}+\omega_0^2 x+\alpha x^2+\beta x^3=0$$

$$x=x^{(1)}+x^{(2)}+x^{(3)}$$

$$\omega=\omega_0+\omega^{(1)}+\omega^{(2)}$$

多尺度分析:

$$\ddot{x}+\omega_0^2 x+\underset{\sim}{\varepsilon}\alpha x^2=0.$$

小量和大量分清楚.

$$x=x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdots$$

$$\omega^2=\omega_0^2+\varepsilon\omega_1+\varepsilon^2\omega_2$$

$$\frac{d^2x}{dt^2}=\omega^2\frac{d^2x}{d\tau^2},\quad \underline{\tau=\omega t}.$$

待定.原则 $\begin{cases} 低阶 \xrightarrow{驱动} 高阶. \\ 驱动 \Rightarrow 共振. \end{cases}$

消除发散. $x\to\infty$

$$\omega^2 \frac{d^2x}{d\tau^2} + \omega_0^2 x + \varepsilon\alpha x^2 = 0$$

$$(\omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2)(\ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2) + \omega_0^2(x_0 + \varepsilon x_1 + \varepsilon^2 x_2)$$

$$+ \varepsilon\alpha(x_0 + \varepsilon x_1 + \varepsilon^2 x_2)^2 = 0 ;$$

Landau:

用 ε^k, $k=0,1,2,\cdots$ 进行大量和小量的分类.

ε^0: $\omega_0^2 \ddot{x}_0 + \omega_0^2 x_0 = 0 \Rightarrow x_0 = A\cos(\omega_0 t)$.

ε^1: $\omega_0^2 \ddot{x}_1 + \omega_1 \ddot{x}_0 + \omega_0^2 x_1 + \alpha x_0^2 = 0$.

$$\Rightarrow \quad \underbrace{\omega_0^2 \ddot{x}_1 + \omega_0^2 x_1}_{} + \underbrace{\omega_1 \ddot{x}_0 + \alpha x_0^2}_{\text{低阶为高阶的驱动}} = 0$$

$$\underbrace{\ddot{x}_1 + x_1}_{} + \underbrace{\frac{\omega_1}{\omega_0^2}\ddot{x}_0}_{\cos(\tau)} + \underbrace{\frac{\alpha}{\omega_0^2} x_0^2}_{1+\cos(2\tau)} = 0$$

共振.

$1+\cos(2\tau)$, 不会共振. 受迫振动.

$\omega_1 = 0$ 时. $\Rightarrow$ eq.(28.11). P90;

$$\Rightarrow \quad x_1 = \text{Const.} + f_1 \cos(2\tau)$$

ε^2项:

$$\omega_0^2\ddot{x}_2 + \underbrace{\omega_1 \ddot{x}_1}_{0} + \omega_2\ddot{x}_0 + \omega_0^2 x_2 + \alpha x_1^2 + 2x_0 x_2 \alpha = 0$$

$$\underline{\ddot{x}_2 + x_2} + \underbrace{\omega_2 \ddot{x}_0 + 2\alpha x_0 x_1}_{\text{驱动.(低阶 } 0+1)} = 0.$$

$\cos\tau.$ $\begin{cases} x_0 = A\cos\tau \\ x_1 = C_0 + C_1\cos 2\tau \end{cases}$

让 $\cos\tau[\underbrace{\omega_2 + \text{来自 } 2\alpha x_0 x_1}_{0.}] = 0.$

(补充)多尺度分析.(Mutiscale analysis method).

1) 比 Landau书更简单.更系统.

2) 推广到高阶. Landau书只计算到3阶 ⇒ 高阶复杂度增加.

3) 更系统.更明白.(没有丢弃任何一项).

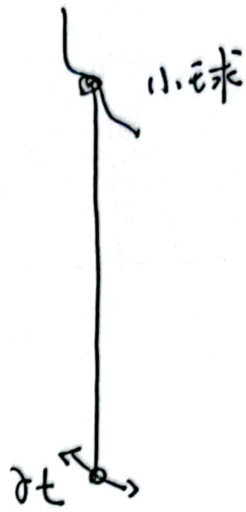
4) 可以推广到 $\forall$ 非线性模型.

5) 可以用 MMA 求解. DSolve[…]

下面讲 Karpitza单摆.

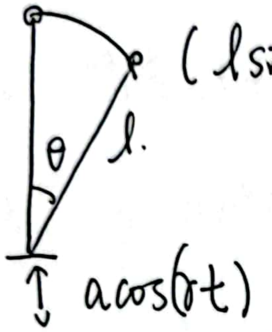
1900年左右. ⇒ Karpitza.

Nobel. Liquid Helium.



Model.

☆ Landau书. 抽象, 细节很少;



$(l\sin\theta, l\cos\theta + a\cos\gamma t)$

$$T = \frac{1}{2} m \left(l^2\dot{\theta}^2 + a^2\gamma^2\sin^2(\gamma t) + 2 l a \gamma \dot{\theta} \sin\theta \sin\gamma t \right)$$

$$U = mg(l\cos\theta + a\cos\gamma t)$$

$$L = T - U$$

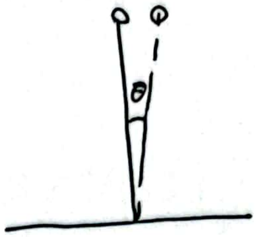
$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} + m l a \gamma \sin\theta \sin\gamma t$$

$$\frac{\partial L}{\partial \theta} = m l a \gamma \dot{\theta} \cos\theta \sin\gamma t + m g l \sin\theta$$

$$\Rightarrow \quad m l^2 \ddot{\theta} = mg\sin\theta - m l a \gamma^2 \sin\theta \cos(\gamma t)$$

$$l\ddot{\theta} = \sin\theta \left(g - a\gamma^2 \cos\gamma t \right).$$

$$\ddot{\theta} = \frac{\sin\theta}{l}\left(g - a\gamma^2\cos\gamma t\right).$$



☆ if $a \to 0$.

$$\ddot{\theta} \simeq \frac{g}{l}\sin\theta \simeq \frac{g}{l}\theta$$

1) $a=0$. 不稳定.

2) $g=0$. 稳定.(无外力) $\Rightarrow$ 给出数值模拟.

3) γ 大. $g - a\gamma^2\cos(\gamma t) < 0$.

4) Mathieu eq

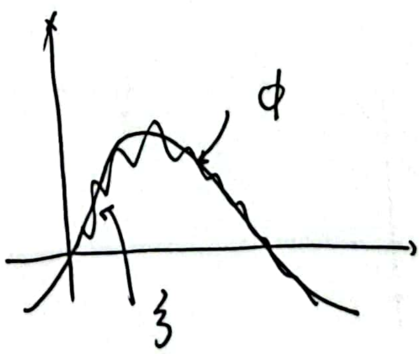
$$\frac{d^2y}{dt^2} + (a - b\cos 2t)\,y = 0, \quad \theta \to y. \ \sin\theta = y.$$

Hill eq.

$$\frac{d^2y}{dt^2} + f(t)\,y = 0. \quad f(t) = f(t+\pi).$$

解: $\theta = \underset{\text{slow}}{\phi} + \underset{\text{fast}}{\xi}$

$$\|\xi\| \ll \|\phi\|$$



$$\ddot{\phi} + \ddot{\xi} = \frac{\sin(\phi+\xi)}{\ell}\left(g - a\gamma^2\cos\gamma t\right)$$

$$= \frac{\sin\phi\cos\xi + \cos\phi\sin\xi}{\ell}\left(g - a\gamma^2\cos\gamma t\right)$$

$$= \frac{\sin\phi + (\cos\phi)\xi}{\ell}\left(g - a\gamma^2\cos\gamma t\right)$$

$$= \underbrace{\frac{g}{\ell}\sin\phi}_{\text{slow}} + \underbrace{\frac{g}{\ell}\xi\cos\phi}_{\text{fast}} - \underbrace{\frac{(\sin\phi)a\gamma^2}{\ell}\cos\gamma t}_{\text{fast.}}$$

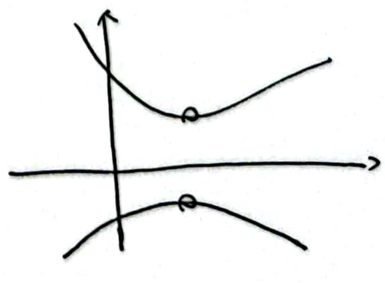
$$- \frac{a\gamma^2}{\ell}(\cos\phi)\xi\cos\gamma t$$

展开:

$$\ddot{\phi} = \frac{g}{\ell}\sin\phi - \frac{a^2\gamma^2}{\ell^2}\sin\phi\cos\phi\cos^2(\gamma t) + O(a^4)$$

$$= \frac{g}{\ell}\overset{\sin}{\cancel{\cos}}\phi - \frac{a^2\gamma^2}{2\ell^2}\sin\phi\cos\phi + \text{fast} + O(a^4) + \dots$$

$$\ddot{\phi} = \frac{g}{\ell}\overset{\sin}{\cancel{\cos}}\phi - \frac{a^2\gamma^2}{2\ell^2}\sin\phi\cos\phi$$



$$= \left(\frac{g}{\ell} - \frac{a^2\gamma^2}{2\ell^2}\right)\phi$$

$$= \frac{g}{\ell}\Big(\underbrace{1 - \frac{a^2\gamma^2}{2g\ell}}_{<0}\Big)\phi \quad . \text{ Stable.}$$

$$= -\frac{\partial U}{\partial \phi} \Leftrightarrow U \propto \cos\phi + \frac{a^2\gamma^2}{4g\ell}\sin^2\phi$$

作业：用多尺度方法计算

$$\begin{cases} \ddot{x} + \omega_0 x + \varepsilon(\alpha x^2 + \beta x^3) = 0 \text{ 的解.} \\ \text{高阶 } x^{(6)}.\ \varepsilon \to 0. \end{cases}$$

ω^1, ω^2, …, ω^5, ω^6.

作业：求解：

$$\ddot{\theta} = \frac{\sin\theta}{l}\left(g - a^2\gamma^2\cos\gamma t\right).$$

(1) 求不稳定 → 稳定点的转变. a. γ

(2) $2gl = a^2\gamma^2$. 不完全准确. 误差大概多大？

作业： Landau书.

P97 – P98.

习题 1. 2. 自己求解.