2022.09.02. 第1周第2次课;

牛顿方程，在直角坐标中更加直观.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z \end{cases}$$

$$F_r = m\ddot{r} = -\frac{\partial U}{\partial r}$$

$$F_\theta = m\ddot{\theta} = -\frac{\partial U}{\partial \theta} \quad ???$$

$$x = r\cos\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$\frac{\partial U}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial U}{\partial \theta}$$

$$\frac{\partial U}{\partial x} = ?$$

$\Rightarrow$ 不能简单地 $x_i \to q_i$ 替换;

但 Lagrange 方程可以.

如何证明 Lagrange Eqn.?

$$S = \int_{t_1}^{t_2} L\,dt \xrightarrow[\delta S = 0]{\text{变分法}} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q};$$

$$\text{Const.}(t) \longleftrightarrow \text{PDE}(t).\ 2\text{阶}$$

$$\updownarrow$$

$$\text{Out of Mechanics} \longleftrightarrow L = ?$$

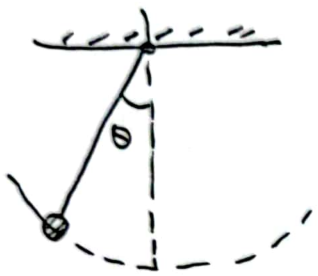
2022.0902. 第1周第2次课;

- 广义坐标
- L体系的逻辑 ⇒ 任何坐标都成立;
- 例子(理解L)
- 推导

广义坐标. 为什么? ⇒ 描述一个运动所需参数 = 直角坐标维度 − 受限条件数

① 自由空间 ⇒ 直角坐标有天然优势;

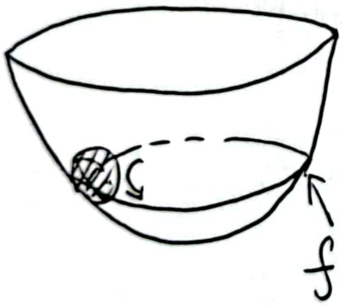
② 受限系统.



2d系统 (x,y)

↓约束. $x^2+y^2=r^2$

1d运动 (θ)



3d系统 (x,y,z)

↓约束 $f(x,y,z)=0$.

①. 坐标选择 (因问题而异, 非唯一);

②. 选择会影响计算的复杂程度;

③. 非唯一, 有约束 ⇒ 一般不用 $m\ddot{\vec{r}} = -\frac{\partial U}{\partial \vec{r}}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$, 要求它和坐标无关;

和牛顿力学等价;
考虑受限条件;
和坐标无关;

广义坐标：任意 $q=(q_1,\cdots,q_N)$

generalized coordinates，可能没有量纲. eg: θ

广义速度：generalized velocity $\Rightarrow \dot{q}_i$

广义动量：… momentum $\Rightarrow \dot{p}_i$

广义力：$\frac{\partial L}{\partial q_i}=F_i$

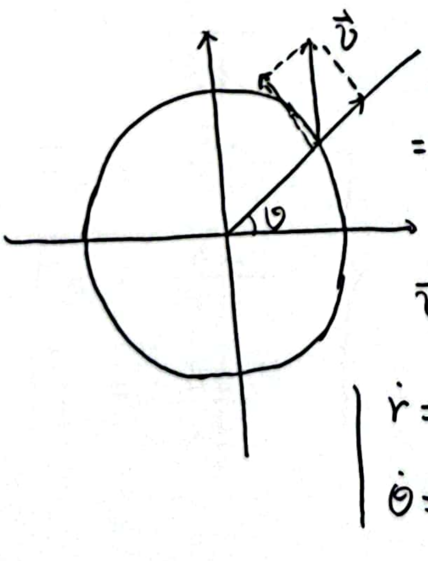
$(\dot{p}_i=F_i)$

例子1：L的计算：

L $\Rightarrow$ 运动方程.

2阶微分方程 $\Rightarrow \tilde{L}$

1) 自由空间，(x,y,z) $\longrightarrow$ 球坐标（or 柱坐标）.



$\frac{1}{2}mv^2$
$=\frac{m}{2}(\dot{r}^2+r^2\dot{\theta}^2)$

$\vec{v}=v_r\hat{r}+v_\theta\hat{\theta}$

$\dot{r}=0 \Rightarrow$ 圆周.

$\dot{\theta}=0 \Rightarrow \dot{r}$

$r.\ \theta.\ \phi \Leftrightarrow$ 3d 球：

$x = r\sin\theta\cos\phi$

$y = r\cos\theta\sin\phi$

$z = r\sin\theta$

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\,\dot{\phi}^2) - U(r)$$

广义坐标：$(r.\theta.\phi) = q$

广义速度：$(\dot{r},\dot{\theta},\dot{\phi})$

广义动量：$P = \frac{\partial L}{\partial \dot{q}}$

广义力：$\cancel{F = \frac{\partial L}{\partial \dot{q}}}$ $F = \frac{\partial L}{\partial q}$

$$P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}}$$

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}}$$

$$F_i = \dot{P}_i$$

$$\left.\begin{aligned} F_r &= \frac{\partial L}{\partial r} \\ F_\theta &= \frac{\partial L}{\partial \theta} \\ F_\phi &= \frac{\partial L}{\partial \phi} \end{aligned}\right\}$$

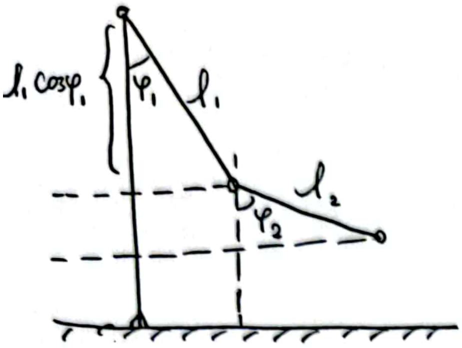
$U=0$. 只有T，无力

换一下坐标 $\Leftarrow$ 坐标不合适，则表达式复杂

可能有广义力（与速度有关～）反之，则简单（守恒量）

例2：平面双摆，double pendulum：

（有名的 chaos pendulum）



$$L = T - U$$

$$U = -m_1 g l_1 \cos\varphi_1 - m_2 g (l_1\cos\varphi_1 + l_2\cos\varphi_2)$$

$$T = \frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2$$

v的求解：

① 直接计算

$$\vec{r}_1 = (l_1\cos\varphi_1,\ l_1\sin\varphi_1)$$

$$\vec{r}_2 = (l_1\cos\varphi_1 + l_2\cos\varphi_2,\ l_1\sin\varphi_1 + l_2\sin\varphi_2)$$

$$v_1 = \dot{\vec{r}}_1 = l_1(-\sin\varphi_1,\ \cos\varphi_1)\dot{\varphi}_1$$

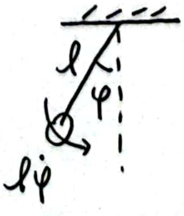
$$|\dot{\vec{r}}_1|^2 = l_1^2\dot{\varphi}_1^2$$

$$v_2 = \dot{\vec{r}}_2 = (-\sin\varphi_1 l_1\dot{\varphi}_1 - \sin\varphi_2 l_2\dot{\varphi}_2,\ \cos\varphi_1 l_1\dot{\varphi}_1 + \cos\varphi_2 l_2\dot{\varphi}_2)$$

$$|\dot{\vec{r}}_2|^2 = (\sin\varphi_1 l_1\dot{\varphi}_1 - \sin\varphi_2 l_2\dot{\varphi}_2)^2 + (\cos\varphi_1 l_1\dot{\varphi}_1 + \cos\varphi_2 l_2\dot{\varphi}_2)^2$$

$$L = T - U = \sim$$

② 物理图像. $(l\dot{\varphi}_1)^2$



☆ 技巧/图像，方法；

☆ 用 MMA 化简/运算：

1. D算子

D[f, x], D[f, {x, 2}]

2. FullSimplify[f]

$$L = \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2$$

$$+ \frac{1}{2} m_2 \left(l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2\cos(\varphi_1 - \varphi_2) l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \right)$$

$$- \left(m_1 g l_1 \cos\varphi_1 + m_2 g l_2 \cos\varphi_2 \right)$$

广义动量： $P_{\varphi_1} = \dfrac{\partial L}{\partial \dot{\varphi}_1} = m_1 l_1^2 \dot{\varphi}_1 + m_2 \cos(\varphi_1 - \varphi_2) l_1 l_2 \dot{\varphi}_2$

广义力　： $F_{\varphi_1} = \dfrac{\partial L}{\partial \varphi_1} = -m_2 \sin(\varphi_1 - \varphi_2) l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2$

$$+ m_1 g l_1 \sin\varphi_1$$

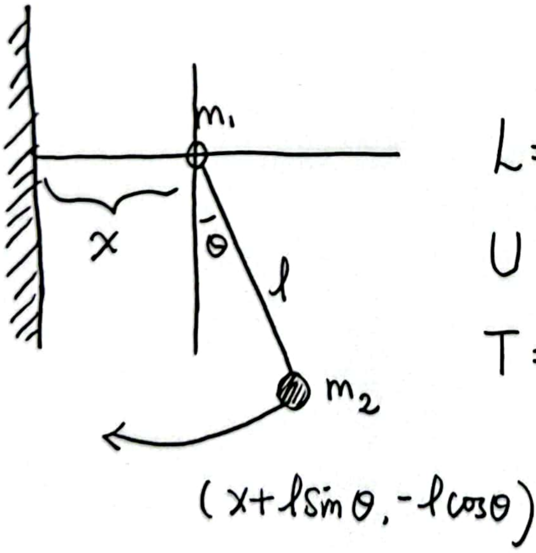
近似： $\varphi_1 \sim 0$

$\varphi_2 \sim 0$

$$L \cong \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \right)$$

$$- m_1 g l_1 \left(1 - \frac{1}{2}\varphi_1^2\right) - m_2 g l_2 \left(1 - \frac{1}{2}\varphi_2^2\right) + \underbrace{O(\varphi_i \varphi_j)}_{\text{小量}}$$

例3：



$L = T - U$

$U = -m_2 g l \cos\theta$

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left[(\dot{x} + l\cos\theta\,\dot{\theta})^2 + l^2 \sin^2\theta\, \dot{\theta}^2 \right]$$

作业.(0902)

1. 求 double pendulum 运动规律;

(用MMA / NDSolve 处理)