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工学硕士学位论文

落猫现象的动力学研究

硕士研究生：郝名望

指 导 教 师：梁立孚 教授

学科、专业：飞行器设计

哈尔滨工程大学

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A Dissertation for the Degree of M. Eng

Dynamical Studying of Falling Cat Phenomenon

Candidate:	Hao Mingwang
Supervisor:	Prof. Liang Lifu
Academic Degree Applied for:	Master of Engineering
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哈尔滨工程大学
学位论文原创性声明

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摘　要

多体系统力学是一般力学的一个新分支。它分为多刚体系统动力学与多柔体系统动力学，有着很强的工程背景与实用价值，在航天器动力学与机器人动力学中有着广泛地应用。

本文首先讲述了多体系统力学在自然科学中所处的位置，介绍了多体系统力学的发展概况，然后介绍了相关的高等刚体力学与分析力学的知识。接着把“落猫”问题化为多刚体模型，运用拉格朗日方法、牛顿—欧拉法、凯恩方程法解题，并比较它们之间的优缺点。本文还建立了落猫的多柔体模型，并用凯恩方程法解多柔体模型。其中拉格朗日方法属于分析力学的方法，牛顿欧拉法属于矢量力学，凯恩方程法既具有分析力学的特点，又具有矢量力学的特征。

本文在解题过程中较多的使用矩阵运算，这不但使行文方便而且便于计算机编程计算。同时将本文的矩阵方法与凯恩的矢量方法进行对比，最后，凯恩文献中没有给出非线性微分方程的解析解，而本文给出了这个非线性微分方程的半解析解。

关键词：多刚体；多柔体；落猫；非线性微分方程

Abstract

Multibody kinetics is one branch of general mechanics.It contains rigid bodies dynamics and flexible multibody system dynamics.It has great of practice values.It is greatly applied in Robotics and Spacecraft dynamics.

At first ,in this paper, we introduce the position which mutibody kinetics in whole science and the development of mutibody kinetics .Second ,the relational knowledge of advanced rigid bodies dynamics and analytical mecanics is introduced. Third, We simplify the 'Falling cat' into mold of mutibody and use Lagrangian Method ,Newton—Euer's Method and Kane's Method are used to make solution and compare their advantages and disadvantages .Forth, we also make the flexible mutibody mold of 'Falling Cat' and use Kane's Method to make solution.

In this paper, matrix method is greatly used ,it is not only make the looking conveniently ,but also make the calculation conveniently . The matrix method is compared to the Kane's vector method. Since Kane didn't give analytical solution of nonlinear differential equation, we will give a series solution of a nonlinear differential equation.

Key words: rigid bodies; flexible mutibody; falling cat; nonlinear differential equation.

目　　录

第1章 绪论

1.1 概述

物理学是自然科学中的一类基础学科，它研究的范围极其广泛，包括力、热、光、电、声、电磁以及近代物理等。其中力学又是物理学中最古老的分支，近代自然科学就是以力学的发展为开端。意大利著名物理学家伽利略，强调力学应与数学紧密相联系。随他之后的众多科学家们都是按照他的做法去做的，最后造成的结果是人们总是把数学与力学联系在一起，而且有了一个新名词——数力。力学发展的时间比较长，因而也比较完善，故有很多方法都推广到物理学其他分支中去了。现在，一般人们把力学分成：一般力学、固体力学、流体力学等。其中一般力学中包含古老的经典力学，它的研究对象多是质点、刚体等，给我们的感受是它比固体力学、流体力学更具体，更实用。只研究整个物体的宏观运动，而不去研究物质内部的力学现象（如应力、应变等）。然而一般力学的基本定理以及解题方法正是我们用以研究连续介质力学的手段、工具。一般力学包括经典的质点力学、刚体动力学、分析力学、多体系统力学、变质量力学、运动稳定性以及非线性振动等。其中多体系统力学是60年代在经典力学基础上发展起来的力学新分支，多体系统动力学产生的工程背景是宇宙航行技术和机器人的飞速发展，以及数字式电子计算机的出现。多体系统力学又跟建模时是否考虑物体的柔性又可分为多刚体系统力学与多柔体系统力学。多体系统可分为两类：树形系统与非树形系

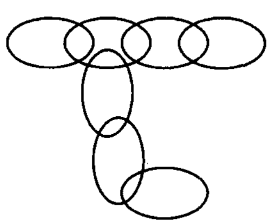

图1.1 树形系统

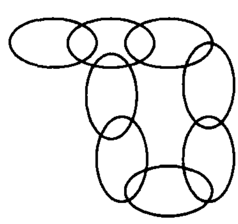

图1.2 非树形系统

统，树形系统即开环系统，非树形系统即闭环系统，这两个概念通过图 1.1 和图 1.2 一目了然，不必细说。

在实际问题中还会遇到两种运动条件不同的多体系统。若系统有一个或者一个以上的刚体用铰链接在系统以外的物体上，而此物体在惯性空间中的位置是预先给定的时间函数。这种系统为非自由刚体系统（有根树）。另一种情况是系统中没有一个物体与预先给定的外界物体相连接，称这种系统为自由多体系统（无根树）。

多刚体系统是指任意有限个刚体的组合，而各刚体之间又以某种形式的接头连接起来的系统。

关于多柔体系统，目前尚无确切的、众所接受的定义。人们通常将多柔体系统理解为由多个柔性体(柔性部件)通过铰链(又称关节)连接而成的一个系统；相邻两个柔性部件之间可以有较大的相对刚体位移发生。因此，多柔体系统和由多个柔体组成的结构是不同的。传统意义下的结构，应该具有几何不可变性，就是说相邻两部件间除弹性变形外，不允许有相对的刚体位移发生。

目前公认，多柔体系统动力学理论的发展是以三个重要的工程领域作为背景的。这就是航天器、机器人(机械臂)和高速精密机构，尤以前两者的推动作用最大。

航天器工程领域是推动多柔体系统动力学发展的一个重要领域。航天器上都需要带有大型天线和太阳帆板。这些我们都通称为柔性附件。由于航天器在空间轨道上是在接近于零的过载下(微重力、无机动情况)工作的，因此，这些附件都设计得尽可能的轻柔。又由于航天器在发射入轨过程中，要承受很大过载，因此，通常在发射时，这些附件是以紧凑形式折叠安装于航天器上，在入轨后，再展开到工作状态。

机器人(机械臂)必然被模化为多体系统是显而易见的。目前的地面应用的机器人，它的臂杆还多是刚性的。但是用于美国航天飞机上的空间遥控机械臂已经具有相当的柔性。而机器人(机械臂)的轻型化、高速化要求的需要，

特别是用于空间环境的空间机械人的发展需要，使得机器人工程领域成为推动多柔体系统动力学发展的一个重要的工程领域。

1.2 多体系统力学发展简介

多体系统动力学的发展概况从动力学角度讲，刚体是柔性体的特殊情况。因此，在讨论多体系统动力学发展时，必然会首先涉及到多刚体系统动力学。而且正像人们所能想象的那样，首先被发展起来的正是多刚体系统动力学。最早的工作是 H. J. Fletcher 等人在 1963 年做的。他们讨论的是由两个刚体组成的系统，使用的是向量力学中的 Newton-Euler 法[1]。稍晚，在 1965 年 W. W. Hooker 和 G. Margulies 讨论了由 n 个刚体组成的多刚体系统[2][3]。1968 年 R. E. Roberson 和 J. W. Wittenburg 又把图论中的概念和数学工具(关联矩阵和通路矩阵等)创造性地用于对多刚体系统的描述和动力学方程的建立中[4]。他们的研究工作把多刚体系统动力学的研究推进到一个新阶段，并有了关于多刚体系统动力学的一本专著[5]。到目前，多刚体系统动力学发展已较成熟，国内外都有专著[4]。

对多柔体系统动力学的研究开展得略晚一点。这方面早期工作有 P. W. Likins 在 70 年代初的研究工作[5][6]。他首先研究的是带有弹性附件的卫星的动力学问题。P. W. Likins 采用了由 L Meirovitch 和 H. D. Nelson 最先提出的混合坐标[7]，这一概念后来被广泛应用于多柔体系统动力学中。到此，可以认为多柔体系统动力学的发展已初具规模，它作为力学的一个重要分支的地位已稳固树立了。事实上，1977 年国际理论力学与应用力学联合会(International Union Of Theoretical and Applied Mechanics，IU-TAM)曾组织了多体系统动力学的专题学术会议[16]。

在多体系统动力学发展的初期，差不多向量力学[42]和分析力学的一些方法都被用来建立动力学方程。最早被采用的是向量力学中的 Newton-Euler 法[1]。它以物理概念鲜明、建立方程直接而著称。在分析过程中，若需要增加体的数目，只需续增加方程数目，无需重新另行建立动力学方程组。有些文

献称之为具有良好的开放性。这是 Newton-Euler 法的又一个优点。但消除约束力的困难则是它的一个大的弱点。以后人们还发现，在采用递推型式时，递推的 Newton-Euler 法运算量最小。因此，Newton-Euler 法一直受到一些作者的注意。更多文献在建立多柔体系统动力学方程时，采用的大多是分析力学的各种方法。P. W. Likins 曾用虚功原理[5]，这种方法繁而不难，思路清晰。C. S. Bodley 等人用 Lagrange 乘子法。采用这种方法处理有约束条件的多柔体系统时特别方便，但是它的求导数过程较繁琐，必须十分小心。R. K. Cavin 和 A. R. Dusto 用 Hamilton 原理建立动力学方程。此外，还有 T. R. Kane 提出的 Kane 方程，它兼有向量力学的形式简洁和分析力学的不必考虑理想约束的约束反力的作用，以及推导规范等优点，被广泛应用于多柔体系统动力学方程的建立[10-12]还有一些力学工作者深入讨论了 Kane 方程的概念[13]。以后，随着多柔体系统动力学在航天器、机器人和高速精密机构等工程领域内的深入应用，研究工作又向纵深发展。由于机器人的爪端有时要在特定的表面上以特定的条件运动；又有些复杂机构需模化成有回路的多柔体系统，又由于多个机器人的协调操作，因此，约束动力学问题引起人们的注意。J. T. Wang 和 R. L. Huston 把待定乘子法推广到 Kane 方程中[14]，建立了带有待定乘子的 Kane 动力学方程，它和 Lagrange 乘子法异曲同工。J. W. Kamman 和 R. L Huston，许宏伟和马兴瑞[15]等先后用正交补矩阵消除待定乘子(Lagrange 乘子)，以降低方程维数。而 Singh 和 Likins 则用奇异值分解技术消除待定乘子。宋培林和马兴瑞还用切空间和法空间正交理论处理约束系统[16]。此外，胡海昌院士与梁立孚，等利用变分法推出一般力学非保守、非完整系统的广义变分原理，这在理论上极大的促进了多体系统力学的发展[28-41]。梅凤翔教授在非完整系统力学的研究中也做出了很大的贡献[44]。其它一些学者的工作参见文献[43,45]。

随着实时模拟多柔体系统动力学过程的需要，人们又致力于研究提高运算速度的方法，于是递推型动力学方程就被提出来了[17,18]。D.S.Bae 和 E.J.Haug 用并行处理技术实施了递推动力学的仿真过程[17]。

人们还对求解动力学方程中的一些实用问题给予了很大的注意。一个重要的、一直吸引了很多作者研究的问题就是动力学模型降阶问题。柔性体的弹性位移的引入使动力学模型的阶数大为增加。即使通过弹性位移的模态展开方法引入模态坐标后，动力学模型的阶数仍然过高。这样高阶的设计模型，其阶数已超出了多变量控制设计技术所能提供的设计能力。因此模型降阶问题(又称为模态截断问题)曾是很多作者研究的热点。

在求解多柔体系统动力学方程时，另一个使实际工作者们深感烦恼的问题，就是解的稳定性的问题。出现解的不稳定，可能是物理不稳定，也可能是数值不稳定。一些作者由方程的非线性特点出发，研究了它的性态，企图弄清楚在哪些情况下可能出现物理不稳定的现象。马兴瑞等人曾以两臂杆的柔性机械臂为例，说明在这里存在有出现参数共振的可能[26]。还有文献指出当机械臂的基座作某种往复运动时，可以出现动力不稳定现象。张嘉钟等的实验研究与分析也说明对非线性系统可能由于“内部共振”而引起不稳定现象。另一些作者则着重于克服数值不稳定问题的研究。在这方面辛(Symplectic)算法受到很大重视[19]，因为这种算法在应用于Hamilton系统时有很好的数值稳定性。辛算法已被应用于多体系统动力学的求解中。

在将多柔体系统动力学应用于机器人(或机械臂)时，对振动的主动抑制也是一个重要的问题。逆动力学方法是近几年提出来的一个振动主动抑制方法。逆动力学方法是用合理设计机器人(臂)的关节驱动力矩的方法，在机器人(臂)达到操作要求的同时，使弹性振动被抑制到最小。显然这是一个很吸引人的方法。在国内黄文虎院士等在这方面作了系统的工作。

另外，近些年来，有些学者在函数空间内，用分布参数理论，对多柔体动力学系统稳定性、镇定与控制进行了研究，从而可以克服由于对无限维柔性系统的“离散化”和“截断”所产生的溢出(spillover)现象以及系统特性的“失真”问题。

国内关于多柔体系统的研究工作大约起步于80年代初。这些研究工作也是先在航天器及机器人领域内进行的[9]。还有一些学者则由一般力学角度进行

了讨论。自80年代后期，国内也连续召开了一些学术讨论会，推动这一学科的发展。在应用方面中国空间技术研究院的总体设计部联合国内一些大学作了很深入的工作[20]，结合航天器动力学仿真，编制了若干关于多柔体系统动力学仿真的软件包，由航天器工程提出的需求推动了国内有关研究工作。

1.3 本文工作简述

这里我们回顾历史上一个著名的问题："落猫现象"。何谓"落猫现象"呢？我们把一只猫举起来在空中释放（任意状态，甚至完全睡着的四脚朝上的猫），猫总能够通过自身的操作，最后安全地四脚着地。这现象曾引起了科学家们的兴趣，也引发了一系列有趣的问题，得到了许多启示。在本文我们主要是从动力学角度去研究它。首先猫的躯体可以看成多体系统，否则，看成单个刚体或柔体，是无论如何也翻不过身的。怎样建立其动力学模型呢？历史上有好多学者专家都提出自己的看法与见解。本文通过三种方法分别对多刚体猫进行建模，并比较其方法的特点，并运用凯恩方程对多柔体猫建模，最后给出一个非线性方程的半解析解，此方程为凯恩在《A DYNAMICAL EXPLANATION OF THE FALLING CAT PHENOMENON》中的结论方程。

第 2 章　基础知识及要点介绍

2.1 方向余弦矩阵

在参考系中固定矢量基$\{\vec{i}\}$它对应于$oxyz$坐标系，同时取固连于刚体上的矢量基$\{\vec{e}\}$，对应于固连在刚体上的转动坐标系$o\xi\eta\zeta$，令$l_{jk}=\vec{e}_j\cdot\vec{i}_k\ (j,k=1,2,3)$，于是有[21][23]

$$\begin{aligned}\vec{e}_1&=l_{11}\vec{i}_1+l_{12}\vec{i}_2+l_{13}\vec{i}_3\\ \vec{e}_2&=l_{21}\vec{i}_1+l_{22}\vec{i}_2+l_{23}\vec{i}_3\\ \vec{e}_3&=l_{31}\vec{i}_1+l_{32}\vec{i}_2+l_{33}\vec{i}_3\end{aligned}\tag{2-1}$$

将上式写为矩阵形式为

$$\{\vec{e}\}=[L]\{\vec{i}\}\tag{2-2}$$

式中

$$[L]=\begin{bmatrix}l_{11}&l_{12}&l_{13}\\ l_{21}&l_{22}&l_{23}\\ l_{31}&l_{32}&l_{33}\end{bmatrix}$$

为方向余弦矩阵。

方向余弦矩阵有如下性质：

① 正交性　$[L]^T=[L]^{-1}$

② $\det[L]=\pm1$

给出如下两种问题的答案

（1）矢量$\vec{r}$在不同矢量基中的坐标变换

$$\{r'\}=[L]\{r\}\tag{2-3}$$

式中

$$\{r'\}=[r_1',r_2',r_3']^T\quad,\quad\{r\}=[r_1,r_2,r_3]^T$$

r_1,r_2,r_3 为矢量 $\vec{r}$ 在矢量基 $\{\vec{i}\}$ 中的坐标，r_1',r_2',r_3' 为矢量 $\vec{r}$ 在矢量基 $\{\vec{e}\}$ 中的坐标。

（2）固连于转动刚体上的矢量在刚体上不同方位时的关系式

若矢量 $\vec{r}$ 是固连在刚体上，例如矢量 $\vec{r}$ 的起点在定点 o 上，则刚体转动到另一位置，矢量 $\vec{r}$ 位于 r'，$\vec{r}'$ 在矢量基 $\{\vec{i}\}$ 的坐标列阵为 $\{r'\}$ 与在固连矢量基 $\{\vec{e}\}$ 的坐标列阵 $\{r\}$ 的关系式为

$$\{r'\}=[L]^T\{r\} \tag{2-4}$$

应用方向余弦矩阵表示角速度矩阵，定义角速度矩阵为

$$[\tilde{\omega}]=\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

有如下关系[21][23]

$$[\tilde{\omega}]=[L][\dot{L}]^T \tag{2-5}$$

2.2 绕定点转动的动量矩和动能

设一质量连续分布的刚体绕定点 o 转动，根据动量矩定义，刚体对固定点 o 的动量矩为[21][23]

$$\vec{H}_0=\int_M \vec{r}\times\vec{v}dm \tag{2-6}$$

式中 $\vec{r}$，$\vec{v}$ 及 dm 分别为刚体的任一微元体的矢径、速度及质量，经过推演可以得

$$\begin{Bmatrix} H_\xi \\ H_\eta \\ H_\zeta \end{Bmatrix}=\begin{bmatrix} -J_\xi & -J_{\xi\eta} & -J_{\xi\zeta} \\ -J_{\eta\xi} & -J_\eta & -J_{\eta\xi} \\ -J_{\zeta\zeta} & -J_{\zeta\eta} & -J_\zeta \end{bmatrix}\begin{Bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{Bmatrix}$$

或

$$\{H\}=[J]\{\omega\} \tag{2-7}$$

式中$[H_\xi,H_\eta,H_\zeta]^T$为刚体对定点o的动量矩$\vec{H}_0$的坐标列阵，$[\omega_\xi,\omega_\eta,\omega_\zeta]^T$为角速度$\vec{\omega}$的坐标列阵，$[J]$为惯量矩阵，其元素为

$$J_\xi=\int_M(\eta^2+\zeta^2)dm \quad , \qquad J_{\xi\eta}=\int_M\xi\eta dm, \qquad J_{\xi\zeta}=\int_M\xi\zeta dm \tag{2-8}$$

余仿此，略写。

现在推导刚体定点运动的动能表达式。根据动能定义，刚体动能为

$$T=\int_M\frac{1}{2}(\vec{v}\bullet\vec{v})dm=\frac{1}{2}\int_M(\vec{\omega}\times\vec{r})\bullet\vec{v}dm \tag{2-9}$$

根据混合积的性质上式可写成

$$T=\int_M\frac{1}{2}\vec{\omega}\bullet(\vec{r}\times\vec{v})dm=\frac{1}{2}\vec{\omega}\bullet\int_M\vec{r}\times\vec{v}dm$$

$$T=\vec{\omega}\bullet\vec{H}_0$$

写成矩阵形式有：

$$T=[\omega]\{H\}=[\omega][J]\{\omega\} \tag{2-10}$$

展开写成

$$T=\frac{1}{2}(J_\xi\omega_\xi^2+J_\eta\omega_\eta^2+J_\zeta\omega_\zeta^2-2J_{\xi\eta}\omega_\xi\omega_\eta-2J_{\eta\xi}\omega_\xi\omega_\eta-2J_{\zeta\xi}\omega_\zeta\omega_\xi) \tag{2-11}$$

2.3 惯量矩阵的移轴公式

惯量矩阵存在着类似于转动惯量平行轴定理的移轴公式。对于固连于刚体上的$o\xi'\eta'\zeta'$坐标系的惯量矩阵可以通过与之平行的，原点在质量中心C的$c\xi\eta\zeta$坐标系的惯量矩阵来表达[21][23]。

$$\vec{H}_0=\vec{H}_c+\vec{r}\times M\vec{v}_c \tag{2-12}$$

由于$\vec{v}_c=\vec{\omega}\times\vec{r}$ ，于是

$$\vec{H}_0=\vec{H}_c+\vec{r}\times M(\vec{\omega}\times\vec{r})$$

或

$$\vec{H}_0 = \vec{H}_c - M\vec{r} \times (\vec{r} \times \vec{\omega}) \tag{2-13}$$

写成矩阵形式：

$$[J_0] = [J_c] - M[\tilde{r}][\tilde{r}]$$

若矢径 $\vec{r}$ 的坐标为 (a,b,c)，展开上式，得

$$[J_0] = [J_c] + M\begin{bmatrix} c^2+b^2 & -ab & -ac \\ -ab & a^2+c^2 & -cb \\ -ac & -cb & a^2+b^2 \end{bmatrix} \tag{2-14}$$

2.4 欧拉动力学方程

设刚体有固定点 o，受外力矩 $\vec{M}_0$ 作用，取坐标系 $o\xi\eta\zeta$，沿 o 点得三个主轴，即取固连矢量基 $\{e\}$ 沿刚体 o 点的主轴，于是刚体对 o 点的动量矩有简单关系式[21][23]

$$\begin{Bmatrix} H_\xi \\ H_\eta \\ H_\zeta \end{Bmatrix} = \begin{bmatrix} J_\xi & & \\ & J_\eta & \\ & & J_\zeta \end{bmatrix} \begin{Bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{Bmatrix}$$

由绝对导数与矢量基 $\{e\}$ 中的相对导数关系，得

$$\frac{d\vec{H}_0}{dt} = \frac{d\widetilde{H_0}}{dt} + \vec{\omega} \times \vec{H}_0$$

$$\frac{d\widetilde{H_0}}{dt} + \vec{\omega} \times \vec{H}_0 = \vec{M}_0$$

即

$$[J]\{\dot{\omega}\} + \{\tilde{\omega}\}[J]\{\omega\} = \{M\} \tag{2-15}$$

由于角速度的相对导数等于绝对导数，故而上式中 $\{\dot{\omega}\}$ 是角速度绝对导数的坐标列阵。式（2-15）展开为

$$J_\xi \dot{\omega}_\eta + (J_\zeta - J_\eta)\omega_\eta\omega_\zeta = M_\xi$$

$$J_\eta \dot{\omega}_\eta + (J_\xi - J_\zeta)\omega_\xi\omega_\eta = M_\eta \qquad (2\text{-}16)$$

$$J_\zeta \omega_\zeta + (J_\eta - J_\xi)\omega_\eta\omega_\zeta = M_\zeta$$

2.5 刚体运动的控制方程

设刚体的质量为M，质心为c，取固定坐标系$oxyz$，通过质心c取一固连坐标系$c\xi\eta\zeta$，ξ、η和ζ轴为惯性主轴，刚体运动的控制方程为：

质心运动学方程

$$\begin{cases} \dot{x}_c = v_{cx} \\ \dot{y}_c = v_{cy} \\ \dot{z}_c = v_{cz} \end{cases} \qquad (2\text{-}17)$$

质心动力学方程

$$\begin{cases} M\ddot{x}_c = \sum F_x \\ M\ddot{y}_c = \sum F_y \\ M\ddot{z}_c = \sum F_z \end{cases} \qquad (2\text{-}18)$$

欧拉运动学方程

$$\begin{cases} \omega_x = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\ \omega_y = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi \\ \omega_z = \dot{\psi}\cos\theta + \dot{\varphi} \end{cases} \qquad (2\text{-}19)$$

欧拉动力学方程

$$J_\xi \dot{\omega}_\xi + (J_\zeta - J_\eta)\omega_\eta\omega_\zeta = M_\xi$$

$$J_\eta \dot{\omega}_\eta + (J_\xi - J_\zeta)\dot{\omega}_\xi\omega_\eta = M_\eta \qquad (2\text{-}20)$$

$$J_\zeta \dot{\omega}_\zeta + (J_\eta - J_\xi)\omega_\eta\omega_\zeta = M_\zeta$$

以及边值条件，初始条件，约束条件。

2.6 分析力学的几个方程

以上我们所列的基本上是牛顿—欧拉体系的力学内容，它的数学形式都是以微分方程为主，几何性很强，属于矢量力学范畴。而分析力学则以数学分析的手段去研究质点系方程，把它化为标量形式，这种方程突出的优点是不考虑内部约束力，以广义坐标为其描述坐标，以变分为其数学基础。当然它也有其缺点，当计算复杂系统时，由于要求导运算，显然比较麻烦。最近出现了凯恩方程弥补了它的不足，只要四则运算，不必求导，很适宜在计算机上进行。

（1）动力学普遍方程[22]

$$\sum_{i=1}^{N}(\vec{F}_i - m_i\ddot{\vec{r}}_i)\bullet\delta\vec{r}_i = 0 \qquad (2-21)$$

该方程表示：具有理想约束的质点系，在任一瞬时作用于系统上所有的主动力和假象的惯性力，在任一位移上的虚元功之和为零。

（2）以动力学普遍方程为基础，将方程式中各点的坐标及其虚位移做广义坐标变换找出广义坐标的动力学普遍方程式，把方程式写成物理概念清楚而便于运算的形式。第二类拉格朗日方程就属于这样一种情况[22]。

$$Q_j - \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}\right) = 0 \quad (j = 1,2,3,\cdots,n) \qquad (2-22)$$

式中：T——广义坐标表示的动能

Q_j——广义力

（3）哈密尔顿正则方程[22]

$$\dot{q}_j = \frac{\partial H}{\partial p_j}$$
$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \quad (j = 1,2,3,\cdots,n) \qquad (2-23)$$
$$\frac{\partial L}{\partial t_j} = -\frac{\partial H}{\partial t}$$

式中：

$p_j=\frac{\partial L}{\partial \dot{q}_j}=\frac{\partial T}{\partial \dot{q}_j}$——广义动量

$H=\sum_{j=1}^{n} p_j\dot{q}_j-L$——哈密尔顿函数

L——拉格朗日函数或动势，$L=T-V$，V 为势函数

（4）哈密尔顿原理

设有一完整的保守的力学系统，在具有相同的时间间隔和始末位置的一切可能运动中，对于真实运动来说，哈密顿作用量[22]，即

$$I=\int_{1}^{0} L(q_1\cdots q_n,\dot{q}_1\cdots \dot{q}_n,t)dt$$

应具有极值，即必须满足

$$\delta I=\delta\int_{1}^{0} L(q_1\cdots q_n,\dot{q}_1\cdots \dot{q}_n,t)dt=0 \tag{2-24}$$

（5）凯恩方程表达式为[21]

$$F_R'+F_R^*=0 \quad (r=1,2,\cdots n-l) \tag{2-25}$$

式中 n，l，R 分别为广义坐标数，非完整约束数，独立广义速度数目。F_R' 为广义主动力，F_R^* 为广义惯性力。

$$F_R'=\vec{R}_0\bullet\vec{v}_{oR}+\vec{L}_0\bullet\vec{\omega}_{vR} \tag{2-26}$$

第一项为主矢与简化中心 o 对应于广义速率 u_R 的偏速度的标积，第二项为主矩与对应与广义速率 u_R 的偏角速度的标积。

$$F_R^*=\overrightarrow{R_c^*}\bullet\vec{v}_{cR}+\overrightarrow{L_c^*}\bullet\vec{\omega}_{vR} \tag{2-27}$$

$\overrightarrow{L_c^*}$ 为诸质点相对与原点在质心的平动坐标系的惯性力与质心 c 的主矩，即惯性主矩，$\overrightarrow{R_c^*}$ 为系统惯性力主矩，$\vec{v}_{cR}$ 为偏速度，$\vec{\omega}_{vR}$ 为偏角速度。

2.7 本章小结

本章介绍了高等刚体运动学、动力学有关的知识和分析力学的几个公式。具体内容包括：

（1）方向余弦矩阵的定义及性质、坐标变换及矢量旋转公式角速度的表达公式、绕定点转动的动量矩和动能表达式、惯量矩阵及移轴公式、欧拉动力学方程和刚体运动的控制方程。

（2）动力学普遍方程、第二类拉格朗日方程、哈密尔顿正则方程、哈密尔顿原理、凯恩方程。

第 3 章　落猫问题的多刚体建模

3.1 第二类拉格朗日方程法建模

我们把猫的躯体简化成两个圆柱体相连接的模型，假设每个刚体均是均质的。刚体 1 代表猫的前躯，刚体 2 代表猫的后躯，两刚体的中间接头有万向铰与球铰两种取法. 本文考虑万向接头铰的情况。

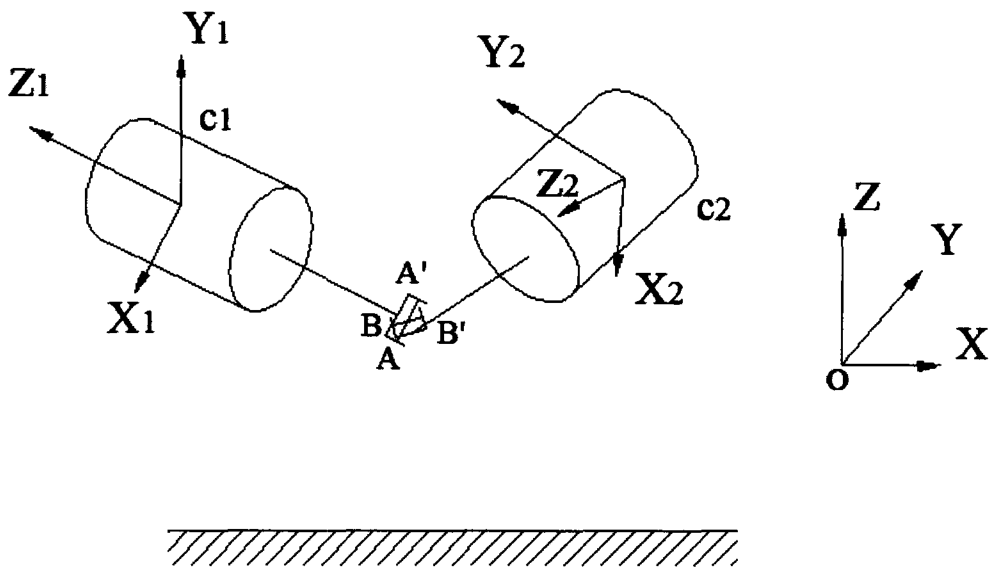

图 3.1 猫体的多刚体简化模型及相关的坐标系

如图 3.1，物体 1 可以绕着 AA' 轴旋转，物体 2 可以绕着 BB' 轴旋转，建立体坐标系 $C_1X_1Y_1Z_1$，$C_2X_2Y_2Z_2$，C_1,C_2 分别为物体 1，2 的质心，且满足 C_1Z_1 平行与物体 1 的纵向对称轴，C_1Y_1 平行与 AA' 轴，C_1X_1 与它们组成右手坐标系。$C_2X_2Y_2Z_2$ 由如下旋转而得：假令猫体伸直时，$C_1X_1Y_1Z_1$，$C_2X_2Y_2Z_2$ 相互平行，接着 2 物体先绕 AA' 轴旋转 α，再绕 BB' 轴旋转 β，最后得到 $C_2X_2Y_2Z_2$，1 物体的质心在 $OXYZ$ 坐标系中的坐标为 x_{c1}，y_{c1}，z_{c1}，物体 1 相对于地面坐标系中的姿态用三个欧拉角表示，进动角 ψ，章动角 θ，自转角 φ。取广义坐标

为 $x_1, y_1, z_1, \alpha, \beta, \psi, \theta, \varphi$。万向接头的十字中点 C_3 到 C_1，C_2 的距离分别为 s_1，s_2。$C_1X_1Y_1Z_1$ 相对于 $OXYZ$ 的方向余弦矩阵为

$$[L]_{1,0} = \begin{bmatrix} \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi & \sin\psi\cos\varphi + \cos\psi\cos\theta\sin\varphi & \sin\theta\sin\varphi \\ -\cos\psi\sin\varphi - \sin\psi\cos\theta\cos\varphi & -\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi & \sin\theta\cos\varphi \\ \sin\psi\sin\theta & -\cos\psi\sin\theta & \cos\theta \end{bmatrix} \tag{3-1}$$

式中 $[L]_{1,0}$ 中的下标 1，0 表示坐标系 $C_1X_1Y_1Z_1$ 、$OXYZ$ ，下文符号意义同此。

$$[L]_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ \sin\alpha\sin\beta & \cos\beta & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta & -\sin\beta & \cos\alpha\cos\beta \end{bmatrix} \tag{3-2}$$

$$[L]_{2,0} = [L]_{1,0}[L]_{2,1}$$

$$= \begin{bmatrix} \cos\beta(\cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi) - \sin\alpha\sin\beta(\sin\psi\cos\varphi + \cos\psi\cos\theta\sin\varphi) - \sin\beta\cos\alpha\sin\theta\sin\varphi & \cos\alpha(\sin\psi\cos\varphi + \cos\psi\cos\theta\sin\varphi) + \sin\alpha\sin\theta\sin\varphi & -\sin\beta(\cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi) + \sin\alpha\cos\beta(\sin\psi\cos\varphi + \cos\psi\cos\theta\sin\varphi) + \cos\alpha\cos\beta\sin\theta\sin\varphi \\ \cos\beta(-\cos\psi\sin\varphi - \sin\psi\cos\theta\cos\varphi) - \sin\alpha\sin\beta(-\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi) - \sin\beta\cos\alpha\sin\theta\cos\varphi & \cos\alpha(-\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi) - \sin\alpha\sin\theta\cos\varphi & -\sin\beta(-\cos\psi\sin\varphi - \sin\psi\cos\theta\cos\varphi) + \sin\alpha\cos\beta(-\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi) + \sin\theta\cos\varphi\cos\alpha\cos\beta \\ \cos\beta\sin\theta\sin\varphi + \sin\alpha\sin\beta\cos\psi\sin\theta - \sin\beta\cos\alpha\cos\theta & -\cos\alpha\cos\varphi\sin\theta - \sin\alpha\cos\theta & -\sin\beta\sin\psi\sin\theta - \sin\alpha\cos\beta\cos\psi\sin\theta + \cos\alpha\cos\beta\cos\theta \end{bmatrix} \tag{3-3}$$

2. 求物体 2 的质心相对与地面的坐标

$$\vec{oc_2} = \vec{oc_1} + \vec{c_1c_3} + \vec{c_3c_2}$$

$$= x_1\vec{i} + y_1\vec{j} + z_1\vec{k} - s_1\vec{k_1} - s_2\vec{k_2}$$

$$= x_1\vec{i} + y_1\vec{j} + z_1\vec{k} - s_1 \times \{[L]_{1,0}\text{的第三行}\}^T\{\vec{e}_0\} - s_2 \times \{[L]_{2,0}\text{的第三行}\}^T\{\vec{e}_0\}$$

$$= x_1\vec{i} + y_1\vec{j} + z_1\vec{k} - s_1(\sin\psi\sin\theta\vec{i} - \cos\psi\sin\theta\vec{j} + \cos\theta\vec{k}) -$$
$$s_2\{(\cos\beta\sin\theta\sin\varphi + \sin\alpha\sin\beta\cos\psi\sin\theta - \sin\beta\cos\alpha\cos\theta)\vec{i} -$$
$$(\cos\alpha\cos\psi\sin\theta - \sin\alpha\cos\theta)\vec{j} + (-\sin\beta\sin\psi\sin\theta - \sin\alpha\cos\beta\cos\psi\sin\theta + \cos\alpha\cos\beta\cos\theta)\vec{k}\}$$

（3-4）

求物体 1 的角速度可以按公式

$$[\tilde{\omega}] = [L]_{1,0}[\dot{L}]_{1,0}^T$$

计算，但是比较繁，且容易出错，最好按经典的欧拉连续转动的方法求得

$$\begin{bmatrix}\omega_{1x}\\ \omega_{1y}\\ \omega_{1z}\end{bmatrix} = \begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}\begin{Bmatrix}0\\ 0\\ \dot{\psi}\end{Bmatrix} + \begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\dot{\theta}\\ 0\\ 0\end{Bmatrix} + \begin{Bmatrix}0\\ 0\\ \dot{\varphi}\end{Bmatrix}$$
$$= \begin{bmatrix}\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi\\ \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi\\ \dot{\psi}\cos\theta + \dot{\varphi}\end{bmatrix}$$

（3-5）

3．物体 2 的角速度

$$\begin{Bmatrix}\omega_{2x}\\ \omega_{2y}\\ \omega_{2z}\end{Bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\beta & \sin\beta\\ 0 & -\sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ \sin\alpha & 0 & \cos\alpha\end{bmatrix}\begin{Bmatrix}\omega_{1x}\\ \omega_{1y}\\ \omega_{1z}\end{Bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\beta & \sin\beta\\ 0 & -\sin\beta & \cos\beta\end{bmatrix}\begin{Bmatrix}0\\ \dot{\alpha}\\ 0\end{Bmatrix} + \begin{Bmatrix}\dot{\beta}\\ 0\\ 0\end{Bmatrix}$$
$$= \begin{Bmatrix}\cos\alpha\omega_{1x} - \sin\alpha\omega_{1z}\\ \sin\alpha\sin\beta\omega_{1x} + \cos\beta\omega_{1y} + \cos\alpha\sin\beta\omega_{1z}\\ \sin\alpha\cos\beta\omega_{1x} - \sin\beta\omega_{1y} + \cos\alpha\cos\beta\omega_{1z}\end{Bmatrix} + \begin{Bmatrix}0\\ \dot{\alpha}\cos\beta\\ -\dot{\alpha}\sin\beta\end{Bmatrix} + \begin{Bmatrix}\dot{\beta}\\ 0\\ 0\end{Bmatrix}$$

$$
=\left\{\begin{array}{l}\cos\alpha(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\alpha(\dot{\psi}\cos\theta+\dot{\varphi})+\dot{\beta}\\ \sin\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\\ \dot{\alpha}\cos\beta+\cos\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})\\ \sin\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\\ \cos\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})-\dot{\alpha}\sin\beta\end{array}\right\}
\tag{3-6}
$$

物体 1 的动能和势能

$$
\begin{aligned}
T_1&=\frac{1}{2}[\omega_1][J_1]\{\omega_2\}+\frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2+\dot{z}_1)^2\\
&=\frac{1}{2}(J_{1x}\omega_{1x}^2+J_{1y}\omega_{1y}^2+J_{1z}\omega_{1z}^2)+\frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2+\dot{z}_1^2)
\end{aligned}
\tag{3-7}
$$

$$
V_1=m_1gz_1 \tag{3-8}
$$

物体 2 的动能和势能

$$
T_2=\frac{1}{2}[\omega_2][J_2]\{\omega_2\}=\frac{1}{2}(J_{2x}\omega_{2x}^2+J_{2y}\omega_{2y}^2+J_{2z}\omega_{2z}^2) \tag{3-9}
$$

$$
V_2=m_2gz_2 \tag{3-10}
$$

4.求广义力

沿 AA' 的控制力矩 M_α 和沿 BB' 方向的控制力矩 M_β 它们与广义坐标 α,β 相对应，故广义力

$$
Q_\alpha=M_\alpha,\qquad Q_\beta=M_\beta; \tag{3-11}
$$

其余的广义力均为零。

拉格朗日函数为

$$
\begin{aligned}
L&=T-V\\
&=T_1+T_2-V_1-V_2
\end{aligned}
\tag{3-12}
$$

拉格朗日方程可以写成

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j})-\frac{\partial L}{\partial q_j}=Q_j \tag{3-13}$$

式（3-13）中

$$\frac{\partial L}{\partial q_j}=J_{1x}\omega_{1x}\frac{\partial \omega_{1x}}{\partial q_j}+J_{1y}\omega_{1y}\frac{\partial \omega_{1y}}{\partial q_j}+J_{1z}\omega_{1z}\frac{\partial \omega_{1z}}{\partial q_j}+J_{2x}\omega_{2x}\frac{\partial \omega_{2x}}{\partial q_j}+J_{2y}\omega_{2y}\frac{\partial \omega_{2y}}{\partial q_j}+ J_{2z}\omega_{2z}\frac{\partial \omega_{2z}}{\partial q_j}+m_1 g\frac{\partial z_1}{\partial q_j}+m_2 g\frac{\partial z_2}{\partial q_j}$$

求$\frac{\partial \omega_{1x}}{\partial q_j}$ $(j=1\sim 8)$:

$$\frac{\partial \omega_{1x}}{\partial x_1}=0,\frac{\partial \omega_{1x}}{\partial y_1}=0,\frac{\partial \omega_{1x}}{\partial z_1}=0,\frac{\partial \omega_{1x}}{\partial \alpha}=0,\frac{\partial \omega_{1x}}{\partial \beta}=0$$
$$\frac{\partial \omega_{1x}}{\partial \psi}=0,\frac{\partial \omega_{1x}}{\partial \theta}=\dot{\psi}\cos\theta\sin\varphi,\frac{\partial \omega_{1x}}{\partial \varphi}=-\dot{\psi}\sin\theta\sin\varphi-\dot{\theta}\cos\varphi \tag{3-14}$$

求$\frac{\partial \omega_{1y}}{\partial q_j}$ $(j=1\sim 8)$:

$$\frac{\partial \omega_{1y}}{\partial x_1}=0,\frac{\partial \omega_{1y}}{\partial y_1}=0,\frac{\partial \omega_{1y}}{\partial z_1}=0,\frac{\partial \omega_{1y}}{\partial \alpha}=0,\frac{\partial \omega_{1y}}{\partial \beta}=0,\frac{\partial \omega_{1y}}{\partial \psi}=0,$$
$$\frac{\partial \omega_{1y}}{\partial \theta}=\dot{\psi}\cos\theta\cos\varphi-\dot{\theta}\sin\varphi$$
$$\frac{\partial \omega_{1y}}{\partial \varphi}=-\dot{\psi}\sin\theta\sin\varphi-\dot{\theta}\cos\varphi \tag{3-15}$$

求$\frac{\partial \omega_{1z}}{\partial q_j}$ $(j=1\sim 8)$:

$$\frac{\partial \omega_{1z}}{\partial x_1}=0,\frac{\partial \omega_{1z}}{\partial y_1}=0,\frac{\partial \omega_{1z}}{\partial z_1}=0,\frac{\partial \omega_{1z}}{\partial \alpha}=0,\frac{\partial \omega_{1z}}{\partial \beta}=0,\frac{\partial \omega_{1z}}{\partial \psi}=0,\frac{\partial \omega_{1z}}{\partial \theta}=-\dot{\psi}\sin\theta,\frac{\partial \omega_{1z}}{\partial \varphi}=0 \tag{3-16}$$

求 $\frac{\partial \omega_{2x}}{\partial q_j} \quad (j=1\sim 8)$：

$$\frac{\partial \omega_{2x}}{\partial x_1}=0,\frac{\partial \omega_{2x}}{\partial y_1}=0,\frac{\partial \omega_{2x}}{\partial z_1}=0,\frac{\partial \omega_{2x}}{\partial \beta}=0,\frac{\partial \omega_{2x}}{\partial \psi}=0$$

$$\frac{\partial \omega_{2x}}{\partial \alpha}=-\sin\alpha(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\cos\alpha(\dot{\psi}\cos\theta+\dot{\varphi})+\dot{\beta}$$

$$\frac{\partial \omega_{2x}}{\partial \theta}=\cos\alpha\dot{\psi}\cos\theta\sin\varphi+\sin\alpha\dot{\psi}\sin\theta$$

$$\frac{\partial \omega_{2x}}{\partial \varphi}=\cos\alpha(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)$$

（3-17）

求 $\frac{\partial \omega_{2y}}{\partial q_j} \quad (j=1\sim 8)$：

$$\frac{\partial \omega_{2y}}{\partial x_1}=0,\frac{\partial \omega_{2y}}{\partial y_1}=0,\frac{\partial \omega_{2y}}{\partial z_1}=0,\frac{\partial \omega_{2y}}{\partial \psi}=0$$

$$\frac{\partial \omega_{2y}}{\partial \alpha}=\cos\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})$$

$$\frac{\partial \omega_{2y}}{\partial \beta}=\sin\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)-\dot{\alpha}\sin\beta+\cos\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})$$

$$\frac{\partial \omega_{2y}}{\partial \theta}=\sin\alpha\sin\beta\dot{\psi}\cos\theta\sin\varphi+\cos\beta\dot{\psi}\cos\theta\cos\varphi-\cos\alpha\sin\beta\dot{\psi}\sin\theta$$

$$\frac{\partial \omega_{2y}}{\partial \varphi}=\sin\alpha\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\cos\beta(-\dot{\psi}\sin\theta\sin\varphi-\dot{\theta}\cos\varphi)$$

（3-18）

求：$\frac{\partial \omega_{2z}}{\partial q_j} \quad (j=1\sim 8)$

$$\frac{\partial \omega_{2z}}{\partial x_1}=0,\frac{\partial \omega_{2z}}{\partial y_1}=0,$$

$$\frac{\partial \omega_{2z}}{\partial z_1}=0,\frac{\partial \omega_{2z}}{\partial \psi}=0$$

$$\frac{\partial\omega_{2z}}{\partial\alpha}=\cos\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})$$

$$\frac{\partial\omega_{2z}}{\partial\beta}=-\sin\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)-$$
$$\cos\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})-\dot{\alpha}\cos\beta$$

$$\frac{\partial\omega_{2z}}{\partial\theta}=\sin\alpha\cos\beta\dot{\psi}\cos\theta-\sin\beta\dot{\psi}\cos\theta\cos\varphi-\cos\alpha\cos\beta\dot{\psi}\sin\theta$$

$$\frac{\partial\omega_{2z}}{\partial\varphi}=\sin\alpha\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)-\sin\beta(-\dot{\psi}\sin\theta\sin\varphi-\dot{\theta}\cos\varphi)$$

（3-19）

求$\frac{\partial z_1}{\partial q_j}$　$(j=1\sim 8)$

$$\frac{\partial z_1}{\partial z_1}=1,其余\frac{\partial z}{\partial q_j}=0$$ （3-20）

求$\frac{\partial z_2}{\partial q_j}$　$(j=1\sim 8)$

由（3-4）得

$$x_2=x_1-s_1\sin\psi\sin\theta-s_2(\cos\beta\sin\theta\sin\varphi+\sin\alpha\sin\beta\cos\psi\sin\theta)$$
$$y_2=y_1+s_1\cos\psi\sin\theta+s_2(\cos\alpha\cos\psi\sin\theta-\sin\alpha\cos\theta)$$
$$z_2=z_1-s_1\cos\theta-s_2(-\sin\beta\sin\psi\sin\theta-\sin\alpha\cos\beta\cos\psi\sin\theta+$$
$$\cos\alpha\cos\beta\cos\theta)$$

（3-21）

所以有

$$\frac{\partial z_2}{\partial x_1}=0,\frac{\partial z_2}{\partial y_1}=0,$$
$$\frac{\partial z_2}{\partial z_1}=1,\frac{\partial z_2}{\partial\varphi}=0$$
$$\frac{\partial z_2}{\partial\alpha}=s_2(\cos\alpha\cos\beta\cos\psi\sin\theta+\sin\alpha\cos\beta\cos\theta)$$
$$\frac{\partial z_2}{\partial\beta}=-s_2(\sin\alpha\sin\beta\cos\psi\sin\theta-\cos\alpha\sin\beta\cos\theta)$$

$$\frac{\partial z_2}{\partial \psi}=-s_2(-\sin\beta\cos\psi\sin\theta+\sin\alpha\cos\beta\sin\psi\sin\theta)$$

$$\frac{\partial z_2}{\partial \theta}=s_1\sin\theta+s_2(\sin\beta\sin\psi\cos\theta+\sin\alpha\cos\beta\cos\psi\cos\theta+\cos\alpha\cos\beta\sin\theta)$$

（3-22）

式（3-13）中

$$\frac{\partial L}{\partial \dot{q}_j}=J_{1x}\omega_{1x}\frac{\partial\omega_{1x}}{\partial \dot{q}_j}+J_{1y}\omega_{2y}\frac{\partial\omega_{2y}}{\partial \dot{q}_j}+J_{1z}\omega_{1z}\frac{\partial\omega_{1z}}{\partial \dot{q}_j}+J_{2z}\omega_{2x}\frac{\partial\omega_{2x}}{\partial \dot{q}_j}+J_{2y}\omega_{2y}\frac{\partial\omega_{2y}}{\partial \dot{q}_j}+J_{2z}\omega_{2z}\frac{\partial\omega_{2z}}{\partial \dot{q}_j}+m_1(\dot{x}_1\frac{\partial\dot{x}_1}{\partial \dot{q}_j}+y_1\frac{\partial\dot{y}_1}{\partial \dot{q}_j}+\dot{z}_1\frac{\partial\dot{z}_1}{\partial \dot{q}_j})+m_2(\dot{x}_2\frac{\partial\dot{x}_2}{\partial \dot{q}_j}+\dot{y}_2\frac{\partial\dot{y}_2}{\partial \dot{q}_j}+\dot{z}_2\frac{\partial\dot{z}_2}{\partial \dot{q}_j})$$

（3-23）

求$\frac{\partial\omega_{1x}}{\partial \dot{q}_j}$　$(j=1\sim8)$:

$$\frac{\partial\omega_{1x}}{\partial\dot{x}_1}=0,\frac{\partial\omega_{1x}}{\partial\dot{y}_1}=0,\frac{\partial\omega_{1x}}{\partial\dot{z}_1}=0,\frac{\partial\omega_{1x}}{\partial\dot{\alpha}}=0,\frac{\partial\omega_{1x}}{\partial\dot{\beta}}=0$$
$$\frac{\partial\omega_{1x}}{\partial\dot{\psi}}=\sin\theta\sin\varphi,\frac{\partial\omega_{1x}}{\partial\theta}=\cos\varphi,\frac{\partial\omega_{1x}}{\partial\dot{\varphi}}=0$$

（3-24）

求$\frac{\partial\omega_{1y}}{\partial \dot{q}_j}$　$(j=1\sim8)$

$$\frac{\partial\omega_{1y}}{\partial\dot{x}_1}=0,\frac{\partial\omega_{1y}}{\partial\dot{y}_1}=0,\frac{\partial\omega_{1y}}{\partial\dot{z}_1}=0,\frac{\partial\omega_{1y}}{\partial\dot{\alpha}}=0,\frac{\partial\omega_{1y}}{\partial\dot{\beta}}=0,$$
$$\frac{\partial\omega_{1y}}{\partial\dot{\psi}}=\sin\theta\cos\varphi,\frac{\partial\omega_{1y}}{\partial\dot{\theta}}=-\sin\varphi,\frac{\partial\omega_{1y}}{\partial\dot{\varphi}}=0$$

（3-25）

求：$\frac{\partial\omega_{1z}}{\partial \dot{q}_j}$　$(j=1\sim8)$:

$$\frac{\partial w_{1z}}{\partial\dot{x}_1}=0,\frac{\partial w_{1z}}{\partial\dot{y}_1}=0,\frac{\partial w_{1z}}{\partial\dot{z}_1}=0,\frac{\partial w_{1z}}{\partial\dot{\alpha}}=0,\frac{\partial w_{1z}}{\partial\dot{\beta}}=0,$$

$$\frac{\partial w_{1z}}{\partial \dot{\psi}}=\cos\theta,\frac{\partial w_{1z}}{\partial \dot{\theta}}=0,\frac{\partial w_{1z}}{\partial \dot{\varphi}}=0$$

（3-26）

求$\frac{\partial \omega_{2x}}{\partial \dot{q}_j}$ $(j=1\sim 8)$：

$$\frac{\partial \omega_{2x}}{\partial \dot{x}_1}=0,\frac{\partial \omega_{2x}}{\partial \dot{y}_1}=0,\frac{\partial \omega_{2x}}{\partial \dot{z}_1}=0,\frac{\partial \omega_{2x}}{\partial \dot{\alpha}}=0,\frac{\partial \omega_{2x}}{\partial \dot{\beta}}=1,$$

$$\frac{\partial \omega_{2x}}{\partial \dot{\psi}}=\cos\alpha\sin\theta\sin\varphi-\sin\alpha\cos\theta$$

$$\frac{\partial \omega_{2x}}{\partial \dot{\theta}}=\cos\alpha\cos\varphi$$

$$\frac{\partial \omega_{2x}}{\partial \dot{\varphi}}=-\sin\alpha$$

（3-27）

求$\frac{\partial \omega_{2y}}{\partial \dot{q}_j}$ $(j=1\sim 8)$：

$$\frac{\partial \omega_{2y}}{\partial \dot{x}_1}=0,\frac{\partial \omega_{2y}}{\partial \dot{y}_1}=0,\frac{\partial \omega_{2y}}{\partial \dot{z}_1}=0,\frac{\partial \omega_{2y}}{\partial \dot{\alpha}}=\cos\beta,\frac{\partial \omega_{2y}}{\partial \dot{\beta}}=0,$$

$$\frac{\partial \omega_{2y}}{\partial \dot{\psi}}=\sin\alpha\sin\beta\sin\theta\sin\varphi+\cos\beta\sin\theta\cos\varphi+\cos\alpha\sin\beta\cos\theta$$

$$\frac{\partial \omega_{2y}}{\partial \dot{\theta}}=\sin\alpha\sin\beta\cos\varphi-\cos\beta\sin\varphi$$

$$\frac{\partial \omega_{2y}}{\partial \dot{\varphi}}=\cos\alpha\sin\beta$$

（3-28）

求$\frac{\partial \omega_{2z}}{\partial \dot{q}_j}$ $(j=1\sim 8)$：

$$\frac{\partial \omega_{2z}}{\partial \dot{x}_1}=0,\frac{\partial \omega_{2z}}{\partial \dot{y}_1}=0,\frac{\partial \omega_{2z}}{\partial \dot{z}_1}=0,\frac{\partial \omega_{2z}}{\partial \dot{\alpha}}=-\sin\beta,\frac{\partial \omega_{2z}}{\partial \dot{\beta}}=0,$$

$$\frac{\partial \omega_{2z}}{\partial \dot{\psi}}=\sin\alpha\cos\beta\sin\theta\sin\varphi-\sin\beta\sin\theta\cos\varphi+\cos\alpha\cos\beta\cos\theta$$

$$\frac{\partial \omega_{2z}}{\partial \dot{\theta}} = \sin\alpha\cos\beta\cos\varphi + \sin\varphi$$
$$\frac{\partial \omega_{2z}}{\partial \dot{\varphi}} = \cos\alpha\cos\beta \qquad (3\text{-}29)$$

求$\dfrac{\partial \dot{x}_1}{\partial \dot{q}_j} \quad (j=1\sim8)$：

$$\frac{\partial \dot{x}_1}{\partial \dot{x}_1} = 1, \text{其余} \frac{\partial \dot{x}_1}{\partial \dot{q}_j} = 0 \qquad (3\text{-}30)$$

求$\dfrac{\partial \dot{y}_1}{\partial \dot{q}_j} \quad (j=1\sim8)$：

$$\frac{\partial \dot{y}_1}{\partial \dot{y}_1} = 1, \text{其余} \frac{\partial \dot{y}_1}{\partial \dot{q}_j} = 0 \qquad (3\text{-}31)$$

求$\dfrac{\partial \dot{z}_1}{\partial q_j} \quad (j=1\sim8)$：

$$\frac{\partial \dot{z}_1}{\partial \dot{z}_1} = 1, \text{其余} \frac{\partial \dot{z}_1}{\partial \dot{q}_j} = 0 \qquad (3\text{-}32)$$

由式（3-21）中的x_2, y_2, z_2,求出$\dot{x}_2, \dot{y}_2, \dot{z}_2$分别为

$$\begin{aligned}\dot{x}_2 = \dot{x}_1 - s_1(\dot{\psi}\cos\psi\sin\theta + \dot{\theta}\sin\psi\cos\theta) - s_2(-\dot{\beta}\sin\beta\sin\theta\sin\varphi +\\ \dot{\theta}\cos\beta\cos\theta\sin\varphi + \dot{\varphi}\cos\beta\sin\theta\cos\varphi + \dot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta +\\ \dot{\beta}\sin\alpha\cos\beta\cos\psi\sin\theta - \dot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta + \dot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta)\end{aligned}$$

(3-33)

$$\begin{aligned}\dot{y}_2 = \dot{y}_1 + s_1(-\dot{\psi}\sin\psi\sin\theta + \dot{\theta}\cos\psi\cos\theta) + s_2(-\dot{\alpha}\sin\alpha\cos\psi\sin\theta -\\ \dot{\psi}\cos\alpha\sin\psi\sin\theta + \dot{\theta}\cos\alpha\cos\psi\cos\theta - \dot{\alpha}\cos\alpha\cos\theta + \dot{\theta}\sin\alpha\sin\theta)\end{aligned}$$

(3-34)

$$\begin{aligned}\dot{z}_2 = \dot{z}_1 + s_1\sin\theta - s_2(-\dot{\beta}\cos\beta\sin\psi\sin\theta - \dot{\psi}\sin\beta\cos\psi\sin\theta - \dot{\theta}\sin\beta\sin\psi\cos\theta -\\ \dot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta + \dot{\beta}\sin\alpha\sin\beta\cos\psi\sin\theta + \dot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta -\\ \dot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta - \dot{\alpha}\sin\alpha\cos\beta\cos\theta - \dot{\beta}\cos\alpha\sin\beta\cos\theta -\\ \dot{\theta}\cos\alpha\cos\beta\sin\theta)\end{aligned}$$

(3-35)

求$\dfrac{\partial \dot{x}_2}{\partial \dot{q}_j}$ $(j=1\sim 8)$

$$\frac{\partial \dot{x}_2}{\partial \dot{x}_1}=1,\frac{\partial \dot{x}_2}{\partial \dot{y}_1}=0,\frac{\partial \dot{x}_2}{\partial \dot{z}_1}=0$$

$$\frac{\partial \dot{x}_2}{\partial \dot{\alpha}}=-s_2\cos\alpha\sin\beta\cos\psi\sin\theta$$

$$\frac{\partial \dot{x}_2}{\partial \dot{\beta}}=-s_2(-\sin\beta\sin\theta\sin\varphi+\sin\alpha\cos\beta\cos\psi\sin\theta)$$

$$\frac{\partial \dot{x}_2}{\partial \dot{\psi}}=-s_1\cos\psi\sin\theta+s_2\sin\alpha\sin\beta\sin\psi\sin\theta$$

$$\frac{\partial \dot{x}_2}{\partial \dot{\theta}}=-s_1\cos\theta\sin\psi-s_2(\cos\beta\cos\theta\sin\varphi+\sin\alpha\sin\beta\cos\psi\cos\theta)$$

$$\frac{\partial \dot{x}_2}{\partial \dot{\varphi}}=-s_2\cos\beta\sin\theta\cos\varphi$$

（3-36）

求$\dfrac{\partial \dot{y}_2}{\partial \dot{q}_j}$ $(j=1\sim 8)$

$$\frac{\partial \dot{y}_2}{\partial \dot{x}_1}=\frac{\partial \dot{y}_2}{\partial \dot{z}_1}=0,\frac{\partial \dot{y}_2}{\partial \dot{y}_1}=1$$

$$\frac{\partial \dot{y}_2}{\partial \dot{\alpha}}=s_2(-\sin\alpha\cos\psi\sin\theta-\cos\alpha\cos\theta)$$

$$\frac{\partial \dot{y}_2}{\partial \dot{\beta}}=0$$

$$\frac{\partial \dot{y}_2}{\partial \dot{\psi}}=-s_1\sin\psi\sin\theta-s_2\cos\alpha\sin\psi\sin\theta$$

$$\frac{\partial \dot{y}_2}{\partial \dot{\theta}}=s_1\cos\psi\cos\theta+s_2(\cos\alpha\cos\psi\cos\theta+\sin\alpha\sin\theta)$$

$$\frac{\partial \dot{y}_2}{\partial \dot{\varphi}}=0$$

（3-37）

求$\frac{\partial \dot{z}_2}{\partial \dot{q}_j}$ $(j=1\sim8)$

$$\frac{\partial \dot{z}_2}{\partial \dot{x}_1}=\frac{\partial \dot{z}_2}{\partial \dot{y}_1}=0,\frac{\partial \dot{z}_2}{\partial \dot{z}_1}=1$$

$$\frac{\partial \dot{z}_2}{\partial \dot{\alpha}}=s_2[\cos\alpha\cos\beta\cos\psi\sin\theta+\cos\theta\sin\alpha\cos\beta]$$

$$\frac{\partial \dot{z}_2}{\partial \dot{\beta}}=-s_2[-\cos\beta\sin\psi\sin\theta+\sin\alpha\sin\beta\cos\psi\sin\theta-\cos\theta\cos\alpha\sin\beta]$$

$$\frac{\partial \dot{z}_2}{\partial \dot{\psi}}=-s_2(-\sin\beta\cos\psi\sin\theta+\sin\alpha\cos\beta\sin\psi\sin\theta)$$

$$\frac{\partial \dot{z}_2}{\partial \dot{\theta}}=-s_2(-\sin\beta\sin\psi\cos\theta-\sin\alpha\cos\beta\cos\psi\cos\theta-\cos\alpha\cos\beta\sin\theta)$$

$$\frac{\partial \dot{z}_2}{\partial \dot{\varphi}}=0$$

（3-38）

式（3-13）中的$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j})$ $(j=1\sim8)$展开为

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j})=J_{1x}[\dot{\omega}_{1x}\frac{\partial \omega_{1x}}{\partial \dot{q}_j}+\omega_{1x}(\frac{\partial \omega_{1x}}{\partial \dot{q}_j})_t]+J_{1y}[\dot{\omega}_{1y}\frac{\partial \omega_{1y}}{\partial \dot{q}_j}+\omega_{1y}(\frac{\partial \omega_{1y}}{\partial \dot{q}_j})_t]+J_{1z}[\dot{\omega}_{1z}\frac{\partial \omega_{1z}}{\partial \dot{q}_j}+\omega_{1z}(\frac{\partial \omega_{1z}}{\partial \dot{q}_j})_t]+J_{2x}[\dot{\omega}_{2x}\frac{\partial \omega_{2x}}{\partial \dot{q}_j}+\omega_{2x}(\frac{\partial \omega_{2x}}{\partial \dot{q}_j})_t]+J_{2y}[\dot{\omega}_{2y}\frac{\partial \omega_{2y}}{\partial \dot{q}_j}+\omega_{2y}(\frac{\partial \omega_{2y}}{\partial \dot{q}_j})_t]+J_{2z}[\dot{\omega}_{2z}\frac{\partial \omega_{2z}}{\partial \dot{q}_j}+\omega_{2z}(\frac{\partial \omega_{2z}}{\partial \dot{q}_j})_t]+m_1\{[\ddot{x}_1(\frac{\partial \dot{x}_1}{\partial \dot{q}_j})+\dot{x}_1(\frac{\partial \dot{x}_1}{\partial \dot{q}_j})t]+[\ddot{y}_1\frac{\partial \dot{y}_1}{\partial \dot{q}_j}+\dot{y}_1(\frac{\partial \dot{y}_1}{\partial \dot{q}_j})_t]+[\ddot{z}_1\frac{\partial \dot{z}_1}{\partial \dot{q}_j}+\dot{z}_1(\frac{\partial \dot{z}_1}{\partial \dot{q}_j})_t]+m_2\{[\ddot{x}_2\frac{\partial \dot{x}_2}{\partial \dot{q}_j}+\dot{x}_2(\frac{\partial \dot{x}_2}{\partial \dot{q}_j})_t]+[\ddot{y}_2\frac{\partial \dot{y}_2}{\partial \dot{q}_j}+\dot{y}_2(\frac{\partial \dot{y}_2}{\partial \dot{q}_j})_t]+[\ddot{z}_2\frac{\partial \dot{z}_2}{\partial \dot{q}_j}+\dot{z}_2(\frac{\partial \dot{z}_2}{\partial \dot{q}_j})_t]\}$$

（3-39）

式中$(*)_t$表示对函数（*）关于时间 t 的导数。

求出式（3-39）中各角速度分量的导数

$$\dot{\omega}_{1x}=\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}(\dot{\theta}\cos\theta\sin\varphi+\dot{\varphi}\sin\theta\cos\varphi)+\ddot{\theta}\cos\varphi-\dot{\varphi}\dot{\theta}\sin\varphi$$

$$\dot{\omega}_{1y}=\ddot{\psi}\sin\theta\cos\varphi+\dot{\psi}(\dot{\theta}\cos\theta\cos\varphi-\dot{\varphi}\sin\theta\sin\varphi)-\ddot{\theta}\sin\varphi-\dot{\varphi}\dot{\theta}\cos\varphi$$

$$\dot{\omega}_{1z}=\ddot{\psi}\cos\theta-\dot{\theta}\dot{\psi}\sin\theta+\ddot{\varphi}$$

（3-40）

$$\begin{aligned}\dot{\omega}_{2x}=&-\dot{\alpha}\sin\alpha(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi+\ddot{\psi}\sin\theta\sin\varphi)+\\&\cos\alpha(\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\ddot{\psi}\sin\theta\sin\varphi+\dot{\varphi}\dot{\psi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\sin\varphi)-\\&\dot{\alpha}\cos\alpha(\dot{\psi}\cos\theta+\dot{\varphi})-\sin\alpha(-\dot{\psi}\dot{\theta}\sin\theta+\ddot{\psi}\cos\theta+\ddot{\varphi})+\ddot{\beta}\end{aligned}$$

$$\begin{aligned}\dot{\omega}_{2y}=&(\dot{\alpha}\cos\alpha\sin\beta+\dot{\beta}\sin\alpha\cos\beta)(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+\\&\sin\alpha\sin\beta(\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\dot{\psi}\dot{\varphi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\dot{\varphi}\sin\varphi)-\\&\dot{\beta}\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\cos\beta(\ddot{\psi}\sin\theta\cos\varphi+\dot{\psi}\dot{\theta}\cos\theta\cos\varphi-\\&\dot{\varphi}\dot{\psi}\sin\theta\sin\varphi-\ddot{\theta}\sin\varphi-\dot{\theta}\dot{\varphi}\cos\varphi)+\ddot{\alpha}\cos\beta-\dot{\alpha}\dot{\beta}\sin\beta+(-\dot{\alpha}\sin\alpha\sin\beta+\\&\dot{\beta}\cos\alpha\cos\beta)(\dot{\psi}\cos\theta+\dot{\varphi})+\cos\alpha\sin\beta(\ddot{\psi}\cos\theta+\ddot{\varphi}-\dot{\psi}\dot{\theta}\sin\theta)\end{aligned}$$

$$\begin{aligned}\dot{\omega}_{2z}=&(\dot{\alpha}\cos\beta\cos\alpha-\dot{\beta}\cos\beta\sin\alpha)(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+\\&\cos\beta\sin\alpha(\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\dot{\psi}\dot{\varphi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\dot{\varphi}\sin\varphi)-\\&\dot{\beta}\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)-\sin\beta(\ddot{\psi}\sin\theta\cos\varphi+\dot{\psi}\dot{\theta}\cos\theta\cos\varphi-\\&\dot{\psi}\dot{\varphi}\sin\theta\sin\varphi-\ddot{\theta}\sin\varphi-\dot{\theta}\dot{\varphi}\cos\varphi)+(-\dot{\alpha}\sin\alpha\cos\beta-\dot{\beta}\cos\alpha\sin\beta)(\dot{\psi}\cos\theta+\dot{\varphi})+\\&\cos\alpha\cos\beta(\ddot{\psi}\cos\theta-\dot{\psi}\dot{\theta}\sin\theta+\ddot{\varphi})-\ddot{\alpha}\sin\beta-\dot{\alpha}\dot{\beta}\cos\beta\end{aligned}$$

（3-41）

求$\left(\dfrac{\partial\dot{\omega}_{1x}}{\partial\dot{q}_j}\right)_t$　　$(j=1\sim8)$

$$(\frac{\partial\omega_{1x}}{\partial\dot{x}_1})_t=(\frac{\partial\omega_{1x}}{\partial\dot{y}_1})_t=(\frac{\partial\omega_{1x}}{\partial\dot{z}_1})_t=0$$

$$(\frac{\partial\omega_{1x}}{\partial\dot{\varphi}})_t=(\frac{\partial\omega_{1x}}{\partial\dot{\alpha}})_t=(\frac{\partial\omega_{1x}}{\partial\dot{\beta}})_t=0$$

$$(\frac{\partial\omega_{1x}}{\partial\dot{\psi}})_t=\dot{\theta}\cos\theta\sin\varphi+\dot{\varphi}\sin\theta\cos\varphi$$

$$(\frac{\partial\omega_{1x}}{\partial\dot{\theta}})_t=-\dot{\varphi}\sin\varphi$$

$$(\frac{\partial\omega_{1x}}{\partial\dot{\varphi}})_t=0$$

（3-42）

求 $(\frac{\partial\omega_{1y}}{\partial\dot{q}_j})_t=0 \qquad (j=1\sim8)$

$$(\frac{\partial\omega_{1y}}{\partial\dot{x}_1})_t=(\frac{\partial\omega_{1y}}{\partial\dot{y}_1})_t=(\frac{\partial\omega_{1y}}{\partial\dot{z}_1})_t=(\frac{\partial\omega_{1y}}{\partial\dot{\alpha}})_t=(\frac{\partial\omega_{1y}}{\partial\dot{\beta}})_t=(\frac{\partial\omega_{1y}}{\partial\dot{\varphi}})_t=0$$

$$(\frac{\partial\omega_{1y}}{\partial\dot{\psi}})_t=\dot{\theta}\cos\theta\cos\varphi-\dot{\varphi}\sin\theta\sin\varphi$$

$$(\frac{\partial\omega_{1y}}{\partial\dot{\theta}})_t=-\dot{\varphi}\cos\varphi$$

$$(\frac{\partial\omega_{1y}}{\partial\dot{\varphi}})_t=0$$

（3-43）

求 $(\frac{\partial\omega_{1z}}{\partial\dot{q}_j})_t=0 \qquad (j=1\sim8)$

$$(\frac{\partial\omega_{1z}}{\partial\dot{x}_1})_t=(\frac{\partial\omega_{1z}}{\partial\dot{y}_1})_t=(\frac{\partial\omega_{1z}}{\partial\dot{z}_1})_t=(\frac{\partial\omega_{1z}}{\partial\dot{\alpha}})_t=(\frac{\partial\omega_{1z}}{\partial\dot{\beta}})_t=(\frac{\partial\omega_{1z}}{\partial\dot{\theta}})_t=0$$

$$(\frac{\partial\omega_{1z}}{\partial\dot{\psi}})_t=-\dot{\theta}\sin\theta,(\frac{\partial\omega_{1z}}{\partial\dot{\varphi}})_t=0$$

（3-44）

求 $(\frac{\partial\omega_{2x}}{\partial\dot{q}_j})_t \qquad (j=1\sim8)$

$$(\frac{\partial\omega_{2x}}{\partial\dot{x}_1})_t=(\frac{\partial\omega_{2x}}{\partial\dot{y}_1})_t=(\frac{\partial\omega_{2x}}{\partial\dot{z}_1})_t=0$$

$$(\frac{\partial\omega_{2y}}{\partial\dot{\beta}})_t=(\frac{\partial\omega_{2x}}{\partial\dot{\alpha}})_t=0$$

$$(\frac{\partial\omega_{2x}}{\partial\dot{\psi}})_t=-\dot{\alpha}\sin\alpha\sin\theta\sin\varphi+\dot{\theta}\cos\alpha\cos\theta\sin\varphi+\dot{\varphi}\cos\alpha\sin\theta\cos\varphi-$$

$$\dot{\alpha}\cos\alpha\cos\theta+\dot{\theta}\sin\alpha\sin\theta$$

$$(\frac{\partial\omega_{2x}}{\partial\dot{\theta}})_t=-\dot{\alpha}\sin\alpha\cos\varphi-\dot{\varphi}\cos\alpha\sin\varphi$$

$$(\frac{\partial\omega_{2x}}{\partial\dot{\varphi}})_t=-\dot{\alpha}\cos\alpha\text{，}\ (\frac{\partial\omega_{2x}}{\partial\dot{\beta}})_t=0$$

（3-45）

求 $(\frac{\partial\dot{\omega}_{2y}}{\partial\dot{q}_j})_t=0\qquad(j=1\sim8)$

$$(\frac{\partial\omega_{2y}}{\partial\dot{x}_1})_t=(\frac{\partial\omega_{2y}}{\partial\dot{y}_1})_t=(\frac{\partial\omega_{2y}}{\partial\dot{z}_1})_t=(\frac{\partial\omega_{2y}}{\partial\dot{\beta}})_t=-0\text{，}\ (\frac{\partial\omega_{2y}}{\partial\dot{\alpha}})_t=-\dot{\beta}\sin\beta$$

$$(\frac{\partial\omega_{2y}}{\partial\dot{\psi}})_t=\dot{\alpha}\cos\alpha\sin\beta\sin\theta\sin\varphi+\dot{\beta}\sin\alpha\cos\beta\sin\theta\sin\varphi+\dot{\theta}\sin\alpha\sin\beta\cos\theta\sin\varphi+$$

$$\dot{\varphi}\sin\alpha\sin\beta\sin\theta\cos\varphi-\dot{\beta}\sin\beta\sin\theta\cos\varphi+\dot{\theta}\cos\beta\cos\theta\cos\varphi-$$

$$\dot{\varphi}\cos\beta\sin\theta\sin\varphi-\dot{\alpha}\sin\alpha\sin\beta\cos\theta+\dot{\beta}\cos\alpha\cos\beta\cos\theta-\dot{\theta}\cos\alpha\sin\beta\sin\theta$$

$$(\frac{\partial\omega_{2y}}{\partial\dot{\theta}})_t=\dot{\alpha}\cos\alpha\sin\beta\cos\varphi+\dot{\beta}\sin\alpha\cos\beta\cos\varphi-\dot{\varphi}\sin\alpha\sin\beta\sin\varphi+$$

$$\dot{\beta}\sin\beta\sin\varphi-\dot{\varphi}\cos\beta\cos\varphi$$

$$(\frac{\partial\omega_{2y}}{\partial\dot{\varphi}})_t=-\dot{\alpha}\sin\alpha\sin\beta+\dot{\beta}\cos\alpha\cos\beta$$

（3-46）

求 $(\frac{\partial\dot{\omega}_{2z}}{\partial\dot{q}_j})_t=0\qquad(j=1\sim8)$

$$(\frac{\partial\omega_{2z}}{\partial\dot{x}_1})_t=(\frac{\partial\omega_{2z}}{\partial\dot{y}_1})_t=(\frac{\partial\omega_{2z}}{\partial\dot{z}_1})_t=0$$

$$(\frac{\partial\omega_{2z}}{\partial\dot{\alpha}})_t=-\dot{\beta}\cos\beta\text{，}\ (\frac{\partial\omega_{2z}}{\partial\dot{\beta}})_t=0$$

$$(\frac{\partial\omega_{2z}}{\partial\dot{\psi}})_t=\dot{\alpha}\cos\beta\cos\alpha\sin\varphi\sin\theta-\dot{\beta}\sin\beta\sin\alpha\sin\theta\sin\varphi+$$

$$\dot{\theta}\cos\beta\sin\alpha\cos\theta\sin\varphi+\dot{\varphi}\cos\beta\sin\alpha\sin\theta\cos\varphi-$$

$$\dot{\beta}\cos\beta\sin\theta\cos\varphi-\dot{\theta}\sin\beta\cos\theta\cos\varphi+\dot{\varphi}\sin\beta\sin\theta\sin\varphi-$$
$$\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\beta}\cos\alpha\sin\beta\cos\theta-\dot{\theta}\cos\alpha\cos\beta\sin\theta$$

$$(\frac{\partial\omega_{2z}}{\partial\dot{\theta}})_t=\dot{\alpha}\cos\beta\cos\alpha\cos\varphi-\dot{\beta}\sin\beta\sin\alpha\cos\varphi-\dot{\varphi}\cos\beta\sin\alpha\sin\varphi+$$
$$\dot{\varphi}\cos\varphi$$

$$(\frac{\partial\omega_{2z}}{\partial\dot{\varphi}})_t=-\dot{\alpha}\sin\alpha\cos\beta-\dot{\beta}\cos\alpha\sin\beta$$

（3-47）

求$(\frac{\partial\dot{x}_1}{\partial\dot{q}_j})_t \quad (j=1\sim8)$

$$(\frac{\partial\dot{x}_1}{\partial\dot{q}_j})_t=0 \qquad (j=1\sim8) \tag{3-48}$$

求$(\frac{\partial\dot{y}_1}{\partial\dot{q}_j})_t \quad (j=1\sim8)$

$$(\frac{\partial\dot{y}_1}{\partial\dot{q}_j})_t=0\ (j=1\sim8) \tag{3-49}$$

求$(\frac{\partial\dot{z}_1}{\partial\dot{q}_j})_t \quad (j=1\sim8)$

$$(\frac{\partial\dot{z}_1}{\partial\dot{q}_j})_t=0 \quad (j=1\sim8) \tag{3-50}$$

求$(\frac{\partial\dot{x}_2}{\partial\dot{q}_j})_t \qquad (j=1\sim8)$

$$(\frac{\partial\dot{x}_2}{\partial\dot{x}_1})_t=0\ ,\quad (\frac{\partial\dot{x}_2}{\partial\dot{y}_1})_t=0\ ,\quad (\frac{\partial\dot{x}_2}{\partial\dot{z}_1})_t=0$$

$$\left(\frac{\partial\dot{x}_2}{\partial\dot{\alpha}}\right)_t=-s_2(-\dot{\alpha}\sin\alpha\sin\beta\cos\psi\sin\theta+\dot{\beta}\cos\alpha\cos\beta\cos\psi\sin\theta-$$
$$\dot{\psi}\cos\alpha\sin\beta\sin\theta\sin\psi+\dot{\theta}\cos\psi\cos\theta\sin\beta\cos\alpha)$$

$$(\frac{\partial\dot{x}_2}{\partial\dot{\beta}})_t=-s_2[-\dot{\beta}\cos\beta\sin\theta\sin\varphi-\dot{\theta}\sin\beta\cos\theta\sin\varphi-\dot{\varphi}\sin\beta\sin\theta\cos\varphi+$$
$$\dot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta-\dot{\beta}\sin\alpha\sin\beta\cos\psi\sin\theta-$$
$$\dot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta+\dot{\theta}\cos\psi\cos\theta\sin\alpha\cos\beta]$$

$$(\frac{\partial \dot{x}_2}{\partial \dot{\psi}})_t = -s_1(-\dot{\psi}\sin\psi\sin\theta + \dot{\theta}\cos\psi\cos\theta) + s_2(\dot{\alpha}\cos\alpha\sin\beta\sin\theta\sin\psi +$$
$$\dot{\beta}\sin\alpha\cos\beta\sin\theta\sin\psi + \dot{\psi}\sin\alpha\sin\beta\sin\theta\cos\psi + \dot{\theta}\sin\alpha\sin\beta\sin\psi\cos\theta)$$
$$(\frac{\partial \dot{x}_2}{\partial \dot{\theta}})_t = -s_1(-\dot{\psi}\cos\psi\cos\theta - \dot{\theta}\sin\theta\sin\psi) - s_2(-\dot{\beta}\sin\beta\cos\theta\sin\varphi -$$
$$\dot{\theta}\cos\beta\sin\theta\sin\varphi + \dot{\varphi}\cos\beta\cos\theta\sin\varphi + \dot{\alpha}\cos\alpha\sin\beta\cos\psi\cos\theta +$$
$$\dot{\beta}\sin\alpha\cos\beta\cos\theta\cos\psi - \dot{\psi}\sin\alpha\sin\beta\sin\psi\cos\theta - \dot{\theta}\sin\alpha\sin\beta\cos\psi\sin\theta)$$
$$(\frac{\partial \dot{x}_2}{\partial \dot{\varphi}})_t = -s_2(-\dot{\beta}\sin\beta\sin\theta\cos\varphi + \dot{\theta}\cos\beta\cos\theta\cos\varphi - \dot{\varphi}\cos\beta\sin\theta\sin\varphi)$$

（3-51）

求$(\frac{\partial \dot{y}_2}{\partial \dot{q}_j})_t$：

$$(\frac{\partial \dot{y}_2}{\partial \dot{x}_1})_t = (\frac{\partial \dot{y}_2}{\partial \dot{z}_1})_t = (\frac{\partial \dot{y}_2}{\partial \dot{y}_1})_t = 0，\ (\frac{\partial \dot{y}_2}{\partial \dot{\beta}})_t = 0，\ (\frac{\partial \dot{y}_2}{\partial \dot{\varphi}})_t = 0 \qquad (3\text{-}52)$$

$$(\frac{\partial \dot{y}_2}{\partial \dot{\alpha}})_t = s_2(-\dot{\alpha}\cos\alpha\cos\psi\sin\theta + \dot{\psi}\sin\alpha\sin\psi\sin\theta - \dot{\theta}\sin\alpha\cos\psi\cos\theta +$$
$$\dot{\alpha}\sin\alpha\cos\theta + \dot{\theta}\cos\alpha\sin\theta)$$
$$(\frac{\partial \dot{y}_2}{\partial \dot{\psi}})_t = -s_1(\dot{\psi}\cos\psi\sin\theta + \dot{\theta}\sin\psi\cos\theta) - s_2(-\dot{\alpha}\sin\alpha\sin\psi\sin\theta +$$
$$\dot{\psi}\cos\alpha\cos\psi\sin\theta + \dot{\theta}\cos\alpha\sin\psi\cos\theta)$$
$$(\frac{\partial \dot{y}_2}{\partial \dot{\theta}})_t = s_1(-\dot{\psi}\sin\psi\cos\theta - \dot{\theta}\cos\psi\sin\theta) + s_2(-\dot{\alpha}\sin\alpha\cos\psi\cos\theta -$$
$$\dot{\psi}\cos\alpha\sin\psi\cos\theta - \dot{\theta}\cos\alpha\cos\psi\sin\theta + \dot{\alpha}\cos\alpha\sin\theta + \dot{\theta}\sin\alpha\cos\theta)$$

（3-53）

求：$(\frac{\partial \dot{z}_2}{\partial \dot{q}_j})_t$：

$$(\frac{\partial \dot{z}_2}{\partial \dot{x}_1}) = (\frac{\partial \dot{z}_2}{\partial \dot{y}_1})_t = (\frac{\partial \dot{z}_2}{\partial \dot{z}_1})_t = 0 \qquad (3\text{-}54)$$

$$(\frac{\partial \dot{z}_2}{\partial \dot{\alpha}})_t = s_2(-\dot{\alpha}\sin\alpha\cos\beta\cos\psi\sin\theta - \dot{\beta}\cos\alpha\sin\beta\cos\psi\sin\theta -$$
$$\cos\alpha\cos\beta\sin\psi\sin\theta + \dot{\theta}\cos\alpha\cos\beta\cos\psi\cos\theta + \dot{\theta}\cos\psi\cos\theta\sin\alpha\cos\beta +$$
$$\dot{\alpha}\cos\theta\cos\alpha\cos\beta - \dot{\beta}\cos\theta\sin\alpha\sin\beta - \dot{\theta}\sin\alpha\cos\beta\sin\theta)$$

$$(\frac{\partial \dot{z}_2}{\partial \dot{\beta}})_t = -s_2(\dot{\beta}\sin\beta\sin\psi\sin\theta - \dot{\psi}\cos\beta\cos\psi\sin\theta - \dot{\theta}\cos\beta\sin\psi\cos\theta +$$
$$\dot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta + \dot{\beta}\sin\alpha\cos\beta\cos\psi\sin\theta - \dot{\psi}\sin\alpha\sin\beta\sin\psi\cos\theta +$$
$$\dot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta + \dot{\theta}\sin\theta\cos\alpha\sin\beta + \dot{\alpha}\cos\theta\sin\alpha\sin\beta -$$
$$\dot{\beta}\cos\theta\cos\alpha\cos\beta)$$

$$(\frac{\partial \dot{z}_2}{\partial \dot{\psi}})_t = -s_2(-\dot{\beta}\cos\beta\cos\psi\sin\theta + \dot{\psi}\sin\beta\sin\psi\sin\theta - \dot{\theta}\sin\beta\cos\psi\cos\theta +$$
$$\dot{\alpha}\cos\alpha\cos\beta\sin\psi\sin\theta - \dot{\beta}\sin\alpha\sin\beta\sin\psi\sin\theta + \dot{\psi}\sin\alpha\cos\beta\cos\dot{\psi}\sin\theta +$$
$$\dot{\theta}\sin\alpha\cos\beta\sin\psi\cos\theta)$$

$$(\frac{\partial \dot{z}_2}{\partial \dot{\theta}})_t = -s_2(-\dot{\beta}\cos\beta\sin\psi\cos\theta - \dot{\psi}\sin\beta\cos\psi\cos\theta + \dot{\theta}\sin\beta\sin\psi\sin\theta -$$
$$\dot{\alpha}\cos\alpha\cos\beta\cos\psi\cos\theta + \dot{\beta}\sin\alpha\sin\beta\cos\psi\cos\theta +$$
$$\dot{\psi}\sin\alpha\cos\beta\sin\psi\cos\theta + \dot{\alpha}\sin\alpha\cos\beta\sin\theta + \dot{\beta}\cos\alpha\sin\beta\sin\theta -$$
$$\dot{\theta}\cos\alpha\cos\beta\cos\theta)$$

$$(\frac{\partial \dot{z}_2}{\partial \dot{\varphi}})_t = 0 \qquad (3\text{-}55)$$

由（3-33）、（3-34）、（3-35）得

$$\ddot{x}_2 = \ddot{x}_1 - s_1(\ddot{\psi}\cos\psi\sin\theta - \dot{\psi}\dot{\psi}\sin\psi\sin\theta + \dot{\psi}\dot{\theta}\cos\psi\cos\theta +$$
$$\ddot{\theta}\sin\psi\cos\theta + \dot{\theta}\dot{\psi}\cos\psi\cos\theta - \dot{\theta}\dot{\theta}\sin\psi\sin\theta) -$$
$$s_2(-\ddot{\beta}\sin\beta\sin\theta\sin\varphi - \dot{\beta}\dot{\beta}\cos\beta\sin\theta\sin\varphi - \dot{\beta}\dot{\theta}\sin\beta\cos\theta\sin\varphi - \dot{\beta}\dot{\varphi}\sin\beta\sin\theta\cos\varphi +$$
$$\ddot{\theta}\cos\beta\cos\theta\sin\varphi - \dot{\theta}\dot{\beta}\sin\beta\cos\theta\sin\varphi - \dot{\theta}\dot{\theta}\cos\beta\sin\theta\sin\varphi + \dot{\theta}\dot{\varphi}\cos\beta\cos\theta\cos\varphi +$$
$$\ddot{\varphi}\cos\beta\sin\theta\cos\varphi - \dot{\varphi}\dot{\beta}\sin\beta\sin\theta\cos\varphi + \dot{\varphi}\dot{\theta}\cos\beta\cos\theta\cos\varphi - \dot{\varphi}\dot{\varphi}\cos\beta\sin\theta\sin\varphi +$$
$$\ddot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta - \dot{\alpha}^2\sin\alpha\sin\beta\cos\psi\sin\theta + \dot{\alpha}\dot{\beta}\cos\alpha\cos\beta\cos\psi\sin\theta -$$
$$\dot{\alpha}\dot{\psi}\cos\alpha\sin\beta\sin\psi\sin\theta + \dot{\alpha}\dot{\theta}\cos\alpha\sin\beta\cos\psi\cos\theta +$$
$$\ddot{\beta}\sin\alpha\cos\beta\cos\psi\sin\theta + \dot{\beta}\dot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta - \dot{\beta}^2\sin\alpha\sin\beta\cos\psi\sin\theta -$$
$$\dot{\beta}\dot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta + \dot{\beta}\dot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta -$$
$$\ddot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta + \dot{\psi}\dot{\alpha}\cos\alpha\sin\beta\sin\dot{\psi}\sin\theta + \dot{\psi}\dot{\beta}\sin\alpha\cos\beta\sin\psi\sin\theta +$$
$$\dot{\psi}^2\sin\alpha\sin\beta\cos\psi\sin\theta + \dot{\psi}\dot{\theta}\sin\alpha\sin\beta\sin\psi\cos\theta +$$
$$\ddot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta + \dot{\theta}\dot{\alpha}\cos\alpha\sin\beta\cos\psi\cos\theta + \dot{\theta}\dot{\beta}\sin\alpha\cos\beta\cos\psi\cos\theta -$$
$$\dot{\theta}\dot{\psi}\sin\alpha\sin\beta\sin\psi\cos\theta - \dot{\theta}^2\sin\alpha\sin\beta\cos\psi\sin\theta)$$

(3-56)

$$
\begin{aligned}
\ddot{y}_2 = \ddot{y}_1 + s_1(&-\ddot{\psi}\sin\psi\sin\theta - \dot{\psi}^2\cos\psi\sin\theta - \dot{\psi}\dot{\theta}\sin\psi\cos\theta + \\
&\ddot{\theta}\cos\psi\cos\theta - \dot{\theta}\dot{\psi}\sin\psi\cos\theta - \dot{\theta}^2\cos\psi\sin\theta) + \\
&s_2(-\ddot{\alpha}\sin\alpha\cos\psi\sin\theta - \dot{\alpha}^2\cos\alpha\cos\psi\sin\theta + \dot{\alpha}\dot{\psi}\sin\alpha\sin\psi\sin\theta - \\
&\dot{\alpha}\dot{\theta}\sin\alpha\cos\psi\cos\theta - \ddot{\psi}\cos\alpha\sin\psi\sin\theta - \dot{\psi}\dot{\alpha}\sin\alpha\sin\psi\sin\theta + \\
&\dot{\psi}^2\cos\alpha\cos\psi\sin\theta + \dot{\psi}\dot{\theta}\cos\alpha\sin\psi\cos\theta + \ddot{\theta}\cos\alpha\cos\psi\cos\theta + \\
&\dot{\theta}\dot{\alpha}\sin\alpha\cos\psi\cos\theta + \dot{\theta}\dot{\psi}\cos\alpha\sin\psi\cos\theta + \dot{\theta}^2\cos\alpha\cos\psi\sin\theta - \\
&\ddot{\alpha}\cos\alpha\cos\theta - \dot{\alpha}^2\sin\alpha\cos\theta - \dot{\alpha}\dot{\theta}\cos\alpha\sin\theta + \ddot{\theta}\sin\alpha\sin\theta + \\
&\dot{\theta}\dot{\alpha}\cos\alpha\sin\theta + \dot{\theta}^2\sin\alpha\cos\theta)
\end{aligned}
\tag{3-57}
$$

$$
\begin{aligned}
\ddot{z}_2 = \ddot{z}_1 + s_1\dot{\theta}\cos\theta - \\
&s_2(-\ddot{\beta}\cos\beta\sin\psi\sin\theta + \dot{\beta}^2\sin\beta\sin\psi\sin\theta - \dot{\beta}\dot{\psi}\cos\beta\cos\psi\sin\theta - \\
&\dot{\beta}\dot{\theta}\cos\beta\sin\psi\cos\theta - \\
&\ddot{\psi}\sin\beta\cos\psi\sin\theta - \dot{\psi}\dot{\beta}\cos\beta\cos\psi\sin\theta + \dot{\psi}^2\sin\beta\sin\psi\sin\theta - \\
&\dot{\psi}\dot{\theta}\sin\beta\cos\psi\cos\theta - \\
&\ddot{\theta}\sin\beta\sin\psi\cos\theta - \dot{\theta}\dot{\beta}\sin\beta\sin\psi\cos\theta - \dot{\theta}\dot{\psi}\sin\beta\sin\psi\cos\theta + \\
&\dot{\theta}^2\sin\beta\sin\psi\sin\theta - \\
&\ddot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta + \dot{\alpha}^2\sin\alpha\cos\beta\cos\psi\sin\theta + \dot{\alpha}\dot{\beta}\cos\alpha\sin\beta\cos\psi\sin\theta + \\
&\dot{\alpha}\dot{\psi}\cos\alpha\cos\beta\sin\psi\sin\theta - \dot{\alpha}\dot{\theta}\cos\alpha\cos\beta\cos\psi\cos\theta + \\
&\ddot{\beta}\sin\alpha\sin\beta\cos\psi\sin\theta + \dot{\beta}\dot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta + \dot{\beta}^2\sin\alpha\cos\beta\cos\psi\sin\theta - \\
&\dot{\beta}\dot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta + \dot{\beta}\dot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta + \\
&\ddot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta + \dot{\psi}\dot{\alpha}\cos\alpha\cos\beta\sin\psi\sin\theta - \dot{\psi}\dot{\beta}\sin\alpha\sin\beta\sin\psi\sin\theta + \\
&\dot{\psi}^2\sin\alpha\cos\beta\cos\psi\sin\theta + \dot{\psi}\dot{\theta}\sin\alpha\cos\beta\sin\psi\cos\theta - \\
&\ddot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta - \dot{\theta}\dot{\alpha}\cos\alpha\cos\beta\cos\psi\cos\theta + \dot{\theta}\dot{\beta}\sin\alpha\sin\beta\cos\psi\cos\theta + \\
&\dot{\theta}\dot{\psi}\sin\alpha\cos\beta\sin\psi\cos\theta + \dot{\theta}^2\sin\alpha\cos\beta\cos\psi\sin\theta - \\
&\dot{\alpha}\sin\alpha\cos\beta\cos\theta - \dot{\alpha}\sin\alpha\cos\beta\cos\theta - \dot{\alpha}\sin\alpha\cos\beta\cos\theta - \dot{\alpha}\sin\alpha\cos\beta\cos\theta + \\
&\dot{\alpha}\dot{\theta}\sin\alpha\cos\beta\sin\theta - \\
&\ddot{\beta}\cos\alpha\sin\beta\cos\theta + \dot{\beta}\dot{\alpha}\sin\alpha\sin\beta\cos\theta - \dot{\beta}^2\cos\alpha\cos\beta\cos\theta + \dot{\beta}\dot{\theta}\cos\alpha\sin\beta\sin\theta - \\
&\ddot{\theta}\cos\alpha\cos\beta\sin\theta + \dot{\theta}\dot{\alpha}\sin\alpha\cos\beta\sin\theta + \dot{\theta}\dot{\beta}\cos\alpha\sin\beta\sin\theta - \dot{\theta}^2\cos\alpha\cos\beta\cos\theta)
\end{aligned}
\tag{3-58}
$$

至此，拉格朗日方程中各元素都已求出，将它们代入下面控制方程

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j})+\frac{\partial L}{\partial q_j}=Q_{q_j} \qquad (j=1\sim 8)$$

进行基本的代数运算即得展开式为

$$J_{1x}[\dot{\omega}_{1x}\frac{\partial \omega_{1x}}{\partial \dot{q}_j}+\omega_{1x}(\frac{\partial \omega_{1x}}{\partial \dot{q}_j})_t]+J_{1y}[\dot{\omega}_{1y}\frac{\partial \omega_{1y}}{\partial \dot{q}_j}+\omega_{1y}(\frac{\partial \omega_{1y}}{\partial \dot{q}_j})_t]+J_{1z}[\dot{\omega}_{1z}\frac{\partial \omega_{1z}}{\partial \dot{q}_j}+$$
$$\omega_{1z}(\frac{\partial \omega_{1z}}{\partial \dot{q}_j})_t]+J_{2x}[\dot{\omega}_{2x}\frac{\partial \omega_{2x}}{\partial \dot{q}_j}+\omega_{2x}(\frac{\partial \omega_{2x}}{\partial \dot{q}_j})_t]+J_{2y}[\dot{\omega}_{2y}\frac{\partial \omega_{2y}}{\partial \dot{q}_j}-\omega_{2y}(\frac{\partial \omega_{2y}}{\partial \dot{q}_j})_t]+$$
$$J_{2z}[\dot{\omega}_{2z}\frac{\partial \omega_{2z}}{\partial \dot{q}_j}+\omega_{2z}(\frac{\partial \omega_{2z}}{\partial \dot{q}_j})_t]+m_1\{[\ddot{x}_1\frac{\partial \dot{x}_1}{\partial \dot{q}_j}+\dot{x}_1(\frac{\partial \dot{x}_1}{\partial \dot{q}_j})_t]+[\ddot{y}_1\frac{\partial \dot{y}_1}{\partial \dot{q}_j}+\dot{y}_1(\frac{\partial \dot{y}_1}{\partial \dot{q}_j})_t]+$$
$$[\ddot{z}_1\frac{\partial \dot{z}_1}{\partial \dot{q}_j}+\dot{z}_1(\frac{\partial \dot{z}_1}{\partial \dot{q}_j})_t]\}+m_2\{[\ddot{x}_2\frac{\partial \dot{x}_2}{\partial \dot{q}_j}+\dot{x}_2(\frac{\partial \dot{x}_2}{\partial \dot{q}_j})_t]+[\ddot{y}_2\frac{\partial \dot{y}_2}{\partial \dot{q}_j}+\dot{y}_2(\frac{\partial \dot{y}_2}{\partial \dot{q}_j})_t]+$$
$$[\ddot{z}_2\frac{\partial \dot{z}_2}{\partial \dot{q}_j}+\dot{z}_2(\frac{\partial \dot{z}_2}{\partial \dot{q}_j})_t]\}+J_{1x}\omega_{1x}\frac{\partial \omega_{1x}}{\partial q_j}+J_{1y}\omega_{1y}\frac{\partial \omega_{1y}}{\partial q_j}+J_{1z}\omega_{1z}\frac{\partial \omega_{1z}}{\partial q_j}+$$
$$J_{2x}\omega_{2x}\frac{\partial \omega_{2x}}{\partial q_j}+J_{2y}\omega_{2y}\frac{\partial \omega_{2y}}{\partial q_j}+J_{2z}\omega_{2z}\frac{\partial \omega_{2z}}{\partial q_j}+m_1 g\frac{\partial z_1}{\partial q_j}+m_2 g\frac{\partial z_2}{\partial q_j}=Q_{q_j}$$

（3-59）

对广义坐标 x_1

$$m_1\ddot{x}_1+m_2\ddot{x}_2=0 \qquad (3\text{-}60a)$$

$$m_1\ddot{x}_1+m_2\{\ddot{x}_1-s_1(\ddot{\psi}\cos\psi\sin\theta-\dot{\psi}\dot{\psi}\sin\psi\sin\theta+$$
$$\dot{\psi}\dot{\theta}\cos\psi\cos\theta+\ddot{\theta}\sin\psi\cos\theta+\dot{\theta}\dot{\psi}\cos\psi\cos\theta-$$
$$\dot{\theta}\dot{\theta}\sin\psi\sin\theta)-s_2(-\ddot{\beta}\sin\beta\sin\theta\sin\varphi-\dot{\beta}\dot{\beta}\cos\beta\sin\theta\sin\varphi-$$
$$\dot{\beta}\dot{\theta}\sin\beta\cos\theta\sin\varphi-\dot{\beta}\dot{\varphi}\sin\beta\sin\theta\cos\varphi+\ddot{\theta}\cos\beta\cos\theta\sin\varphi-$$
$$\dot{\theta}\dot{\beta}\sin\beta\cos\theta\sin\varphi-\dot{\theta}\dot{\theta}\cos\beta\sin\theta\sin\varphi+\dot{\theta}\dot{\varphi}\cos\beta\cos\theta\cos\varphi+$$
$$\ddot{\varphi}\cos\beta\sin\theta\cos\varphi-\dot{\varphi}\dot{\beta}\sin\beta\sin\theta\cos\varphi+\dot{\varphi}\dot{\theta}\cos\beta\cos\theta\cos\varphi-$$
$$\dot{\varphi}\dot{\varphi}\cos\beta\sin\theta\sin\varphi+\ddot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta-$$
$$\dot{\alpha}^2\sin\alpha\sin\beta\cos\psi\sin\theta+\dot{\alpha}\dot{\beta}\cos\alpha\cos\beta\cos\psi\sin\theta-$$
$$\dot{\alpha}\dot{\psi}\cos\alpha\sin\beta\sin\psi\sin\theta+\dot{\alpha}\dot{\theta}\cos\alpha\sin\beta\cos\psi\cos\theta+$$
$$\ddot{\beta}\sin\alpha\cos\beta\cos\psi\sin\theta+\dot{\beta}\dot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta-$$
$$\dot{\beta}^2\sin\alpha\sin\beta\cos\psi\sin\theta-\dot{\beta}\dot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta+$$
$$\dot{\beta}\dot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta-\ddot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta+$$

$$\dot{\psi}\dot{\alpha}\cos\alpha\sin\beta\sin\psi\sin\theta+\dot{\psi}\dot{\beta}\sin\alpha\cos\beta\sin\psi\sin\theta+$$
$$\dot{\psi}^2\sin\alpha\sin\beta\cos\psi\sin\theta+\dot{\psi}\dot{\theta}\sin\alpha\sin\beta\sin\psi\cos\theta+$$
$$\ddot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta+\dot{\theta}\dot{\alpha}\cos\alpha\sin\beta\cos\psi\cos\theta+$$
$$\dot{\theta}\dot{\beta}\sin\alpha\cos\beta\cos\psi\cos\theta-\dot{\theta}\dot{\psi}\sin\alpha\sin\beta\sin\psi\cos\theta-$$
$$\dot{\theta}^2\sin\alpha\sin\beta\cos\psi\sin\theta)\}=0$$

（3-61a）

对广义坐标 y_1

$$m_1\ddot{y}_1+m_2\ddot{y}_2=0 \quad \text{（3-60b）}$$

$$m_1\ddot{y}_1+m_2\{\ddot{y}_1+s_1(-\ddot{\psi}\sin\psi\sin\theta-\dot{\psi}^2\cos\psi\sin\theta-\dot{\psi}\dot{\theta}\sin\psi\cos\theta+$$
$$\ddot{\theta}\cos\psi\cos\theta-\dot{\theta}\dot{\psi}\sin\psi\cos\theta-\dot{\theta}^2\cos\psi\sin\theta)+s_2(-\ddot{\alpha}\sin\alpha\cos\psi\sin\theta-$$
$$\dot{\alpha}^2\cos\alpha\cos\psi\sin\theta+\dot{\alpha}\dot{\psi}\sin\alpha\sin\psi\sin\theta-\dot{\alpha}\dot{\theta}\sin\alpha\cos\psi\cos\theta-\ddot{\psi}\cos\alpha\sin\psi\sin\theta-$$
$$\dot{\psi}\dot{\alpha}\sin\alpha\sin\psi\sin\theta+\dot{\psi}^2\cos\alpha\cos\psi\sin\theta+\dot{\psi}\dot{\theta}\cos\alpha\sin\psi\cos\theta+\ddot{\theta}\cos\alpha\cos\psi\cos\theta+$$
$$\dot{\theta}\dot{\alpha}\sin\alpha\cos\psi\cos\theta+\dot{\theta}\dot{\psi}\cos\alpha\sin\psi\cos\theta+\dot{\theta}^2\cos\alpha\cos\psi\sin\theta-\ddot{\alpha}\cos\alpha\cos\theta-$$
$$\dot{\alpha}^2\sin\alpha\cos\theta-\dot{\alpha}\dot{\theta}\cos\alpha\sin\theta+\ddot{\theta}\sin\alpha\sin\theta+\dot{\theta}\dot{\alpha}\cos\alpha\sin\theta+\dot{\theta}^2\sin\alpha\cos\theta)\}=0$$

（3-61b）

对广义坐标 z_1

$$m_1\ddot{z}_1+m_2\ddot{z}_2+m_1g+m_2g=0 \quad \text{（3-60c）}$$

$$m_1\ddot{z}_1+m_2\ddot{z}_1+s_1\dot{\theta}\cos\theta-$$
$$s_2(-\ddot{\beta}\cos\beta\sin\psi\sin\theta+\dot{\beta}^2\sin\beta\sin\psi\sin\theta-\dot{\beta}\dot{\psi}\cos\beta\cos\psi\sin\theta-$$
$$\dot{\beta}\dot{\theta}\cos\beta\sin\psi\cos\theta-\ddot{\psi}\sin\beta\cos\psi\sin\theta-\dot{\psi}\dot{\beta}\cos\beta\cos\psi\sin\theta+\dot{\psi}^2\sin\beta\sin\psi\sin\theta-$$
$$\dot{\psi}\dot{\theta}\sin\beta\cos\psi\cos\theta-\ddot{\theta}\sin\beta\sin\psi\cos\theta-\dot{\theta}\dot{\beta}\sin\beta\sin\psi\cos\theta-\dot{\theta}\dot{\psi}\sin\beta\sin\psi\cos\theta+$$
$$\dot{\theta}^2\sin\beta\sin\psi\sin\theta-\ddot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta+\dot{\alpha}^2\sin\alpha\cos\beta\cos\psi\sin\theta+$$
$$\dot{\alpha}\dot{\beta}\cos\alpha\sin\beta\cos\psi\sin\theta+\dot{\alpha}\dot{\psi}\cos\alpha\cos\beta\sin\psi\sin\theta-\dot{\alpha}\dot{\theta}\cos\alpha\cos\beta\cos\psi\cos\theta+$$
$$\ddot{\beta}\sin\alpha\sin\beta\cos\psi\sin\theta+\dot{\beta}\dot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta+\dot{\beta}^2\sin\alpha\cos\beta\cos\psi\sin\theta-$$
$$\dot{\beta}\dot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta+\dot{\beta}\dot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta+\ddot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta+$$
$$\dot{\psi}\dot{\alpha}\cos\alpha\cos\beta\sin\psi\sin\theta-\dot{\psi}\dot{\beta}\sin\alpha\sin\beta\sin\psi\sin\theta+\dot{\psi}^2\sin\alpha\cos\beta\cos\psi\sin\theta+$$
$$\dot{\psi}\dot{\theta}\sin\alpha\cos\beta\sin\psi\cos\theta-\ddot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta-\dot{\theta}\dot{\alpha}\cos\alpha\cos\beta\cos\psi\cos\theta+$$
$$\dot{\theta}\dot{\beta}\sin\alpha\sin\beta\cos\psi\cos\theta+\dot{\theta}\dot{\psi}\sin\alpha\cos\beta\sin\psi\cos\theta+\dot{\theta}^2\sin\alpha\cos\beta\cos\psi\sin\theta-$$
$$\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\alpha}\sin\alpha\cos\beta\cos\theta+$$

$$\dot{\alpha}\dot{\theta}\sin\alpha\cos\beta\sin\theta-\ddot{\beta}\cos\alpha\sin\beta\cos\theta+\dot{\beta}\dot{\alpha}\sin\alpha\sin\beta\cos\theta-\dot{\beta}^2\cos\alpha\cos\beta\cos\theta+$$
$$\dot{\beta}\dot{\theta}\cos\alpha\sin\beta\sin\theta-\ddot{\theta}\cos\alpha\cos\beta\sin\theta+\dot{\theta}\dot{\alpha}\sin\alpha\cos\beta\sin\theta+\dot{\theta}\dot{\beta}\cos\alpha\sin\beta\sin\theta-$$
$$\dot{\theta}^2\cos\alpha\cos\beta\cos\theta)+m_1g+m_2g=0$$

（3-61c）

对广义坐标 α

$$J_{2y}[\dot{\omega}_{2y}\frac{\partial\omega_{2y}}{\partial\dot{q}_j}-\omega_{2y}(\frac{\partial\omega_{2y}}{\partial\dot{q}_j})_t]+J_{2z}[\omega_{2z}\frac{\partial\omega_{2z}}{\partial\dot{q}_j}+\dot{\omega}_{2z}(\frac{\partial\omega_{2z}}{\partial\dot{q}_j})_t]+$$
$$m_2\{[\ddot{x}_2\frac{\partial\dot{x}_2}{\partial\dot{q}_j}+\dot{x}_2(\frac{\partial\dot{x}_2}{\partial\dot{q}_j})_t]+[\ddot{y}_2\frac{\partial\dot{y}_2}{\partial\dot{q}_j}+\dot{y}_2(\frac{\partial\dot{y}_2}{\partial\dot{q}_j})_t]+[\ddot{z}_2\frac{\partial\dot{z}_2}{\partial\dot{q}_j}+\dot{z}_2(\frac{\partial\dot{z}_2}{\partial\dot{q}_j})_t]\}+$$
$$J_{1x}\omega_{2x}\frac{\partial\omega_{2x}}{\partial q_j}+J_{2y}\omega_{2y}\frac{\partial\omega_{2y}}{\partial q_j}+J_{2z}\omega_{2z}\frac{\partial\omega_{2z}}{\partial q_j}+m_2g\frac{\partial z_2}{\partial q_j}=M_\alpha$$

（3-60d）

$$J_{2y}\{[(\dot{\alpha}\cos\alpha\sin\beta+\dot{\beta}\sin\alpha\cos\beta)(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+$$
$$\sin\alpha\sin\beta(\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\dot{\psi}\dot{\varphi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\dot{\varphi}\sin\varphi)-$$
$$\dot{\beta}\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\cos\beta(\ddot{\psi}\sin\theta\cos\varphi+\dot{\psi}\dot{\theta}\cos\theta\cos\varphi-\dot{\varphi}\dot{\psi}\sin\theta\sin\varphi-$$
$$\ddot{\theta}\sin\varphi-\dot{\theta}\dot{\varphi}\cos\varphi)+\ddot{\alpha}\cos\beta-\dot{\alpha}\dot{\beta}\sin\beta+(-\dot{\alpha}\sin\alpha\sin\beta+\dot{\beta}\cos\alpha\cos\beta)(\dot{\psi}\cos\theta+$$
$$\dot{\varphi})+\cos\alpha\sin\beta(\ddot{\psi}\cos\theta+\ddot{\varphi}-\dot{\psi}\dot{\theta}\sin\theta)]\cos\beta-[\sin\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+$$
$$\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\dot{\alpha}\cos\beta+\cos\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})](-\dot{\beta}\sin\beta)\}+$$
$$J_{2z}\{[(\dot{\alpha}\cos\beta\cos\alpha-\dot{\beta}\cos\beta\sin\alpha)(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+\cos\beta\sin\alpha(\ddot{\psi}\sin\theta\sin\varphi+$$
$$\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\dot{\psi}\dot{\varphi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\dot{\varphi}\sin\varphi)-\dot{\beta}\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)-$$
$$\sin\beta(\ddot{\psi}\sin\theta\cos\varphi+\dot{\psi}\dot{\theta}\cos\theta\cos\varphi-\dot{\psi}\dot{\varphi}\sin\theta\sin\varphi-\ddot{\theta}\sin\varphi-\dot{\theta}\dot{\varphi}\cos\varphi)+$$
$$(-\dot{\alpha}\sin\alpha\cos\beta-\dot{\beta}\cos\alpha\sin\beta)(\dot{\psi}\cos\theta+\dot{\varphi})+\cos\alpha\cos\beta(\ddot{\psi}\cos\theta-\dot{\psi}\dot{\theta}\sin\theta+\ddot{\varphi})-$$
$$\ddot{\alpha}\sin\beta-\dot{\alpha}\dot{\beta}\cos\beta](-\sin\beta)+[\sin\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\beta(\dot{\psi}\sin\theta\cos\varphi-$$
$$\dot{\theta}\sin\varphi)+\cos\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})-\dot{\alpha}\sin\beta](-\dot{\beta}\cos\beta)\}+$$
$$m_2\{[\ddot{x}_1-s_1(\ddot{\psi}\cos\psi\sin\theta-\dot{\psi}\dot{\psi}\sin\psi\sin\theta+\dot{\psi}\dot{\theta}\cos\psi\cos\theta+\ddot{\theta}\sin\psi\cos\theta+$$
$$\dot{\theta}\dot{\psi}\cos\psi\cos\theta-\dot{\theta}\dot{\theta}\sin\psi\sin\theta)-s_2(-\ddot{\beta}\sin\beta\sin\theta\sin\varphi-\dot{\beta}\dot{\beta}\cos\beta\sin\theta\sin\varphi-$$
$$\dot{\beta}\dot{\theta}\sin\beta\cos\theta\sin\varphi-\dot{\beta}\dot{\varphi}\sin\beta\sin\theta\cos\varphi+\ddot{\theta}\cos\beta\cos\theta\sin\varphi-\dot{\theta}\dot{\beta}\sin\beta\cos\theta\sin\varphi-$$
$$\dot{\theta}\dot{\theta}\cos\beta\sin\theta\sin\varphi+\dot{\theta}\dot{\varphi}\cos\beta\cos\theta\cos\varphi+\ddot{\varphi}\cos\beta\sin\theta\cos\varphi-\dot{\varphi}\dot{\beta}\sin\beta\sin\theta\cos\varphi+$$

$\dot{\varphi}\dot{\theta}\cos\beta\cos\theta\cos\varphi-\dot{\varphi}\dot{\varphi}\cos\beta\sin\theta\sin\varphi+\ddot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta-$

$\dot{\alpha}^2\sin\alpha\sin\beta\cos\psi\sin\theta+\dot{\alpha}\dot{\beta}\cos\alpha\cos\beta\cos\psi\sin\theta-\dot{\alpha}\dot{\psi}\cos\alpha\sin\beta\sin\psi\sin\theta+$

$\dot{\alpha}\dot{\theta}\cos\alpha\sin\beta\cos\psi\cos\theta+\ddot{\beta}\sin\alpha\cos\beta\cos\psi\sin\theta+\dot{\beta}\dot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta-$

$\dot{\beta}^2\sin\alpha\sin\beta\cos\psi\sin\theta-\dot{\beta}\dot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta+\dot{\beta}\dot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta-$

$\ddot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta+\dot{\psi}\dot{\alpha}\cos\alpha\sin\beta\sin\psi\sin\theta+\dot{\psi}\dot{\beta}\sin\alpha\cos\beta\sin\psi\sin\theta+$

$\dot{\psi}^2\sin\alpha\sin\beta\cos\psi\sin\theta+\dot{\psi}\dot{\theta}\sin\alpha\sin\beta\sin\psi\cos\theta+\ddot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta+$

$\dot{\theta}\dot{\alpha}\cos\alpha\sin\beta\cos\psi\cos\theta+\dot{\theta}\dot{\beta}\sin\alpha\cos\beta\cos\psi\cos\theta-\dot{\theta}\dot{\psi}\sin\alpha\sin\beta\sin\psi\cos\theta-$

$\dot{\theta}^2\sin\alpha\sin\beta\cos\psi\sin\theta)]\times[-s_2\cos\alpha\sin\beta\cos\psi\sin\theta]+[\dot{x}_1-s_1(\dot{\psi}\cos\psi\sin\theta+$

$\dot{\theta}\sin\psi\cos\theta)-s_2(-\dot{\beta}\sin\beta\sin\theta\sin\varphi+\dot{\theta}\cos\beta\cos\theta\sin\varphi+\dot{\varphi}\cos\beta\sin\theta\cos\varphi+$

$\dot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta+\dot{\beta}\sin\alpha\cos\beta\cos\psi\sin\theta-\dot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta+$

$\dot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta)]\times[-s_2(-\dot{\alpha}\sin\alpha\sin\beta\cos\psi\sin\theta+\dot{\beta}\cos\alpha\cos\beta\cos\psi\sin\theta-$

$\dot{\psi}\cos\alpha\sin\beta\sin\theta\sin\psi+\dot{\theta}\cos\psi\cos\theta\sin\beta\cos\alpha)]+[\ddot{y}_1+s_1(-\ddot{\psi}\sin\psi\sin\theta-$

$\dot{\psi}^2\cos\psi\sin\theta-\dot{\psi}\dot{\theta}\sin\psi\cos\theta+\ddot{\theta}\cos\psi\cos\theta-\dot{\theta}\dot{\psi}\sin\psi\cos\theta-\dot{\theta}^2\cos\psi\sin\theta)+$

$s_2(-\ddot{\alpha}\sin\alpha\cos\psi\sin\theta-\dot{\alpha}^2\cos\alpha\cos\psi\sin\theta+\dot{\alpha}\dot{\psi}\sin\alpha\sin\psi\sin\theta-$

$\dot{\alpha}\dot{\theta}\sin\alpha\cos\psi\cos\theta-\ddot{\psi}\cos\alpha\sin\psi\sin\theta-\dot{\psi}\dot{\alpha}\sin\alpha\sin\psi\sin\theta+\dot{\psi}^2\cos\alpha\cos\psi\sin\theta+$

$\dot{\psi}\dot{\theta}\cos\alpha\sin\psi\cos\theta+\ddot{\theta}\cos\alpha\cos\psi\cos\theta+\dot{\theta}\dot{\alpha}\sin\alpha\cos\psi\cos\theta+\dot{\theta}\dot{\psi}\cos\alpha\sin\psi\cos\theta+$

$\dot{\theta}^2\cos\alpha\cos\psi\sin\theta-\ddot{\alpha}\cos\alpha\cos\theta-\dot{\alpha}^2\sin\alpha\cos\theta-\dot{\alpha}\dot{\theta}\cos\alpha\sin\theta+\ddot{\theta}\sin\alpha\sin\theta+$

$\dot{\theta}\dot{\alpha}\cos\alpha\sin\theta+\dot{\theta}^2\sin\alpha\cos\theta)]\times[s_2(-\sin\alpha\cos\psi\sin\theta-\cos\alpha\cos\theta)]+[\dot{y}_1+$

$s_1(-\dot{\psi}\sin\psi\sin\theta+\dot{\theta}\cos\psi\cos\theta)+s_2(-\dot{\alpha}\sin\alpha\cos\psi\sin\theta-\dot{\psi}\cos\alpha\sin\psi\sin\theta+$

$\dot{\theta}\cos\alpha\cos\psi\cos\theta-\dot{\alpha}\cos\alpha\cos\theta+\dot{\theta}\sin\alpha\sin\theta)]\times[s_2(-\dot{\alpha}\cos\alpha\cos\psi\sin\theta+$

$\dot{\psi}\sin\alpha\sin\psi\sin\theta-\dot{\theta}\sin\alpha\cos\psi\cos\theta+\dot{\alpha}\sin\alpha\cos\theta+\dot{\theta}\cos\alpha\sin\theta)]+$

$[\ddot{z}_1+s_1\dot{\theta}\cos\theta-s_2(-\ddot{\beta}\cos\beta\sin\psi\sin\theta+\dot{\beta}^2\sin\beta\sin\psi\sin\theta-\dot{\beta}\dot{\psi}\cos\beta\cos\psi\sin\theta-$

$\dot{\beta}\dot{\theta}\cos\beta\sin\psi\cos\theta-\ddot{\psi}\sin\beta\cos\psi\sin\theta-\dot{\psi}\dot{\beta}\cos\beta\cos\psi\sin\theta+\dot{\psi}^2\sin\beta\sin\psi\sin\theta-$

$\dot{\psi}\dot{\theta}\sin\beta\cos\psi\cos\theta-\ddot{\theta}\sin\beta\sin\psi\cos\theta-\dot{\theta}\dot{\beta}\sin\beta\sin\psi\cos\theta-\dot{\theta}\dot{\psi}\sin\beta\sin\psi\cos\theta+$

$\dot{\theta}^2\sin\beta\sin\psi\sin\theta-\ddot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta+\dot{\alpha}^2\sin\alpha\cos\beta\cos\psi\sin\theta+$

$\dot{\alpha}\dot{\beta}\cos\alpha\sin\beta\cos\psi\sin\theta+\dot{\alpha}\dot{\psi}\cos\alpha\cos\beta\sin\psi\sin\theta-\dot{\alpha}\dot{\theta}\cos\alpha\cos\beta\cos\psi\cos\theta+$

$\ddot{\beta}\sin\alpha\sin\beta\cos\psi\sin\theta+\dot{\beta}\dot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta+\dot{\beta}^2\sin\alpha\cos\beta\cos\psi\sin\theta-$

$\dot{\beta}\dot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta+\dot{\beta}\dot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta+\ddot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta+$

$\dot{\psi}\dot{\alpha}\cos\alpha\cos\beta\sin\psi\sin\theta-\dot{\psi}\dot{\beta}\sin\alpha\sin\beta\sin\psi\sin\theta+\dot{\psi}^2\sin\alpha\cos\beta\cos\psi\sin\theta+$

$\dot{\psi}\dot{\theta}\sin\alpha\cos\beta\sin\psi\cos\theta-\ddot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta-\dot{\theta}\dot{\alpha}\cos\alpha\cos\beta\cos\psi\cos\theta+$

$\dot{\theta}\dot{\beta}\sin\alpha\sin\beta\cos\psi\cos\theta+\dot{\theta}\dot{\psi}\sin\alpha\cos\beta\sin\psi\cos\theta+\dot{\theta}^2\sin\alpha\cos\beta\cos\psi\sin\theta-$
$\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\alpha}\sin\alpha\cos\beta\cos\theta+$
$\dot{\alpha}\dot{\theta}\sin\alpha\cos\beta\sin\theta-\ddot{\beta}\cos\alpha\sin\beta\cos\theta+\dot{\beta}\dot{\alpha}\sin\alpha\sin\beta\cos\theta-\dot{\beta}^2\cos\alpha\cos\beta\cos\theta+$
$\dot{\beta}\dot{\theta}\cos\alpha\sin\beta\sin\theta-\ddot{\theta}\cos\alpha\cos\beta\sin\theta+\dot{\theta}\dot{\alpha}\sin\alpha\cos\beta\sin\theta+\dot{\theta}\dot{\beta}\cos\alpha\sin\beta\sin\theta-$
$\dot{\theta}^2\cos\alpha\cos\beta\cos\theta)]\times[s_2(\cos\alpha\cos\beta\cos\psi\sin\theta+\cos\theta\sin\alpha\cos\beta)]+[\dot{z}_1+s_1\sin\theta-$
$s_2(-\dot{\beta}\cos\beta\sin\psi\sin\theta-\dot{\psi}\sin\beta\cos\psi\sin\theta-\dot{\theta}\sin\beta\sin\psi\cos\theta-$
$\dot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta+\dot{\beta}\sin\alpha\sin\beta\cos\psi\sin\theta+\dot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta-$
$\dot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta-\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\beta}\cos\alpha\sin\beta\cos\theta-\dot{\theta}\cos\alpha\cos\beta\sin\theta)]\times$
$[s_2(-\dot{\alpha}\sin\alpha\cos\beta\cos\psi\sin\theta-\dot{\beta}\cos\alpha\sin\beta\cos\psi\sin\theta-\cos\alpha\cos\beta\sin\psi\sin\theta+$
$\dot{\theta}\cos\alpha\cos\beta\cos\psi\cos\theta+\dot{\theta}\cos\psi\cos\theta\sin\alpha\cos\beta+\dot{\alpha}\cos\theta\cos\alpha\cos\beta-$
$\dot{\beta}\cos\theta\sin\alpha\sin\beta-\dot{\theta}\sin\alpha\cos\beta\sin\theta)]\}+J_{1x}[\cos\alpha(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-$
$\sin\alpha(\dot{\psi}\cos\theta+\dot{\varphi})+\dot{\beta}]\times[-\sin\alpha(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\cos\alpha(\dot{\psi}\cos\theta+\dot{\varphi})+\dot{\beta}]+$
$J_{2y}[\sin\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\dot{\alpha}\cos\beta+$
$\cos\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})]\times[\cos\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\alpha\sin\beta(\dot{\psi}\cos\theta+$
$\dot{\varphi})]+J_{2z}[\sin\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\dot{\beta}(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+$
$\cos\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})-\dot{\alpha}\sin\beta]\times[\cos\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-$
$\sin\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})]+m_2g\times[s_2(\cos\alpha\cos\beta\cos\psi\sin\theta+\sin\alpha\cos\beta\cos\theta)]=M_\alpha$

（3-61d）

对广义坐标β

$$J_{2x}\dot{\omega}_{2x}+m_2\{[\ddot{x}_2\frac{\partial\dot{x}_2}{\partial\dot{q}_j}+\dot{x}_2(\frac{\partial\dot{x}_2}{\partial\dot{q}_j})_t]+[\ddot{z}_2\frac{\partial\dot{z}_2}{\partial\dot{q}_j}+\dot{z}_2(\frac{\partial\dot{z}_2}{\partial\dot{q}_j})_t]\}+J_{2y}\omega_{2y}\frac{\partial\omega_{2y}}{\partial q_j}+$$

$$J_{2z}\omega_{2z}\frac{\partial\omega_{2z}}{\partial q_j}+m_2g\frac{\partial z_2}{\partial q_j}=M_\beta \qquad (3\text{-}60\text{e})$$

$J_{2x}\times[-\dot{\alpha}\sin\alpha(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi+\dot{\psi}\sin\theta\sin\varphi)+\cos\alpha(\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+$
$\ddot{\psi}\sin\theta\sin\varphi+\dot{\varphi}\dot{\psi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\sin\varphi)-\dot{\alpha}\cos\alpha(\dot{\psi}\cos\theta+\dot{\varphi})-$
$\sin\alpha(-\dot{\psi}\dot{\theta}\sin\theta+\ddot{\psi}\cos\theta+\ddot{\varphi})+\ddot{\beta}]+m_2\{[\ddot{x}_1-s_1(\ddot{\psi}\cos\psi\sin\theta-\dot{\psi}\dot{\psi}\sin\psi\sin\theta+$
$\dot{\psi}\dot{\theta}\cos\psi\cos\theta+\ddot{\theta}\sin\psi\cos\theta+\dot{\theta}\dot{\psi}\cos\psi\cos\theta-\dot{\theta}\dot{\theta}\sin\psi\sin\theta)-$
$s_2(-\ddot{\beta}\sin\beta\sin\theta\sin\varphi-\dot{\beta}\dot{\beta}\cos\beta\sin\theta\sin\varphi-\dot{\beta}\dot{\theta}\sin\beta\cos\theta\sin\varphi-$

$$\dot\beta\dot\varphi\sin\beta\sin\theta\cos\varphi+\ddot\theta\cos\beta\cos\theta\sin\varphi-\dot\theta\dot\beta\sin\beta\cos\theta\sin\varphi-$$
$$\dot\theta\dot\theta\cos\beta\sin\theta\sin\varphi+\dot\theta\dot\varphi\cos\beta\cos\theta\cos\varphi+\ddot\varphi\cos\beta\sin\theta\cos\varphi-$$
$$\dot\varphi\dot\beta\sin\beta\sin\theta\cos\varphi+\dot\varphi\dot\theta\cos\beta\cos\theta\cos\varphi-\dot\varphi\dot\varphi\cos\beta\sin\theta\sin\varphi+$$
$$\ddot\alpha\cos\alpha\sin\beta\cos\psi\sin\theta-\dot\alpha^2\sin\alpha\sin\beta\cos\psi\sin\theta+\dot\alpha\dot\beta\cos\alpha\cos\beta\cos\psi\sin\theta-$$
$$\dot\alpha\dot\psi\cos\alpha\sin\beta\sin\psi\sin\theta+\dot\alpha\dot\theta\cos\alpha\sin\beta\cos\psi\cos\theta+\ddot\beta\sin\alpha\cos\beta\cos\psi\sin\theta+$$
$$\dot\beta\dot\alpha\cos\alpha\cos\beta\cos\psi\sin\theta-\dot\beta^2\sin\alpha\sin\beta\cos\psi\sin\theta-\dot\beta\dot\psi\sin\alpha\cos\beta\sin\psi\sin\theta+$$
$$\dot\beta\dot\theta\sin\alpha\cos\beta\cos\psi\cos\theta-\ddot\psi\sin\alpha\sin\beta\sin\psi\sin\theta+\dot\psi\dot\alpha\cos\alpha\sin\beta\sin\psi\sin\theta+$$
$$\dot\psi\dot\beta\sin\alpha\cos\beta\sin\psi\sin\theta+\dot\psi^2\sin\alpha\sin\beta\cos\psi\sin\theta+\dot\psi\dot\theta\sin\alpha\sin\beta\sin\psi\cos\theta+$$
$$\ddot\theta\sin\alpha\sin\beta\cos\psi\cos\theta+\dot\theta\dot\alpha\cos\alpha\sin\beta\cos\psi\cos\theta+\dot\theta\dot\beta\sin\alpha\cos\beta\cos\psi\cos\theta-$$
$$\dot\theta\dot\psi\sin\alpha\sin\beta\sin\psi\cos\theta-\dot\theta^2\sin\alpha\sin\beta\cos\psi\sin\theta)]\times[-s_2(-\sin\beta\sin\theta\sin\varphi+$$
$$\sin\alpha\cos\beta\cos\psi\sin\theta)]+[\dot x_1-s_1(\dot\psi\cos\psi\sin\theta+\dot\theta\sin\psi\cos\theta)-$$
$$s_2(-\dot\beta\sin\beta\sin\theta\sin\varphi+\dot\theta\cos\beta\cos\theta\sin\varphi+\dot\varphi\cos\beta\sin\theta\cos\varphi+$$
$$\dot\alpha\cos\alpha\sin\beta\cos\psi\sin\theta+\dot\beta\sin\alpha\cos\beta\cos\psi\sin\theta-\dot\psi\sin\alpha\sin\beta\sin\psi\sin\theta+$$
$$\dot\theta\sin\alpha\sin\beta\cos\psi\cos\theta)]\times[-s_2(-\dot\beta\cos\beta\sin\theta\sin\varphi-\dot\theta\sin\beta\cos\theta\sin\varphi-$$
$$\dot\varphi\sin\beta\sin\theta\cos\varphi+\dot\alpha\cos\alpha\cos\beta\cos\psi\sin\theta-\dot\beta\sin\alpha\sin\beta\cos\psi\sin\theta-$$
$$\dot\psi\sin\alpha\cos\beta\sin\psi\sin\theta+\dot\theta\cos\psi\cos\theta\sin\alpha\cos\beta)]+[\ddot z_1+s_1\dot\theta\cos\theta-$$
$$s_2(-\ddot\beta\cos\beta\sin\psi\sin\theta+\dot\beta^2\sin\beta\sin\psi\sin\theta-\dot\beta\dot\psi\cos\beta\cos\psi\sin\theta-$$
$$\dot\beta\dot\theta\cos\beta\sin\psi\cos\theta-\ddot\psi\sin\beta\cos\psi\sin\theta-\dot\psi\dot\beta\cos\beta\cos\psi\sin\theta+$$
$$\dot\psi^2\sin\beta\sin\psi\sin\theta-\dot\psi\dot\theta\sin\beta\cos\psi\cos\theta-\ddot\theta\sin\beta\sin\psi\cos\theta-$$
$$\dot\theta\dot\beta\sin\beta\sin\psi\cos\theta-\dot\theta\dot\psi\sin\beta\sin\psi\cos\theta+\dot\theta^2\sin\beta\sin\psi\sin\theta-$$
$$\ddot\alpha\cos\alpha\cos\beta\cos\psi\sin\theta+\dot\alpha^2\sin\alpha\cos\beta\cos\psi\sin\theta+\dot\alpha\dot\beta\cos\alpha\sin\beta\cos\psi\sin\theta+$$
$$\dot\alpha\dot\psi\cos\alpha\cos\beta\sin\psi\sin\theta-\dot\alpha\dot\theta\cos\alpha\cos\beta\cos\psi\cos\theta+\ddot\beta\sin\alpha\sin\beta\cos\psi\sin\theta+$$
$$\dot\alpha\cos\alpha\sin\beta\cos\psi\sin\theta+\dot\beta^2\sin\alpha\cos\beta\cos\psi\sin\theta-\dot\beta\dot\psi\sin\alpha\sin\beta\sin\psi\sin\theta+$$
$$\dot\beta\dot\theta\sin\alpha\sin\beta\cos\psi\cos\theta+\ddot\psi\sin\alpha\cos\beta\sin\psi\sin\theta+\dot\psi\dot\alpha\cos\alpha\cos\beta\sin\psi\sin\theta-$$
$$\dot\psi\dot\beta\sin\alpha\sin\beta\sin\psi\sin\theta+\dot\psi^2\sin\alpha\cos\beta\cos\psi\sin\theta+\dot\psi\dot\theta\sin\alpha\cos\beta\sin\psi\cos\theta-$$
$$\ddot\theta\sin\alpha\cos\beta\cos\psi\cos\theta-\dot\theta\dot\alpha\cos\alpha\cos\beta\cos\psi\cos\theta+\dot\theta\dot\beta\sin\alpha\sin\beta\cos\psi\cos\theta+$$
$$\dot\theta\dot\psi\sin\alpha\cos\beta\sin\psi\cos\theta+\dot\theta^2\sin\alpha\cos\beta\cos\psi\sin\theta-\dot\alpha\sin\alpha\cos\beta\cos\theta-$$
$$\dot\alpha\sin\alpha\cos\beta\cos\theta-\dot\alpha\sin\alpha\cos\beta\cos\theta-\dot\alpha\sin\alpha\cos\beta\cos\theta+\dot\alpha\dot\theta\sin\alpha\cos\beta\sin\theta-$$
$$\ddot\beta\cos\alpha\sin\beta\cos\theta+\dot\beta\dot\alpha\sin\alpha\sin\beta\cos\theta-\dot\beta^2\cos\alpha\cos\beta\cos\theta+$$
$$\dot\beta\dot\theta\cos\alpha\sin\beta\sin\theta-\ddot\theta\cos\alpha\cos\beta\sin\theta+\dot\theta\dot\alpha\sin\alpha\cos\beta\sin\theta+$$

$\dot{\theta}\dot{\beta}\cos\alpha\sin\beta\sin\theta-\dot{\theta}^2\cos\alpha\cos\beta\cos\theta)]\times[-s_2(-\cos\beta\sin\psi\sin\theta+$

$\sin\alpha\sin\beta\cos\psi\sin\theta-\cos\theta\cos\alpha\sin\beta)]+[\dot{z}_1+s_1\sin\theta-s_2(-\dot{\beta}\cos\beta\sin\psi\sin\theta-$

$\dot{\psi}\sin\beta\cos\psi\sin\theta-\dot{\theta}\sin\beta\sin\psi\cos\theta-\dot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta+$

$\dot{\beta}\sin\alpha\sin\beta\cos\psi\sin\theta+\dot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta-\dot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta-$

$\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\beta}\cos\alpha\sin\beta\cos\theta-\dot{\theta}\cos\alpha\cos\beta\sin\theta)]\times$

$[-s_2(\dot{\beta}\sin\beta\sin\psi\sin\theta-\dot{\psi}\cos\beta\cos\psi\sin\theta-\dot{\theta}\cos\beta\sin\psi\cos\theta+$

$\dot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta+\dot{\beta}\sin\alpha\cos\beta\cos\psi\sin\theta-\dot{\psi}\sin\alpha\sin\beta\sin\psi\cos\theta+$

$\dot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta+\dot{\theta}\sin\theta\cos\alpha\sin\beta+\dot{\alpha}\cos\theta\sin\alpha\sin\beta-$

$\dot{\beta}\cos\theta\cos\alpha\cos\beta)]\}+J_{2y}[\sin\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+$

$\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\dot{\alpha}\cos\beta+\cos\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})]\times$

$[\sin\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)-$

$\dot{\alpha}\sin\beta+\cos\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})]+J_{2z}[\sin\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-$

$\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\cos\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})-\dot{\alpha}\sin\beta]\times$

$[-\sin\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)-$

$\cos\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})-\dot{\alpha}\cos\beta]+m_2g\times[-s_2(\sin\alpha\sin\beta\cos\psi\sin\theta-$

$\cos\alpha\sin\beta\cos\theta)]=M_\beta$

（3-61e）

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$$J_{1x}[\dot{\omega}_{1x}\frac{\partial\dot{\omega}_{1x}}{\partial\dot{q}_j}+\omega_{1x}(\frac{\partial\dot{\omega}_{1x}}{\partial\dot{q}_j})_t]+J_{1y}[\dot{\omega}_{1y}\frac{\partial\omega_{1y}}{\partial\dot{q}_j}+\omega_{1y}(\frac{\partial\omega_{1y}}{\partial\dot{q}_j})_t]+$$

$$J_{1z}[\dot{\omega}_{1z}\frac{\partial\omega_{1z}}{\partial\dot{q}_j}+\omega_{1z}(\frac{\partial\omega_{1z}}{\partial\dot{q}_j})_t]+J_{2x}[\dot{\omega}_{2z}\frac{\partial\omega_{2x}}{\partial\dot{q}_j}+\omega_{2x}(\frac{\partial\omega_{2x}}{\partial\dot{q}_j})_t]+$$

$$J_{2y}[\dot{\omega}_{2y}\frac{\partial\omega_{2y}}{\partial\dot{q}_j}-\omega_{2y}(\frac{\partial\omega_{2y}}{\partial\dot{q}_j})_t]+J_{2z}[\dot{\omega}_{2z}\frac{\partial\omega_{2z}}{\partial\dot{q}_j}+\omega_{2z}(\frac{\partial\omega_{2z}}{\partial\dot{q}_j})_t]+ \quad (3\text{-}60f)$$

$$m_2\{[\ddot{x}_2\frac{\partial\dot{x}_2}{\partial\dot{q}_j}+\dot{x}_2(\frac{\partial\dot{x}_2}{\partial\dot{q}_j})_t]+[\ddot{y}_2\frac{\partial\dot{y}_2}{\partial\dot{q}_j}+\dot{y}_2(\frac{\partial\dot{y}_2}{\partial\dot{q}_j})_t]+[\ddot{z}_2\frac{\partial\dot{z}_2}{\partial\dot{q}_j}+$$

$$\dot{z}_2(\frac{\partial\dot{z}_2}{\partial\dot{q}_j})_t]\}+m_2g\frac{\partial z_2}{\partial q_j}=0$$

$[\sin\theta\sin\varphi]+[\cos\alpha(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\alpha(\dot{\psi}\cos\theta+\dot{\varphi})+\dot{\beta}]\times$

$$
\begin{aligned}
&J_{1x}\{[\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}(\dot{\theta}\cos\theta\sin\varphi+\dot{\varphi}\sin\theta\cos\varphi)+\ddot{\theta}\cos\varphi-\dot{\varphi}\dot{\theta}\sin\varphi]\times\\
&[\dot{\theta}\cos\theta\sin\varphi+\dot{\varphi}\sin\theta\cos\varphi]\}+J_{1y}\{[\ddot{\psi}\sin\theta\cos\varphi+\dot{\psi}(\dot{\theta}\cos\theta\cos\varphi-\\
&\dot{\varphi}\sin\theta\sin\varphi)-\ddot{\theta}\sin\varphi-\dot{\varphi}\dot{\theta}\cos\varphi]\times[\sin\theta\cos\varphi]+[\sin\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\\
&\dot{\theta}\cos\varphi)+\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\dot{\alpha}\cos\beta+\cos\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})]\times\\
&[\dot{\theta}\cos\theta\cos\varphi-\dot{\varphi}\sin\theta\sin\varphi]\}+J_{1z}\{[\ddot{\psi}\cos\theta-\dot{\theta}\dot{\psi}\sin\theta+\ddot{\varphi}]\times[\cos\theta]+\\
&[\sin\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\\
&\cos\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})-\dot{\alpha}\sin\beta]\times[-\dot{\theta}\sin\theta]\}+J_{2x}\{[-\dot{\alpha}\sin\alpha(\dot{\psi}\sin\theta\sin\varphi+\\
&\dot{\theta}\cos\varphi+\ddot{\psi}\sin\theta\sin\varphi)+\cos\alpha(\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\ddot{\psi}\sin\theta\sin\varphi+\dot{\varphi}\dot{\psi}\sin\theta\cos\varphi+\\
&\ddot{\theta}\cos\varphi-\dot{\theta}\sin\varphi)-\dot{\alpha}\cos\alpha(\dot{\psi}\cos\theta+\dot{\varphi})-\sin\alpha(-\dot{\psi}\dot{\theta}\sin\theta+\ddot{\psi}\cos\theta+\ddot{\varphi})+\ddot{\beta}]\times\\
&[\cos\alpha\sin\theta\sin\varphi-\sin\alpha\cos\theta]+[\cos\alpha(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\alpha(\dot{\psi}\cos\theta+\dot{\varphi})+\\
&\dot{\beta}]\times[-\dot{\alpha}\sin\alpha\sin\theta\sin\varphi+\dot{\theta}\cos\alpha\cos\theta\sin\varphi+\dot{\varphi}\cos\alpha\sin\theta\cos\varphi-\dot{\alpha}\cos\alpha\cos\theta+\\
&\dot{\theta}\sin\alpha\sin\theta]\}+J_{2y}\{[(\dot{\alpha}\cos\alpha\sin\beta+\dot{\beta}\sin\alpha\cos\beta)(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+\\
&\sin\alpha\sin\beta(\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\dot{\psi}\dot{\varphi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\dot{\varphi}\sin\varphi)-\\
&\dot{\beta}\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\cos\beta(\ddot{\psi}\sin\theta\cos\varphi+\dot{\psi}\dot{\theta}\cos\theta\cos\varphi-\\
&\dot{\varphi}\dot{\psi}\sin\theta\sin\varphi-\ddot{\theta}\sin\varphi-\dot{\theta}\dot{\varphi}\cos\varphi)+\ddot{\alpha}\cos\beta-\dot{\alpha}\dot{\beta}\sin\beta+(-\dot{\alpha}\sin\alpha\sin\beta+\\
&\dot{\beta}\cos\alpha\cos\beta)(\dot{\psi}\cos\theta+\dot{\varphi})+\cos\alpha\sin\beta(\ddot{\psi}\cos\theta+\ddot{\varphi}-\dot{\psi}\dot{\theta}\sin\theta)]\times\\
&[\sin\alpha\sin\beta\sin\theta\sin\varphi+\cos\beta\sin\theta\cos\varphi+\cos\alpha\sin\beta\cos\theta]-[\sin\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\\
&\dot{\theta}\cos\varphi)+\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\dot{\alpha}\cos\beta+\cos\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})]\times\\
&[\dot{\alpha}\cos\alpha\sin\beta\sin\theta\sin\varphi+\dot{\beta}\sin\alpha\cos\beta\sin\theta\sin\varphi+\dot{\theta}\sin\alpha\sin\beta\cos\theta\sin\varphi+\\
&\dot{\varphi}\sin\alpha\sin\beta\sin\theta\cos\varphi-\dot{\beta}\sin\beta\sin\theta\cos\varphi+\dot{\theta}\cos\beta\cos\theta\cos\varphi-\\
&\dot{\varphi}\cos\beta\sin\theta\sin\varphi-\dot{\alpha}\sin\alpha\sin\beta\cos\theta+\dot{\beta}\cos\alpha\cos\beta\cos\theta-\\
&\dot{\theta}\cos\alpha\sin\beta\sin\theta]\}+J_{2z}\{[(\dot{\alpha}\cos\beta\cos\alpha-\dot{\beta}\cos\beta\sin\alpha)(\dot{\psi}\sin\theta\sin\varphi+\\
&\dot{\theta}\cos\varphi)+\cos\beta\sin\alpha(\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\dot{\psi}\dot{\varphi}\sin\theta\cos\varphi+\\
&\ddot{\theta}\cos\varphi-\dot{\theta}\dot{\varphi}\sin\varphi)-\dot{\beta}\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)-\sin\beta(\ddot{\psi}\sin\theta\cos\varphi+\\
&\dot{\psi}\dot{\theta}\cos\theta\cos\varphi-\dot{\psi}\dot{\varphi}\sin\theta\sin\varphi-\ddot{\theta}\sin\varphi-\dot{\theta}\dot{\varphi}\cos\varphi)+(-\dot{\alpha}\sin\alpha\cos\beta-\\
&\dot{\beta}\cos\alpha\sin\beta)(\dot{\psi}\cos\theta+\dot{\varphi})+\cos\alpha\cos\beta(\ddot{\psi}\cos\theta-\dot{\psi}\dot{\theta}\sin\theta+\ddot{\varphi})-\ddot{\alpha}\sin\beta-\\
&\dot{\alpha}\dot{\beta}\cos\beta]\times[\sin\alpha\cos\beta\sin\theta\sin\varphi-\sin\beta\sin\theta\cos\varphi+\cos\alpha\cos\beta\cos\theta]+\\
&[\sin\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\\
&\cos\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})-\dot{\alpha}\sin\beta][\dot{\alpha}\cos\beta\cos\alpha\sin\varphi\sin\theta-\\
&\dot{\beta}\sin\beta\sin\alpha\sin\theta\sin\varphi+\dot{\theta}\cos\beta\sin\alpha\cos\theta\sin\varphi+\dot{\varphi}\cos\beta\sin\alpha\sin\theta\cos\varphi-
\end{aligned}
$$

$$\dot{\beta}\cos\beta\sin\theta\cos\varphi-\dot{\theta}\sin\beta\cos\theta\cos\varphi+\dot{\varphi}\sin\beta\sin\theta\sin\varphi-$$
$$\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\beta}\cos\alpha\sin\beta\cos\theta-\dot{\theta}\cos\alpha\cos\beta\sin\theta]\}+$$
$$m_2\{[\ddot{x}_1-s_1(\ddot{\psi}\cos\psi\sin\theta-\dot{\psi}\dot{\psi}\sin\psi\sin\theta+\dot{\psi}\dot{\theta}\cos\psi\cos\theta+\ddot{\theta}\sin\psi\cos\theta+$$
$$\dot{\theta}\dot{\psi}\cos\psi\cos\theta-\dot{\theta}\dot{\theta}\sin\psi\sin\theta)-s_2(-\ddot{\beta}\sin\beta\sin\theta\sin\varphi-\dot{\beta}\dot{\beta}\cos\beta\sin\theta\sin\varphi-$$
$$\dot{\beta}\dot{\theta}\sin\beta\cos\theta\sin\varphi-\dot{\beta}\dot{\varphi}\sin\beta\sin\theta\cos\varphi+\ddot{\theta}\cos\beta\cos\theta\sin\varphi-$$
$$\dot{\theta}\dot{\beta}\sin\beta\cos\theta\sin\varphi-\dot{\theta}\dot{\theta}\cos\beta\sin\theta\sin\varphi+\dot{\theta}\dot{\varphi}\cos\beta\cos\theta\cos\varphi+$$
$$\ddot{\varphi}\cos\beta\sin\theta\cos\varphi-\dot{\varphi}\dot{\beta}\sin\beta\sin\theta\cos\varphi+\dot{\varphi}\dot{\theta}\cos\beta\cos\theta\cos\varphi-$$
$$\dot{\varphi}\dot{\varphi}\cos\beta\sin\theta\sin\varphi+\ddot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\dot{\theta}-\dot{\alpha}^2\sin\alpha\sin\beta\cos\psi\sin\theta+$$
$$\dot{\alpha}\dot{\beta}\cos\alpha\cos\beta\cos\psi\sin\theta-\dot{\alpha}\dot{\psi}\cos\alpha\sin\beta\sin\psi\sin\theta+\dot{\alpha}\dot{\theta}\cos\alpha\sin\beta\cos\psi\cos\theta+$$
$$\ddot{\beta}\sin\alpha\cos\beta\cos\psi\sin\theta+\dot{\beta}\dot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta-\dot{\beta}^2\sin\alpha\sin\beta\cos\psi\sin\theta-$$
$$\dot{\beta}\dot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta+\dot{\beta}\dot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta-\ddot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta+$$
$$\dot{\psi}\dot{\alpha}\cos\alpha\sin\beta\sin\psi\sin\theta+\dot{\psi}\dot{\beta}\sin\alpha\cos\beta\sin\psi\sin\theta+\dot{\psi}^2\sin\alpha\sin\beta\cos\psi\sin\theta+$$
$$\dot{\psi}\dot{\theta}\sin\alpha\sin\beta\sin\psi\cos\theta+\ddot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta+\dot{\theta}\dot{\alpha}\cos\alpha\sin\beta\cos\psi\cos\theta+$$
$$\dot{\theta}\dot{\beta}\sin\alpha\cos\beta\cos\psi\cos\theta-\dot{\theta}\dot{\psi}\sin\alpha\sin\beta\sin\psi\cos\theta-\dot{\theta}^2\sin\alpha\sin\beta\cos\psi\sin\theta)]\times$$
$$[-s_1\cos\psi\sin\theta+s_2\sin\alpha\sin\beta\sin\psi\sin\theta]+[\dot{x}_1-s_1(\dot{\psi}\cos\psi\sin\theta+\dot{\theta}\sin\psi\cos\theta)-$$
$$s_2(-\dot{\beta}\sin\beta\sin\theta\sin\varphi+\dot{\theta}\cos\beta\cos\theta\sin\varphi+\dot{\varphi}\cos\beta\sin\theta\cos\varphi+$$
$$\dot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta+\dot{\beta}\sin\alpha\cos\beta\cos\psi\sin\theta-\dot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta+$$
$$\dot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta)]\times[-s_1(-\dot{\psi}\sin\psi\sin\theta+\dot{\theta}\cos\psi\cos\theta)+$$
$$s_2(\dot{\alpha}\cos\alpha\sin\beta\sin\theta\sin\psi+\dot{\beta}\sin\alpha\cos\beta\sin\theta\sin\psi+\dot{\psi}\sin\alpha\sin\beta\sin\theta\cos\psi+$$
$$\dot{\theta}\sin\alpha\sin\beta\sin\psi\cos\theta)]+[\ddot{y}_1+s_1(-\ddot{\psi}\sin\psi\sin\theta-\dot{\psi}^2\cos\psi\sin\theta-$$
$$\dot{\psi}\dot{\theta}\sin\psi\cos\theta+\ddot{\theta}\cos\psi\cos\theta-\dot{\theta}\dot{\psi}\sin\psi\cos\theta-\dot{\theta}^2\cos\psi\sin\theta)+$$
$$s_2(-\ddot{\alpha}\sin\alpha\cos\psi\sin\theta-\dot{\alpha}^2\cos\alpha\cos\psi\sin\theta+\dot{\alpha}\dot{\psi}\sin\alpha\sin\psi\sin\theta-$$
$$\dot{\alpha}\dot{\theta}\sin\alpha\cos\psi\cos\theta-\ddot{\psi}\cos\alpha\sin\psi\sin\theta-\dot{\psi}\dot{\alpha}\sin\alpha\sin\psi\sin\theta+$$
$$\dot{\psi}^2\cos\alpha\cos\psi\sin\theta+\dot{\psi}\dot{\theta}\cos\alpha\sin\psi\cos\theta+\ddot{\theta}\cos\alpha\cos\psi\cos\theta+$$
$$\dot{\theta}\dot{\alpha}\sin\alpha\cos\psi\cos\theta+\dot{\theta}\dot{\psi}\cos\alpha\sin\psi\cos\theta+\dot{\theta}^2\cos\alpha\cos\psi\sin\theta-$$
$$\ddot{\alpha}\cos\alpha\cos\theta-\dot{\alpha}^2\sin\alpha\cos\theta-\dot{\alpha}\dot{\theta}\cos\alpha\sin\theta+\ddot{\theta}\sin\alpha\sin\theta+$$
$$\dot{\theta}\dot{\alpha}\cos\alpha\sin\theta+\dot{\theta}^2\sin\alpha\cos\theta)]\times[-s_1\sin\psi\sin\theta-s_2\cos\alpha\sin\psi\sin\theta]+$$
$$[\dot{y}_1+s_1(-\dot{\psi}\sin\psi\sin\theta+\dot{\theta}\cos\psi\cos\theta)+s_2(-\dot{\alpha}\sin\alpha\cos\psi\sin\theta-$$
$$\dot{\psi}\cos\alpha\sin\psi\sin\theta+\dot{\theta}\cos\alpha\cos\psi\cos\theta-\dot{\alpha}\cos\alpha\cos\theta+\dot{\theta}\sin\alpha\sin\theta)]\times$$
$$[-s_1(\dot{\psi}\cos\psi\sin\theta+\dot{\theta}\sin\psi\cos\theta)-s_2(-\dot{\alpha}\sin\alpha\sin\psi\sin\theta+$$

$$
\begin{aligned}
&\dot\psi\cos\alpha\cos\psi\sin\theta+\dot\theta\cos\alpha\sin\psi\cos\theta)]+[\ddot z_1+s_1\dot\theta\cos\theta-\\
&s_2(-\ddot\beta\cos\beta\sin\psi\sin\theta+\dot\beta^2\sin\beta\sin\psi\sin\theta-\dot\beta\dot\psi\cos\beta\cos\psi\sin\theta-\\
&\dot\beta\dot\theta\cos\beta\sin\psi\cos\theta-\ddot\psi\sin\beta\cos\psi\sin\theta-\dot\psi\dot\beta\cos\beta\cos\psi\sin\theta+\\
&\dot\psi^2\sin\beta\sin\psi\sin\theta-\dot\psi\dot\theta\sin\beta\cos\psi\cos\theta-\ddot\theta\sin\beta\sin\psi\cos\theta-\\
&\dot\theta\dot\beta\sin\beta\sin\psi\cos\theta-\dot\theta\dot\psi\sin\beta\sin\psi\cos\theta+\dot\theta^2\sin\beta\sin\psi\sin\theta-\\
&\ddot\alpha\cos\alpha\cos\beta\cos\psi\sin\theta+\dot\alpha^2\sin\alpha\cos\beta\cos\psi\sin\theta+\dot\alpha\dot\beta\cos\alpha\sin\beta\cos\psi\sin\theta+\\
&\dot\alpha\dot\psi\cos\alpha\cos\beta\sin\psi\sin\theta-\dot\alpha\dot\theta\cos\alpha\cos\beta\cos\psi\cos\theta+\ddot\beta\sin\alpha\sin\beta\cos\psi\sin\theta+\\
&\dot\beta\dot\alpha\cos\alpha\sin\beta\cos\psi\sin\theta+\dot\beta^2\sin\alpha\cos\beta\cos\psi\sin\theta-\dot\beta\dot\psi\sin\alpha\sin\beta\sin\psi\sin\theta+\\
&\dot\beta\dot\theta\sin\alpha\sin\beta\cos\psi\cos\theta+\ddot\psi\sin\alpha\cos\beta\sin\psi\sin\theta+\dot\psi\dot\alpha\cos\alpha\cos\beta\sin\psi\sin\theta-\\
&\dot\psi\dot\beta\sin\alpha\sin\beta\sin\psi\sin\theta+\dot\psi^2\sin\alpha\cos\beta\cos\psi\sin\theta+\dot\psi\dot\theta\sin\alpha\cos\beta\sin\psi\cos\theta-\\
&\ddot\theta\sin\alpha\cos\beta\cos\psi\cos\theta-\dot\theta\dot\alpha\cos\alpha\cos\beta\cos\psi\cos\theta+\dot\theta\dot\beta\sin\alpha\sin\beta\cos\psi\cos\theta+\\
&\dot\theta\dot\psi\sin\alpha\cos\beta\sin\psi\cos\theta+\dot\theta^2\sin\alpha\cos\beta\cos\psi\sin\theta-\ddot\alpha\sin\alpha\cos\beta\cos\theta-\\
&\dot\alpha\sin\alpha\cos\beta\cos\theta-\dot\alpha\sin\alpha\cos\beta\cos\theta-\dot\alpha\sin\alpha\cos\beta\cos\theta+\dot\alpha\dot\theta\sin\alpha\cos\beta\sin\theta-\\
&\ddot\beta\cos\alpha\sin\beta\cos\theta+\dot\beta\dot\alpha\sin\alpha\sin\beta\cos\theta-\dot\beta^2\cos\alpha\cos\beta\cos\theta+\\
&\dot\beta\dot\theta\cos\alpha\sin\beta\sin\theta-\ddot\theta\cos\alpha\cos\beta\sin\theta+\dot\theta\dot\alpha\sin\alpha\cos\beta\sin\theta+\dot\theta\dot\beta\cos\alpha\sin\beta\sin\theta-\\
&\dot\theta^2\cos\alpha\cos\beta\cos\theta)]\times[-s_2(-\sin\beta\cos\psi\sin\theta+\sin\alpha\cos\beta\sin\psi\sin\theta)]+\\
&[\dot z_1+s_1\sin\theta-s_2(-\dot\beta\cos\beta\sin\psi\sin\theta-\dot\psi\sin\beta\cos\psi\sin\theta-\dot\theta\sin\beta\sin\psi\cos\theta-\\
&\dot\alpha\cos\alpha\cos\beta\cos\psi\sin\theta+\dot\beta\sin\alpha\sin\beta\cos\psi\sin\theta+\dot\psi\sin\alpha\cos\beta\sin\psi\sin\theta-\\
&\dot\theta\sin\alpha\cos\beta\cos\psi\cos\theta-\dot\alpha\sin\alpha\cos\beta\cos\theta-\dot\beta\cos\alpha\sin\beta\cos\theta-\\
&\dot\theta\cos\alpha\cos\beta\sin\theta)]\times[-s_2(-\dot\beta\cos\beta\cos\psi\sin\theta+\dot\psi\sin\beta\sin\psi\sin\theta-\\
&\dot\theta\sin\beta\cos\psi\cos\theta+\dot\alpha\cos\alpha\cos\beta\sin\psi\sin\theta-\dot\beta\sin\alpha\sin\beta\sin\psi\sin\theta+\\
&\dot\psi\sin\alpha\cos\beta\cos\dot\psi\sin\theta+\dot\theta\sin\alpha\cos\beta\sin\psi\cos\theta)]\}+\\
&m_2g[-s_2(-\sin\beta\cos\psi\sin\theta+\sin\alpha\cos\beta\sin\psi\sin\theta)]=0
\end{aligned}
\tag{3-61f}
$$

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$$
\begin{aligned}
&J_{1x}[\dot\omega_{1x}\frac{\partial\omega_{1x}}{\partial\dot q_j}+\omega_{1x}(\frac{\partial\omega_{1x}}{\partial\dot q_j})_t]+J_{1y}[\dot\omega_{1y}\frac{\partial\omega_{1y}}{\partial\dot q_j}+\omega_{1y}(\frac{\partial\omega_{1y}}{\partial\dot q_j})_t]+J_{2x}[\dot\omega_{2z}\frac{\partial\omega_{2x}}{\partial\dot q_j}+\omega_{2x}(\frac{\partial\omega_{2x}}{\partial\dot q_j})_t]+\\
&J_{2y}[\dot\omega_{2y}\frac{\partial\omega_{2y}}{\partial\dot q_j}-\omega_{2y}(\frac{\partial\omega_{2y}}{\partial\dot q_j})_t]+J_{2z}[\dot\omega_{2z}\frac{\partial\omega_{2z}}{\partial\dot q_j}+\omega_{2z}(\frac{\partial\omega_{2z}}{\partial\dot q_j})_t]+m_2\{[\ddot x_2\frac{\partial\dot x_2}{\partial\dot q_j}+\dot x_2(\frac{\partial\dot x_2}{\partial\dot q_j})_t]+
\end{aligned}
$$

$$[\ddot{y}_2\frac{\partial\dot{y}_2}{\partial\dot{q}_j}+\dot{y}_2(\frac{\partial\dot{y}_2}{\partial\dot{q}_j})_t]+[\ddot{z}_2\frac{\partial\dot{z}_2}{\partial\dot{q}_j}+\dot{z}_2(\frac{\partial\dot{z}_2}{\partial\dot{q}_j})_t]\}+J_{1x}\omega_{2y}\frac{\partial\omega_{1x}}{\partial q_j}+J_{1y}\omega_{1y}\frac{\partial\omega_{1y}}{\partial q_j}+J_{1z}\omega_{1z}\frac{\partial\omega_{1z}}{\partial q_j}+$$

$$J_{2x}\omega_{2x}\frac{\partial\omega_{2x}}{\partial q_j}+J_{2y}\omega_{2y}\frac{\partial\omega_{2y}}{\partial q_j}+J_{2z}\omega_{2z}\frac{\partial\omega_{2z}}{\partial q_j}+m_2g\frac{\partial z_2}{\partial q_j}=0$$

（3-60g）

$$\begin{aligned}
&J_{1x}\{[\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}(\dot{\theta}\cos\theta\sin\varphi+\dot{\varphi}\sin\theta\cos\varphi)+\ddot{\theta}\cos\varphi-\dot{\varphi}\dot{\theta}\sin\varphi]\times\\
&[\cos\varphi]+[\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi]\times[-\dot{\varphi}\sin\varphi]\}+J_{1y}\{[\ddot{\psi}\sin\theta\cos\varphi+\\
&\dot{\psi}(\dot{\theta}\cos\theta\cos\varphi-\dot{\varphi}\sin\theta\sin\varphi)-\ddot{\theta}\sin\varphi-\dot{\varphi}\dot{\theta}\cos\varphi]\times[-\sin\varphi]+\\
&[\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi][-\dot{\varphi}\cos\varphi]\}+J_{2x}\{[-\dot{\alpha}\sin\alpha(\dot{\psi}\sin\theta\sin\varphi+\\
&\dot{\theta}\cos\varphi+\ddot{\psi}\sin\theta\sin\varphi)+\cos\alpha(\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\ddot{\psi}\sin\theta\sin\varphi+\\
&\dot{\varphi}\dot{\psi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\sin\varphi)-\dot{\alpha}\cos\alpha(\dot{\psi}\cos\theta+\dot{\varphi})-\\
&\sin\alpha(-\dot{\psi}\dot{\theta}\sin\theta+\ddot{\psi}\cos\theta+\ddot{\varphi})+\ddot{\beta}]\times[\cos\alpha\cos\varphi]+[\cos\alpha(\dot{\psi}\sin\theta\sin\varphi+\\
&\dot{\theta}\cos\varphi)-\sin\alpha(\dot{\psi}\cos\theta+\dot{\varphi})+\dot{\beta}]\times[-\dot{\alpha}\sin\alpha\cos\varphi-\dot{\varphi}\cos\alpha\sin\varphi]\}+\\
&J_{2y}\{[(\dot{\alpha}\cos\alpha\sin\beta+\dot{\beta}\sin\alpha\cos\beta)(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+\\
&\sin\alpha\sin\beta(\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\dot{\psi}\dot{\varphi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\dot{\varphi}\sin\varphi)-\\
&\dot{\beta}\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\cos\beta(\ddot{\psi}\sin\theta\cos\varphi+\dot{\psi}\dot{\theta}\cos\theta\cos\varphi-\\
&\dot{\beta}\cos\alpha\cos\beta)(\dot{\psi}\cos\theta+\dot{\varphi})+\cos\alpha\sin\beta(\ddot{\psi}\cos\theta+\ddot{\varphi}-\dot{\psi}\dot{\theta}\sin\theta)]\times\\
&[\sin\alpha\sin\beta\cos\varphi-\cos\beta\sin\varphi]-[\sin\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+\\
&\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\dot{\alpha}\cos\beta+\cos\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})]\times\\
&[\dot{\alpha}\cos\alpha\sin\beta\cos\varphi+\dot{\beta}\sin\alpha\cos\beta\cos\varphi-\dot{\varphi}\sin\alpha\sin\beta\sin\varphi+\\
&\dot{\beta}\sin\beta\sin\varphi-\dot{\varphi}\cos\beta\cos\varphi]\}+J_{2z}\{[(\dot{\alpha}\cos\beta\cos\alpha-\dot{\varphi}\dot{\psi}\sin\theta\sin\varphi-\\
&\ddot{\theta}\sin\varphi-\dot{\theta}\dot{\varphi}\cos\varphi)+\ddot{\alpha}\cos\beta-\dot{\alpha}\dot{\beta}\sin\beta+(-\dot{\alpha}\sin\alpha\sin\beta+\\
&\dot{\beta}\cos\beta\sin\alpha)(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+\cos\beta\sin\alpha(\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\\
&\dot{\psi}\dot{\varphi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\dot{\varphi}\sin\varphi)-\dot{\beta}\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)-\\
&\sin\beta(\ddot{\psi}\sin\theta\cos\varphi+\dot{\psi}\dot{\theta}\cos\theta\cos\varphi-\dot{\psi}\dot{\varphi}\sin\theta\sin\varphi-\ddot{\theta}\sin\varphi-\dot{\theta}\dot{\varphi}\cos\varphi)+\\
&(-\dot{\alpha}\sin\alpha\cos\beta-\dot{\beta}\cos\alpha\sin\beta)(\dot{\psi}\cos\theta+\dot{\varphi})+\cos\alpha\cos\beta(\ddot{\psi}\cos\theta-\dot{\psi}\dot{\theta}\sin\theta+\\
&\ddot{\varphi})-\ddot{\alpha}\sin\beta-\dot{\alpha}\dot{\beta}\cos\beta]\times[\sin\alpha\cos\beta\cos\varphi+\sin\varphi]+[\sin\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\\
&\dot{\theta}\cos\varphi)-\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\cos\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})-\dot{\alpha}\sin\beta]\times\\
&[\dot{\alpha}\cos\beta\cos\alpha\cos\varphi-\dot{\beta}\sin\beta\sin\alpha\cos\varphi-\dot{\varphi}\cos\beta\sin\alpha\sin\varphi+\dot{\varphi}\cos\varphi]\}+
\end{aligned}$$

$$m_2\{[\ddot{x}_1 - s_1(\ddot{\psi}\cos\psi\sin\theta - \dot{\psi}\dot{\psi}\sin\psi\sin\theta + \dot{\psi}\dot{\theta}\cos\psi\cos\theta + \ddot{\theta}\sin\psi\cos\theta +$$

$$\dot{\theta}\psi\cos\psi\cos\theta - \dot{\theta}\dot{\theta}\sin\psi\sin\theta) - s_2(-\ddot{\beta}\sin\beta\sin\theta\sin\varphi - \dot{\beta}\dot{\beta}\cos\beta\sin\theta\sin\varphi -$$

$$\dot{\beta}\dot{\theta}\sin\beta\cos\theta\sin\varphi - \dot{\beta}\dot{\varphi}\sin\beta\sin\theta\cos\varphi + \ddot{\theta}\cos\beta\cos\theta\sin\varphi -$$

$$\dot{\theta}\dot{\beta}\sin\beta\cos\theta\sin\varphi - \dot{\theta}\dot{\theta}\cos\beta\sin\theta\sin\varphi + \dot{\theta}\dot{\varphi}\cos\beta\cos\theta\cos\varphi +$$

$$\ddot{\varphi}\cos\beta\sin\theta\cos\varphi - \dot{\varphi}\dot{\beta}\sin\beta\sin\theta\cos\varphi + \dot{\varphi}\dot{\theta}\cos\beta\cos\theta\cos\varphi -$$

$$\dot{\varphi}\dot{\varphi}\cos\beta\sin\theta\sin\varphi + \ddot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta - \dot{\alpha}^2\sin\alpha\sin\beta\cos\psi\sin\theta +$$

$$\dot{\alpha}\dot{\beta}\cos\alpha\cos\beta\cos\psi\sin\theta - \dot{\alpha}\dot{\psi}\cos\alpha\sin\beta\sin\psi\sin\theta + \dot{\alpha}\dot{\theta}\cos\alpha\sin\beta\cos\psi\cos\theta +$$

$$\ddot{\beta}\sin\alpha\cos\beta\cos\psi\sin\theta + \dot{\beta}\dot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta - \dot{\beta}^2\sin\alpha\sin\beta\cos\psi\sin\theta -$$

$$\dot{\beta}\dot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta + \dot{\beta}\dot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta - \ddot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta +$$

$$\dot{\psi}\dot{\alpha}\cos\alpha\sin\beta\sin\psi\sin\theta + \dot{\psi}\dot{\beta}\sin\alpha\cos\beta\sin\psi\sin\theta + \dot{\psi}^2\sin\alpha\sin\beta\cos\psi\sin\theta +$$

$$\dot{\psi}\dot{\theta}\sin\alpha\sin\beta\sin\psi\cos\theta + \ddot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta + \dot{\theta}\dot{\alpha}\cos\alpha\sin\beta\cos\psi\cos\theta +$$

$$\dot{\theta}\dot{\beta}\sin\alpha\cos\beta\cos\psi\cos\theta - \dot{\theta}\dot{\psi}\sin\alpha\sin\beta\sin\psi\cos\theta - \dot{\theta}^2\sin\alpha\sin\beta\cos\psi\sin\theta)]\times$$

$$[-s_1\cos\theta\sin\psi - s_2(\cos\beta\cos\theta\sin\varphi + \sin\alpha\sin\beta\cos\psi\cos\theta)] + [\dot{x}_1 -$$

$$s_1(\dot{\psi}\cos\psi\sin\theta + \dot{\theta}\sin\psi\cos\theta) - s_2(-\dot{\beta}\sin\beta\sin\theta\sin\varphi + \dot{\theta}\cos\beta\cos\theta\sin\varphi +$$

$$\dot{\varphi}\cos\beta\sin\theta\cos\varphi + \dot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta + \dot{\beta}\sin\alpha\cos\beta\cos\psi\sin\theta -$$

$$\dot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta + \dot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta)]\times[-s_1(-\dot{\psi}\cos\psi\cos\theta -$$

$$\dot{\theta}\sin\theta\sin\psi) - s_2(-\dot{\beta}\sin\beta\cos\theta\sin\varphi - \dot{\theta}\cos\beta\sin\theta\sin\varphi + \dot{\varphi}\cos\beta\cos\theta\sin\varphi +$$

$$\dot{\alpha}\cos\alpha\sin\beta\cos\psi\cos\theta + \dot{\beta}\sin\alpha\cos\beta\cos\theta\cos\psi - \dot{\psi}\sin\alpha\sin\beta\sin\psi\cos\theta -$$

$$\dot{\theta}\sin\alpha\sin\beta\cos\psi\sin\theta)] + [\ddot{y}_1 + s_1(-\ddot{\psi}\sin\psi\sin\theta - \dot{\psi}^2\cos\psi\sin\theta - \dot{\psi}\dot{\theta}\sin\psi\cos\theta +$$

$$\ddot{\theta}\cos\psi\cos\theta - \dot{\theta}\dot{\psi}\sin\psi\cos\theta - \dot{\theta}^2\cos\psi\sin\theta) + s_2(-\ddot{\alpha}\sin\alpha\cos\psi\sin\theta -$$

$$\dot{\alpha}^2\cos\alpha\cos\psi\sin\theta + \dot{\alpha}\dot{\psi}\sin\alpha\sin\psi\sin\theta - \dot{\alpha}\dot{\theta}\sin\alpha\cos\psi\cos\theta -$$

$$\ddot{\psi}\cos\alpha\sin\psi\sin\theta - \dot{\psi}\dot{\alpha}\sin\alpha\sin\psi\sin\theta + \dot{\psi}^2\cos\alpha\cos\psi\sin\theta +$$

$$\dot{\psi}\dot{\theta}\cos\alpha\sin\psi\cos\theta + \ddot{\theta}\cos\alpha\cos\psi\cos\theta + \dot{\theta}\dot{\alpha}\sin\alpha\cos\psi\cos\theta +$$

$$\dot{\theta}\dot{\psi}\cos\alpha\sin\psi\cos\theta + \dot{\theta}^2\cos\alpha\cos\psi\sin\theta - \ddot{\alpha}\cos\alpha\cos\theta - \dot{\alpha}^2\sin\alpha\cos\theta -$$

$$\dot{\alpha}\dot{\theta}\cos\alpha\sin\theta + \ddot{\theta}\sin\alpha\sin\theta + \dot{\theta}\dot{\alpha}\cos\alpha\sin\theta + \dot{\theta}^2\sin\alpha\cos\theta)]\times[s_1\cos\psi\cos\theta +$$

$$s_2(\cos\alpha\cos\psi\cos\theta + \sin\alpha\sin\theta)] + [\dot{y}_1 + s_1(-\dot{\psi}\sin\psi\sin\theta + \dot{\theta}\cos\psi\cos\theta) +$$

$$s_2(-\dot{\alpha}\sin\alpha\cos\psi\sin\theta - \dot{\psi}\cos\alpha\sin\psi\sin\theta + \dot{\theta}\cos\alpha\cos\psi\cos\theta - \dot{\alpha}\cos\alpha\cos\theta +$$

$$\dot{\theta}\sin\alpha\sin\theta)]\times[s_1(-\dot{\psi}\sin\psi\cos\theta - \dot{\theta}\cos\psi\sin\theta) + s_2(-\dot{\alpha}\sin\alpha\cos\psi\cos\theta -$$

$$\dot{\psi}\cos\alpha\sin\psi\cos\theta - \dot{\theta}\cos\alpha\cos\psi\sin\theta + \dot{\alpha}\cos\alpha\sin\theta + \dot{\theta}\sin\alpha\cos\theta)] +$$

$$[\ddot{z}_1 + s_1\dot{\theta}\cos\theta - s_2(-\ddot{\beta}\cos\beta\sin\psi\sin\theta + \dot{\beta}^2\sin\beta\sin\psi\sin\theta - \dot{\beta}\dot{\psi}\cos\beta\cos\psi\sin\theta -$$

$$\dot{\beta}\dot{\theta}\cos\beta\sin\psi\cos\theta-\ddot{\psi}\sin\beta\cos\psi\sin\theta-\dot{\psi}\dot{\beta}\cos\beta\cos\psi\sin\theta+\dot{\psi}^2\sin\beta\sin\psi\sin\theta-$$
$$\dot{\psi}\dot{\theta}\sin\beta\cos\psi\cos\theta-\ddot{\theta}\sin\beta\sin\psi\cos\theta-\dot{\theta}\dot{\beta}\sin\beta\sin\psi\cos\theta-\dot{\theta}\dot{\psi}\sin\beta\sin\psi\cos\theta+$$
$$\dot{\theta}^2\sin\beta\sin\psi\sin\theta-\ddot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta+\dot{\alpha}^2\sin\alpha\cos\beta\cos\psi\sin\theta+$$
$$\dot{\alpha}\dot{\beta}\cos\alpha\sin\beta\cos\psi\sin\theta+\dot{\alpha}\dot{\psi}\cos\alpha\cos\beta\sin\psi\sin\theta-\dot{\alpha}\dot{\theta}\cos\alpha\cos\beta\cos\psi\cos\theta+$$
$$\ddot{\beta}\sin\alpha\sin\beta\cos\psi\sin\theta+\dot{\beta}\dot{\alpha}\cos\alpha\sin\beta\cos\psi\sin\theta+\dot{\beta}^2\sin\alpha\cos\beta\cos\psi\sin\theta-$$
$$\dot{\beta}\dot{\psi}\sin\alpha\sin\beta\sin\psi\sin\theta+\dot{\beta}\dot{\theta}\sin\alpha\sin\beta\cos\psi\cos\theta+\ddot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta+$$
$$\dot{\psi}\dot{\alpha}\cos\alpha\cos\beta\sin\psi\sin\theta-\dot{\psi}\dot{\beta}\sin\alpha\sin\beta\sin\psi\sin\theta+\dot{\psi}^2\sin\alpha\cos\beta\cos\psi\sin\theta+$$
$$\dot{\psi}\dot{\theta}\sin\alpha\cos\beta\sin\psi\cos\theta-\ddot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta-\dot{\theta}\dot{\alpha}\cos\alpha\cos\beta\cos\psi\cos\theta+$$
$$\dot{\theta}\dot{\beta}\sin\alpha\sin\beta\cos\psi\cos\theta+\dot{\theta}\dot{\psi}\sin\alpha\cos\beta\sin\psi\cos\theta+\dot{\theta}^2\sin\alpha\cos\beta\cos\psi\sin\theta-$$
$$\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\alpha}\sin\alpha\cos\beta\cos\theta-\dot{\alpha}\sin\alpha\cos\beta\cos\theta+$$
$$\dot{\alpha}\dot{\theta}\sin\alpha\cos\beta\sin\theta-\ddot{\beta}\cos\alpha\sin\beta\cos\theta+\dot{\beta}\dot{\alpha}\sin\alpha\sin\beta\cos\theta-\dot{\beta}^2\cos\alpha\cos\beta\cos\theta+$$
$$\dot{\beta}\dot{\theta}\cos\alpha\sin\beta\sin\theta-\ddot{\theta}\cos\alpha\cos\beta\sin\theta+\dot{\theta}\dot{\alpha}\sin\alpha\cos\beta\sin\theta+\dot{\theta}\dot{\beta}\cos\alpha\sin\beta\sin\theta-$$
$$\dot{\theta}^2\cos\alpha\cos\beta\cos\theta)]\times[-s_2(-\sin\beta\sin\psi\cos\theta-\sin\alpha\cos\beta\cos\psi\cos\theta-$$
$$\cos\alpha\cos\beta\sin\theta)]+[\dot{z}_1+s_1\sin\theta-s_2(-\dot{\beta}\cos\beta\sin\psi\sin\theta-\dot{\psi}\sin\beta\cos\psi\sin\theta-$$
$$\dot{\theta}\sin\beta\sin\psi\cos\theta-\dot{\alpha}\cos\alpha\cos\beta\cos\psi\sin\theta+\dot{\beta}\sin\alpha\sin\beta\cos\psi\sin\theta+$$
$$\dot{\psi}\sin\alpha\cos\beta\sin\psi\sin\theta-\dot{\theta}\sin\alpha\cos\beta\cos\psi\cos\theta-\dot{\alpha}\sin\alpha\cos\beta\cos\theta-$$
$$\dot{\beta}\cos\alpha\sin\beta\cos\theta-\dot{\theta}\cos\alpha\cos\beta\sin\theta)]\times[-s_2(-\dot{\beta}\cos\beta\sin\psi\cos\theta-$$
$$\dot{\psi}\sin\beta\cos\psi\cos\theta+\dot{\theta}\sin\beta\sin\psi\sin\theta-\dot{\alpha}\cos\alpha\cos\beta\cos\psi\cos\theta+$$
$$\dot{\beta}\sin\alpha\sin\beta\cos\psi\cos\theta+\dot{\psi}\sin\alpha\cos\beta\sin\psi\cos\theta+\dot{\alpha}\sin\alpha\cos\beta\sin\theta+$$
$$\dot{\beta}\cos\alpha\sin\beta\sin\theta-\dot{\theta}\cos\alpha\cos\beta\cos\theta)]\}+J_{1x}[\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi]\times$$
$$[\dot{\psi}\cos\theta\sin\varphi]+J_{1y}[\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi]\times[\dot{\psi}\cos\theta\cos\varphi-\dot{\theta}\sin\varphi]+$$
$$J_{1z}[\dot{\psi}\cos\theta+\dot{\varphi}]\times[-\dot{\psi}\sin\theta]+J_{2x}[\cos\alpha(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-$$
$$\sin\alpha(\dot{\psi}\cos\theta+\dot{\varphi})+\dot{\beta}]\times[\cos\alpha\dot{\psi}\cos\theta\sin\varphi+\sin\alpha\dot{\psi}\sin\theta]+$$
$$J_{2y}[\sin\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+$$
$$\dot{\alpha}\cos\beta+\cos\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})]\times[\sin\alpha\sin\beta\dot{\psi}\cos\theta\sin\varphi+$$
$$\cos\beta\dot{\psi}\cos\theta\cos\varphi-\cos\alpha\sin\beta\dot{\psi}\sin\theta]+J_{2z}[\sin\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+$$
$$\dot{\theta}\cos\varphi)-\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\cos\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})-\dot{\alpha}\sin\beta]\times$$
$$[\sin\alpha\cos\beta\dot{\psi}\cos\theta-\sin\beta\dot{\psi}\cos\theta\cos\varphi-\cos\alpha\cos\beta\dot{\psi}\sin\theta]+m_2g[s_1\sin\theta+$$
$$s_2(\sin\beta\sin\psi\cos\theta+\sin\alpha\cos\beta\cos\psi\cos\theta+\cos\alpha\cos\beta\sin\theta)]=0$$

(3-61g)

对广义坐标φ

$$J_{2x}[\dot{\omega}_{2x}\frac{\partial\omega_{2x}}{\partial\dot{q}_j}+\omega_{2x}(\frac{\partial\omega_{2x}}{\partial\dot{q}_j})_t]+J_{2y}[\dot{\omega}_{2y}\frac{\partial\omega_{2y}}{\partial\dot{q}_j}-\omega_{2y}(\frac{\partial\omega_{2y}}{\partial\dot{q}_j})_t]+$$
$$J_{2z}[\dot{\omega}_{2z}\frac{\partial\omega_{2z}}{\partial\dot{q}_j}+\omega_{2z}(\frac{\partial\omega_{2z}}{\partial\dot{q}_j})_t]+m_2[\ddot{x}_2\frac{\partial\dot{x}_2}{\partial\dot{q}_j}+\dot{x}_2(\frac{\partial\dot{x}_2}{\partial\dot{q}_j})_t]+J_{1x}\omega_{1x}\frac{\partial\omega_{1x}}{\partial q_j}+$$
$$J_{1y}\omega_{1y}\frac{\partial\omega_{1y}}{\partial q_j}+J_{2x}\omega_{2x}\frac{\partial\omega_{2x}}{\partial q_j}+J_{2y}\omega_{2y}\frac{\partial\omega_{2y}}{\partial q_j}+J_{2z}\omega_{2z}\frac{\partial\omega_{2z}}{\partial q_j}=0$$

（3-60h）

$J_{2x}\{[(\dot{\alpha}\cos\beta\cos\alpha-\dot{\beta}\cos\beta\sin\alpha)(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+$
$\cos\beta\sin\alpha(\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\dot{\psi}\dot{\varphi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\dot{\varphi}\sin\varphi)-$
$\dot{\beta}\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)-\sin\beta(\ddot{\psi}\sin\theta\cos\varphi+\dot{\psi}\dot{\theta}\cos\theta\cos\varphi-$
$\dot{\psi}\dot{\varphi}\sin\theta\sin\varphi-\ddot{\theta}\sin\varphi-\dot{\theta}\dot{\varphi}\cos\varphi)+(-\dot{\alpha}\sin\alpha\cos\beta-\dot{\beta}\cos\alpha\sin\beta)(\dot{\psi}\cos\theta+$
$\dot{\varphi})+\cos\alpha\cos\beta(\ddot{\psi}\cos\theta-\dot{\psi}\dot{\theta}\sin\theta+\ddot{\varphi})-\ddot{\alpha}\sin\beta-\dot{\alpha}\dot{\beta}\cos\beta]\times[-\sin\alpha]+$
$[\cos\alpha(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\alpha(\dot{\psi}\cos\theta+\dot{\varphi})+\dot{\beta}]\times[-\dot{\alpha}\cos\alpha]\}+$
$J_{2y}\{[(\dot{\alpha}\cos\alpha\sin\beta+\dot{\beta}\sin\alpha\cos\beta)(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+$
$\sin\alpha\sin\beta(\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\dot{\psi}\dot{\varphi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\dot{\varphi}\sin\varphi)-$
$\dot{\beta}\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+\cos\beta(\ddot{\psi}\sin\dot{\theta}\cos\varphi+\dot{\psi}\dot{\theta}\cos\theta\cos\varphi-$
$\dot{\varphi}\dot{\psi}\sin\theta\sin\varphi-\ddot{\theta}\sin\varphi-\dot{\theta}\dot{\varphi}\cos\varphi)+\ddot{\alpha}\cos\beta-\dot{\alpha}\dot{\beta}\sin\beta+(-\dot{\alpha}\sin\alpha\sin\beta+$
$\dot{\beta}\cos\alpha\cos\beta)(\dot{\psi}\cos\theta+\dot{\varphi})+\cos\alpha\sin\beta(\ddot{\psi}\cos\theta+\ddot{\varphi}-\dot{\psi}\dot{\theta}\sin\theta)]\times$
$[\cos\alpha\sin\beta]-[\sin\alpha\sin\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+$
$\dot{\alpha}\cos\beta+\cos\alpha\sin\beta(\dot{\psi}\cos\theta+\dot{\varphi})]\times[-\dot{\alpha}\sin\alpha\sin\beta+\dot{\beta}\cos\alpha\cos\beta]\}+$
$J_{2z}\{[(\dot{\alpha}\cos\beta\cos\alpha-\dot{\beta}\cos\beta\sin\alpha)(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)+$
$\cos\beta\sin\alpha(\ddot{\psi}\sin\theta\sin\varphi+\dot{\psi}\dot{\theta}\cos\theta\sin\varphi+\dot{\psi}\dot{\varphi}\sin\theta\cos\varphi+\ddot{\theta}\cos\varphi-\dot{\theta}\dot{\varphi}\sin\varphi)-$
$\dot{\beta}\cos\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)-\sin\beta(\ddot{\psi}\sin\theta\cos\varphi+\dot{\psi}\dot{\theta}\cos\theta\cos\varphi-$
$\dot{\psi}\dot{\varphi}\sin\theta\sin\varphi-\ddot{\theta}\sin\varphi-\dot{\theta}\dot{\varphi}\cos\varphi)+(-\dot{\alpha}\sin\alpha\cos\beta-\dot{\beta}\cos\alpha\sin\beta)(\dot{\psi}\cos\theta+\dot{\varphi})+$
$\cos\alpha\cos\beta(\ddot{\psi}\cos\theta-\dot{\psi}\dot{\theta}\sin\theta+\ddot{\varphi})-\ddot{\alpha}\sin\beta-\dot{\alpha}\dot{\beta}\cos\beta]\times[\cos\alpha\cos\beta]+$
$[\sin\alpha\cos\beta(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)-\sin\beta(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)+$
$\cos\alpha\cos\beta(\dot{\psi}\cos\theta+\dot{\varphi})-\dot{\alpha}\sin\beta]\times[-\dot{\alpha}\sin\alpha\cos\beta-\dot{\beta}\cos\alpha\sin\beta]\}+$
$m_2\{[\ddot{x}_1-s_1(\ddot{\psi}\cos\psi\sin\theta-\dot{\psi}\dot{\psi}\sin\psi\sin\theta+\dot{\psi}\dot{\theta}\cos\psi\cos\theta+\ddot{\theta}\sin\psi\cos\theta+$

$$\dot\theta\dot\psi\cos\psi\cos\theta-\dot\theta\dot\theta\sin\psi\sin\theta)-s_2(-\ddot\beta\sin\beta\sin\theta\sin\varphi-\dot\beta\dot\beta\cos\beta\sin\theta\sin\varphi-$$
$$\dot\beta\dot\theta\sin\beta\cos\theta\sin\varphi-\dot\beta\dot\varphi\sin\beta\sin\theta\cos\varphi+\ddot\theta\cos\beta\cos\theta\sin\varphi-\dot\theta\dot\beta\sin\beta\cos\theta\sin\varphi-$$
$$\dot\theta\dot\theta\cos\beta\sin\theta\sin\varphi+\dot\theta\dot\varphi\cos\beta\cos\theta\cos\varphi+\ddot\varphi\cos\beta\sin\theta\cos\varphi-$$
$$\dot\varphi\dot\beta\sin\beta\sin\theta\cos\varphi+\dot\varphi\dot\theta\cos\beta\cos\theta\cos\varphi-\dot\varphi\dot\varphi\cos\beta\sin\theta\sin\varphi+$$
$$\ddot\alpha\cos\alpha\sin\beta\cos\psi\sin\theta-\dot\alpha^2\sin\alpha\sin\beta\cos\psi\sin\theta+\dot\alpha\dot\beta\cos\alpha\cos\beta\cos\psi\sin\theta-$$
$$\dot\alpha\dot\psi\cos\alpha\sin\beta\sin\psi\sin\theta+\dot\alpha\dot\theta\cos\alpha\sin\beta\cos\psi\cos\theta+\ddot\beta\sin\alpha\cos\beta\cos\psi\sin\theta+$$
$$\dot\beta\dot\alpha\cos\alpha\cos\beta\cos\psi\sin\theta-\dot\beta^2\sin\alpha\sin\beta\cos\psi\sin\theta-\dot\beta\dot\psi\sin\alpha\cos\beta\sin\psi\sin\theta+$$
$$\dot\beta\dot\theta\sin\alpha\cos\beta\cos\psi\cos\theta-\ddot\psi\sin\alpha\sin\beta\sin\psi\sin\theta+\dot\psi\dot\alpha\cos\alpha\sin\beta\sin\psi\sin\theta+$$
$$\dot\psi\dot\beta\sin\alpha\cos\beta\sin\psi\sin\theta+\dot\psi^2\sin\alpha\sin\beta\cos\psi\sin\theta+\dot\psi\dot\theta\sin\alpha\sin\beta\sin\psi\cos\theta+$$
$$\ddot\theta\sin\alpha\sin\beta\cos\psi\cos\theta+\dot\theta\dot\alpha\cos\alpha\sin\beta\cos\psi\cos\theta+\dot\theta\dot\beta\sin\alpha\cos\beta\cos\psi\cos\theta-$$
$$\dot\theta\dot\psi\sin\alpha\sin\beta\sin\psi\cos\theta-\dot\theta^2\sin\alpha\sin\beta\cos\psi\sin\theta)]\times[-s_2\cos\beta\sin\theta\cos\varphi]+$$
$$[\dot x_1-s_1(\dot\psi\cos\psi\sin\theta+\dot\theta\sin\psi\cos\theta)-s_2(-\dot\beta\sin\beta\sin\theta\sin\varphi+\dot\theta\cos\beta\cos\theta\sin\varphi+$$
$$\dot\varphi\cos\beta\sin\theta\cos\varphi+\dot\alpha\cos\alpha\sin\beta\cos\psi\sin\theta+\dot\beta\sin\alpha\cos\beta\cos\psi\sin\theta-$$
$$\dot\psi\sin\alpha\sin\beta\sin\psi\sin\theta+\dot\theta\sin\alpha\sin\beta\cos\psi\cos\theta)]\times[-s_2(-\dot\beta\sin\beta\sin\theta\cos\varphi+$$
$$\dot\theta\cos\beta\cos\theta\cos\varphi-\dot\varphi\cos\beta\sin\theta\sin\varphi)]\}+J_{1x}[\dot\psi\sin\theta\sin\varphi+\dot\theta\cos\varphi]\times$$
$$[-\dot\psi\sin\theta\sin\varphi-\dot\theta\cos\varphi]+J_{1y}[\dot\psi\sin\theta\cos\varphi-\dot\theta\sin\varphi]\times[-\dot\psi\sin\theta\sin\varphi-\dot\theta\cos\varphi]+$$
$$J_{2x}[\cos\alpha(\dot\psi\sin\theta\sin\varphi+\dot\theta\cos\varphi)-\sin\alpha(\dot\psi\cos\theta+\dot\varphi)+\dot\beta]\times[\cos\alpha(\dot\psi\sin\theta\cos\varphi-$$
$$\dot\theta\sin\varphi)]+J_{2y}[\sin\alpha\sin\beta(\dot\psi\sin\theta\sin\varphi+\dot\theta\cos\varphi)+\cos\beta(\dot\psi\sin\theta\cos\varphi-\dot\theta\sin\varphi)+$$
$$\dot\alpha\cos\beta+\cos\alpha\sin\beta(\dot\psi\cos\theta+\dot\varphi)]\times[\sin\alpha\sin\beta(\dot\psi\sin\theta\cos\varphi-\dot\theta\sin\varphi)+$$
$$\cos\beta(-\dot\psi\sin\theta\sin\varphi-\dot\theta\cos\varphi)]+J_{2z}[\sin\alpha\cos\beta(\dot\psi\sin\theta\sin\varphi+\dot\theta\cos\varphi)-$$
$$\sin\beta(\dot\psi\sin\theta\cos\varphi-\dot\theta\sin\varphi)+\cos\alpha\cos\beta(\dot\psi\cos\theta+\dot\varphi)-\dot\alpha\sin\beta]\times$$
$$[\sin\alpha\cos\beta(\dot\psi\sin\theta\cos\varphi-\dot\theta\sin\varphi)-\sin\beta(-\dot\psi\sin\theta\sin\varphi-\dot\theta\cos\varphi)]=0$$

（3-61h）

有时需要计算系统的质心位置x_c,y_c,z_c，其公式为：

$$\begin{Bmatrix}x_c\\y_c\\z_c\end{Bmatrix}=\frac{1}{m_1+m_2}(m_1\begin{Bmatrix}x_1\\y_1\\z_1\end{Bmatrix}+m_2\begin{Bmatrix}x_2\\y_2\\z_2\end{Bmatrix})$$

（3-62）

为了求解方程组还需给出初始条件。对落猫问题，在给定高度下，必须在给

时间内完成系统规定的操作（四脚着地），故只有通过控制Q_α,Q_β两个广义力即M_α,M_β两个操纵力矩实现。

用拉格朗日方程求解落猫问题的整个步骤详细地列于上段文字，可以看出它有如下几个特点：

（1）主要是用广义坐标表达动能，动能作为一种能量，是标量，有着不以坐标系转移的特性，但是在进行动能计算时，还是用了不少刚体运动学的知识。

（2）求出能量函数以后对广义坐标及其导数求导是一个固定化程序，但过程是繁琐的，且这一过程，不能用计算机代替，整个过程只能够手工计算。但值得注意的是若系统比较简单，如双连杆摆，使用拉格朗日方程还是比较简单的。

（3）在本文假设的理想约束下，约束力在方程组中没有出现，在实际的工程强度计算中必须给出约束力，以便对各个部件进行强度、刚度、稳定性校核。

（4）本文没有以系统质心为基点，把运动方程组分解成质心的平移运动和绕质心的转动（这一做法是经典方法）。经典方法的相关矢量如图 3.2[25]，图中体坐标与本文相同。仔细观察发现图（3.2）中比图（3.1）中多出了两个未

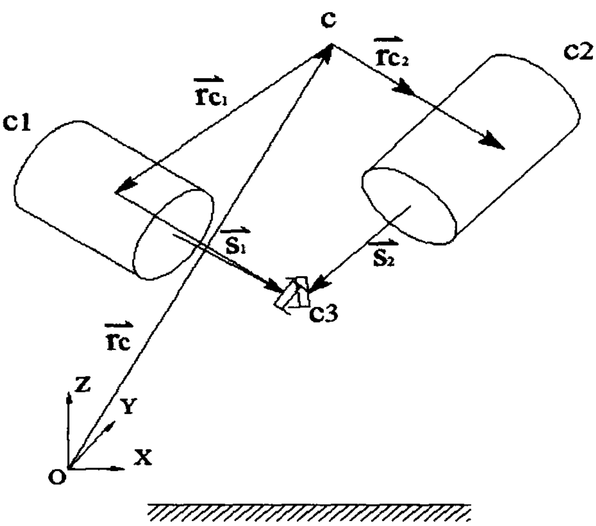

图 3.2 相关矢量图

知矢量，故必须给出以下两个约束方程

$$\begin{cases} \vec{r}_{c1}+\vec{s}_1=\vec{r}_{c2}+\vec{s}_2 \\ m_1\vec{r}_{c1}+m_2\vec{r}_{c1}+(m_1+m_2)\vec{r}_c=0 \end{cases} \tag{3-63}$$

其它的与本文同。

3.2 用牛顿-欧拉法解题

3.2.1 用牛顿-欧拉法解题

牛顿—欧拉法是一种古老的方法，但又是一种有效的方法。在多刚体动力学中，有一种称作维滕伯格的方法，它就是以牛顿—欧拉法为力学基础的，但本节不准备用这种方法解题，而是以传统的牛顿—欧拉法解题，以方便比较诸方法的特点。

看图 3-1，转轴 $\overline{AA'}=\vec{j}_1$，$\overline{BB'}=-\vec{i}_2$。但与第一节不同的是：$C_2X_2Y_2Z_2$ 由如下旋转而得：假令猫体伸直时，$C_1X_1Y_1Z_1$，$C_2X_2Y_2Z_2$ 相互平行，接着 2 物体先绕 AA' 轴旋转 β，再绕 BB' 轴旋转 α，得到 $C_2X_2Y_2Z_2$。

把 1 物体隔离出来进行受力分析，物体受到一个约束力 $\vec{F}$，方向待定；一个约束力矩 $\vec{M}$，$\vec{M}$ 的方向为 $\vec{j}_1\times\vec{i}_2$；主动力 $-m_1g$，方向 $-\vec{k}$；主动力矩 $\overrightarrow{M_\beta}$，方向 $\vec{j}_1$。同时 2 物体也被相应的隔离出来了，对它进行受力分析。2 物体受到一个约束力 $-\vec{F}$，方向未定；一个约束力矩 $-\vec{M}$，方向为 $-\vec{j}_1\times\vec{i}_2$；主动力为 $-m_2g$，方向 $-\vec{k}$；主动力矩 $\overrightarrow{M_\alpha}$，方向 $-\vec{i}_2$。

除了上面的力与力矩还有两个反作用力矩：1 物体还要受到 2 物体的一个反作用力矩 M_α，方向 $\vec{i}_2$，2 物体也要受到 1 物体的一反作用力矩 M_β，方向 $-\vec{j}_1$。在这里往往初学者们认为既然是主动力矩那就应该没有反作用力矩，

似乎只有约束力才是大小相等方向相反的。这里只要分析一下主动力矩在物理上是如何作用于 1 或 2 物体上的，便明白为什么了。从另外一个方面也可以进行校核：系统（猫体）在空中只受到重力的作用，没有受到任何力矩的作用。现在约束力矩是相互抵消的，如果$\overrightarrow{M_\alpha}$、$\overrightarrow{M_\beta}$无反作用力矩，将出现系统受到力矩作用。两者分析矛盾。

对 1 物体运用牛顿—欧拉法可以写出：

$$\begin{cases} m_1\ddot{x}_1 = F_x \\ m_1\ddot{y}_1 = F_y \\ m_1\ddot{z}_1 = F_z \end{cases} \tag{3-64}$$

$$\begin{cases} J_{x1}\dot{\omega}_{x1} + (J_{z1} - J_{y1})\omega_{z1}\omega_{y1} = M_{x1} \\ J_{y1}\dot{\omega}_{y1} + (J_{x1} - J_{z1})\omega_{x1}\omega_{y1} = M_{y1} \\ J_{z1}\dot{\omega}_{z1} + (J_{y1} - J_{x1})\omega_{y1}\omega_{x1} = M_{z1} \end{cases} \tag{3-65}$$

设 1 物体相对于地面的欧拉角为ψ_1、θ_1、φ_1运动学关系式可以写为：

$$\begin{Bmatrix} \omega_{x1} \\ \omega_{y1} \\ \omega_{z1} \end{Bmatrix} = \begin{bmatrix} \dot{\psi}_1 \sin\theta_1 \sin\varphi_1 + \dot{\theta}_1 \cos\varphi_1 \\ \dot{\psi}_1 \sin\theta_1 \cos\varphi_1 - \dot{\theta}_1 \sin\varphi_1 \\ \dot{\psi}_1 \cos\theta_1 + \dot{\varphi}_1 \end{bmatrix} \tag{3-66}$$

对 2 物体也运用同样方法且设 2 物体相对于地面的三个欧拉角ψ_2、θ_2、φ_2，其动力学方程为

$$\begin{cases} m_2\ddot{x}_2 = -F_x \\ m_2\ddot{y}_2 = -F_y \\ m_2\ddot{z}_2 = -F_z \end{cases} \tag{3-67}$$

$$\begin{cases} J_{x2}\dot{\omega}_{x2} + (J_{z2} - J_{y2})\omega_{z2}\omega_{y2} = M_{x2} \\ J_{y2}\dot{\omega}_{y2} + (J_{x2} - J_{z2})\omega_{x2}\omega_{y2} = M_{y2} \\ J_{z2}\dot{\omega}_{z2} + (J_{y2} - J_{x2})\omega_{y2}\omega_{x2} = M_{z2} \end{cases} \tag{3-68}$$

运动学方程为

$$\begin{Bmatrix} \omega_{x2} \\ \omega_{y2} \\ \omega_{z2} \end{Bmatrix} = \begin{bmatrix} \dot{\psi}_2 \sin\theta_2 \sin\varphi_2 + \dot{\theta}_2 \cos\varphi_2 \\ \dot{\psi}_2 \sin\theta_2 \cos\varphi_2 - \dot{\theta}_2 \sin\varphi_2 \\ \dot{\psi}_2 \cos\theta_2 + \dot{\varphi}_2 \end{bmatrix} \tag{3-69}$$

下面求作用在 1 物体上的总力矩在$OX_1Y_1Z_1$坐标系上的分量M_{x1}、M_{y1}、M_{z1}和作用于 2 物体的总力矩在$OX_2Y_2Z_2$坐标系中的分量 M_{x2}、M_{y2}、M_{z2}的值。

作用于物体 1、2 的合力矩分别为

$$\overrightarrow{M_1} = M\vec{j}_1 \times \vec{i}_2 + M_\alpha \vec{i}_2 + M_\beta \vec{j}_1$$

$$\overrightarrow{M_2} = -M\vec{j}_1 \times \vec{i}_2 - M_\alpha \vec{i}_2 - M_\beta \vec{j}_1 \tag{3-70}$$

因为已经求得

$$\begin{Bmatrix} \vec{i}_2 \\ \vec{j}_2 \\ \vec{k}_2 \end{Bmatrix} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ -\sin\alpha\sin\beta & \cos\alpha & \sin\alpha\cos\beta \\ -\sin\beta\cos\alpha & -\sin\beta & \cos\alpha\cos\beta \end{bmatrix} \begin{Bmatrix} \vec{i}_1 \\ \vec{j}_1 \\ \vec{k}_1 \end{Bmatrix} \tag{3-71}$$

所以有

$$\vec{i}_2 = \cos\beta \vec{i}_1 - \sin\beta \vec{k}_1$$

$$\begin{aligned} \vec{j}_1 \times \vec{i}_2 &= \cos\beta(\vec{j}_1 \times \vec{i}_1) - \sin\beta(\vec{j}_1 \times \vec{k}_1) \\ &= -\cos\beta \vec{k}_1 - \sin\beta \vec{i}_1 \end{aligned}$$

$$\vec{j}_1 = \cos\alpha \vec{j}_2 - \sin\beta \vec{k}_2$$

$$\begin{aligned} \vec{j}_1 \times \vec{i}_2 &= \cos\beta(\vec{j}_2 \times \vec{i}_2) - \sin\beta(\vec{k}_2 \times \vec{i}_2) \\ &= -\cos\alpha \vec{k}_2 - \sin\beta \vec{j}_2 \end{aligned}$$

$$\tag{3-72}$$

把上式代入（3-70）得

$$\begin{Bmatrix} M_{x1} \\ M_{y1} \\ M_{z1} \end{Bmatrix} = M \begin{Bmatrix} -\sin\beta \\ 0 \\ -\cos\beta \end{Bmatrix} + M_{\beta} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} + M_{\alpha} \begin{Bmatrix} \cos\beta \\ 0 \\ -\sin\beta \end{Bmatrix} = \begin{Bmatrix} -M\sin\beta + M_{\alpha}\cos\beta \\ M_{\beta} \\ -M\cos\beta - M_{\alpha}\sin\beta \end{Bmatrix}$$

（3-73）

$$\begin{Bmatrix} M_{x2} \\ M_{y2} \\ M_{z2} \end{Bmatrix} = M \begin{Bmatrix} 0 \\ \sin\beta \\ \cos\alpha \end{Bmatrix} + M_{\alpha} \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix} + M_{\beta} \begin{Bmatrix} 0 \\ -\cos\alpha \\ \cdot\sin\beta \end{Bmatrix} = \begin{Bmatrix} -M_{\alpha} \\ M\sin\beta - M_{\beta}\cos\alpha \\ M\cos\alpha + M_{\beta}\sin\beta \end{Bmatrix}$$

（3-74）

下面求约束方程：

在不失一般性的条件下，假设系统（猫体）在初始时刻相对地面是静止的，由质心运动定理可得：

$$x_c = \frac{1}{m_1 + m_2}(m_1 x_1 + m_2 x_2) = c_1$$

$$y_c = \frac{1}{m_1 + m_2}(m_1 y_1 + m_2 y_2) = c_2 \qquad (3\text{-}75)$$

$$z_c = \frac{1}{m_1 + m_2}(m_1 z_1 + m_2 z_2) = h_0 - \frac{1}{2}gt^2$$

式中 x_c, y_c, z_c 为系统的质心；c_1 是一常量，是系统质心在初始时刻的 x 轴分量；c_2 也是一常量，是系统质心在初始时刻的 y 轴的分量；h_0 是初始时系统质心的 z 轴的分量；t 的取值要使实际系统有意义。

下面要根据角动量守恒这个关系求出另一类型的约束方程。我们将求出系统对十字转轴的中心点 c_3 的角动量。假设初始时刻的角动量为零。

由第一章的基础知识有

$$\left[J_{C_3}\right]_1 = \left[J_{C_1}\right]_1 + m_1 \begin{bmatrix} c_1^2 + b_1^2 & -a_1 b_1 & -a_1 c_1 \\ -a_1 b_1 & a_1^2 + c_1^2 & -c_1 b_1 \\ -a_1 c_1 & -c_1 b_1 & a_1^2 + b_1^2 \end{bmatrix} \qquad (3\text{-}76)$$

式中：$\left[J_{c_3}\right]_1$——1 物体在 $C_3X_1Y_1Z_1$ 中相对于 C_3 点的惯量矩阵

$\left[J_{c_1}\right]_1$——1 物体相对于C_1点的主惯量矩阵

这里$a_1=b_1=0, c_1=s_1$，所以得

$$[J_{c3}]_1=\begin{bmatrix}J_{x1}&&\\&J_{y1}&\\&&J_{z1}\end{bmatrix}+m_1\begin{bmatrix}s_1^2&&\\&s_1^2&\\&&0\end{bmatrix}=\begin{bmatrix}J_{x1}+m_1s_1^2&&\\&J_{y1}+m_1s_1^2&\\&&J_{z1}\end{bmatrix}$$

（3-77）

由公式

$$\{H_{C_3}\}_1=[J_{c3}]_1\{\omega_1\} \tag{3-78}$$

式中：　$\{H_{C_3}\}_1$——1 物体在$C_3X_1Y_1Z_1$中的角动量分量列阵

$\{\omega_1\}$——1 物体在$C_1X_1Y_1Z_1$中的角速度分量列阵

得

$$\begin{Bmatrix}H_{C_3x1}\\H_{C_3y1}\\H_{C_3z1}\end{Bmatrix}=\begin{bmatrix}J_{x1}+m_1s_1^2&&\\&J_{y1}+m_1s_1^2&\\&&J_{z1}\end{bmatrix}\begin{Bmatrix}\omega_{x1}\\\omega_{y1}\\\omega_{z1}\end{Bmatrix} \tag{3-79}$$

把$\{H_{c3}\}_1$变换到地面坐标系上表示

$$\{H_{c3}\}_1'=[L_{1,0}]^T\{H_{c3}\}_1 \tag{3-80}$$

式中　$\{H_{c3}\}_1'$ ——$\{H_{C_3}\}_1$在坐标系$OXYZ$中的分量列阵

$$\begin{Bmatrix}H'_{C_3x1}\\H'_{C_3y1}\\H'_{C_3z1}\end{Bmatrix}=\begin{bmatrix}\cos\psi_1\cos\varphi_1-\sin\psi_1\cos\theta_1\sin\varphi_1&-\cos\psi_1\sin\varphi_1-\sin\psi_1\cos\theta_1\cos\varphi_1&\sin\psi_1\sin\theta_1\\\sin\psi_1\cos\varphi_1+\cos\theta_1\cos\psi_1\sin\varphi_1&-\sin\varphi_1\sin\psi_1+\cos\psi_1\cos\varphi_1\cos\theta_1&-\cos\psi_1\sin\theta_1\\\sin\theta_1\sin\varphi_1&\sin\theta_1\cos\varphi_1&\cos\theta_1\end{bmatrix}$$

$$\begin{Bmatrix}\omega_{x1}(J_{x1}+m_1s_1^2)\\\omega_{y1}(J_{y1}+m_1s_1^2)\\\omega_{z1}J_{z1}\end{Bmatrix} \tag{3-81}$$

同理有 $a_2=b_2=0,c_2=-s_2$

$$[J_{c_3}]_2=[J_{c2}]_2+m_2\begin{bmatrix}c_2^2+b_2^2 & -a_2b_2 & -a_2c_2\\ -a_2b_2 & a_2^2+c_2^2 & -c_2b_2\\ -a_2c_2 & -c_2b_2 & a_2^2+b_2^2\end{bmatrix}$$

$$=\begin{bmatrix}J_{x2} & & \\ & J_{y2} & \\ & & J_{z2}\end{bmatrix}+m_2\begin{bmatrix}s_2^2 & & \\ & s_2^2 & . \\ & & 0\end{bmatrix}=\begin{bmatrix}J_{x2}+m_2s_2^2 & & \\ & J_{y2}+m_2s_2^2 & \\ & & J_{z2}\end{bmatrix}$$

（3-82）

$$\{H_{c3}\}_2=[J_{c3}]_2\{\omega_2\} \tag{3-83}$$

式中 $\{H_{c3}\}_2$——2 物体在 $C_3X_2Y_2Z_2$ 中的角动量分量列阵

$[J_{c3}]_2$——2 物体在 $C_3X_2Y_2Z_2$ 中相对于 C_3 点的惯量矩阵

$$\{H_{c3}\}'_2=[L_{2,0}]^T\{H_{c3}\}_2 \tag{3-84}$$

$$\begin{Bmatrix}H_{x2}\\ H_{y2}\\ H_{z2}\end{Bmatrix}=\begin{bmatrix}\cos\psi_2\cos\varphi_2-\sin\psi_2\cos\theta_2\sin\varphi_2 & -\cos\psi_2\sin\varphi_2-\sin\psi_2\cos\theta_2\cos\varphi_2 & \sin\psi_2\sin\theta_2\\ \sin\psi_2\cos\varphi_2+\cos\psi_2\cos\theta_2\sin\varphi_2 & -\sin\psi_2\sin\varphi_2+\cos\psi_2\cos\theta_2\cos\varphi_2 & -\cos\psi_2\sin\theta_2\\ \sin\theta_2\sin\varphi_2 & \sin\theta_2\cos\varphi_2 & \cos\theta_2\end{bmatrix}$$
$$\begin{Bmatrix}\omega_{x2}(J_{x2}+m_2s_2^2)\\ \omega_{y2}(J_{y2}+m_1s_1^2)\\ \omega_{z2}J_{z2}\end{Bmatrix}$$

（3-85）

由初始假设角速度保持为零得

$$\{H_{c3}\}'_1+\{H_{c3}\}'_2=0 \tag{3-86}$$

方程组的未知量 $x_1,y_1,z_1,x_2,y_2,z_2,\psi_1,\theta_1,\varphi_1,\psi_2,\theta_2,\varphi_2,F_x,F_y,F_z,M,\alpha,\beta$，一共 18 个未知量，需要 18 个方程联解。物体的运动方程加约束方程一共 18

个方程，说明方程组是封闭的。

3.2.2 Kane 对落猫问题的解法

值得一提的是美国著名力学专家 Kane 教授也曾就落猫问题给出了动力学解释，而他仅就角动量守恒这一约束方程论述的，其推演过程是用矢量方法推导的，作为与本文的比较，也写出来[24]。

凯恩作出如下三个假设：

（Ⅰ）猫体的躯干只弯不扭；

（Ⅱ）在释放瞬时，脊柱是向前弯的，接着，脊柱向一侧弯曲，然后向后弯曲，然后向另一侧弯曲，最后又前弯。当猫翻转过来那一时刻，脊柱的形状与初始时是相同的。

（Ⅲ）在整个操作过程中后弯曲远比初始与结束时前弯曲小得多，系统被看成两个刚体组成，A 和 B，它有一公共点 O，见图 3.3

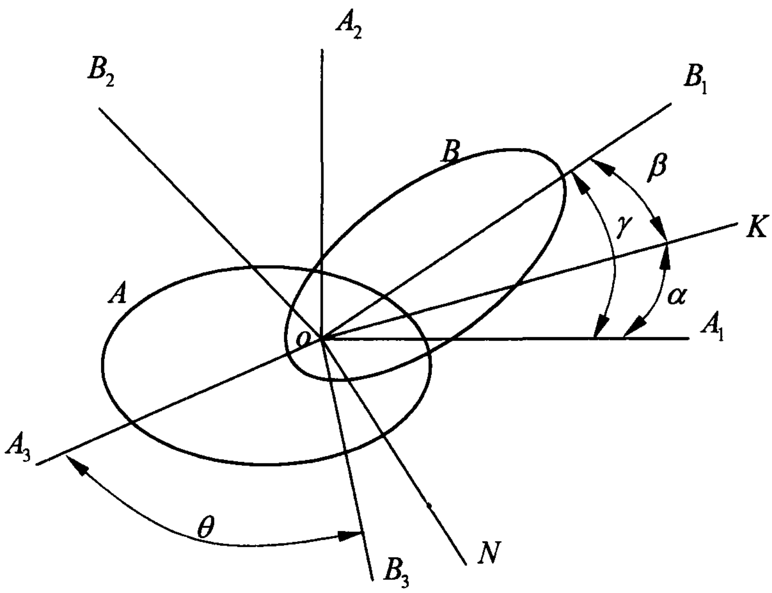

图 3.3 坐标系及其相关的射线与角

A_1, A_2, A_3 ——固定于 A 上的从 O 点发出的相互垂直的射线；

K ——位于由 A_1 和 A_2 决定的平面内；

B_1—— 固定于 B 物体的射线；

B_2 ——垂直于 B_1 同时又位于由 B_1 和 K 决定的平面内，但是不固连于 B 物体。

B_3 ——垂直于 B_1 和 B_2 的射线；

N ——垂直于 A_1 和 B_1 的射线；

α ——A_1 和 K 之间的角；

β ——B_1 和 K 之间的角；

γ ——A_1 和 B_1 之间的夹角；

θ ——A_3 和 B_3 之间的夹角；

$\vec{a_i}$ ——平行于 $\vec{A_i}$ 的单位矢量；

$\vec{b_i}$ ——平行于 $\vec{B_i}$ 的单位矢量；

$\vec{k}$ ——平行于 K 的单位矢量。

$\vec{n}$——平行于 N 的单位矢量。

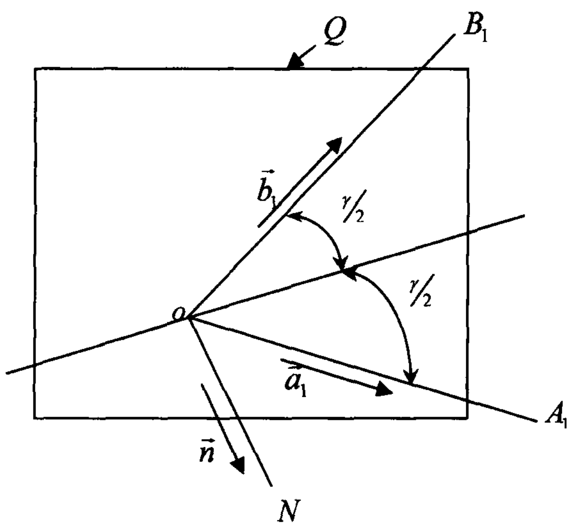

图 3.4 参考平面 Q

A 和 B 代表猫的前半部分和后半部分，A_1，B_1 代表脊柱方向。引进一个

参考平面 Q（见图 3.4），N 和 A_1，B_1 夹角的二等分线被固定住，A 和 B 在 Q 上的角速度记为：${}^{Q}\vec{\omega}^{A}$ 和 ${}^{Q}\vec{\omega}^{B}$

$$ {}^{Q}\vec{\omega}^{A} = u\vec{a}_1 - (\dot{r}/2)\vec{n} \tag{3-87} $$

$$ {}^{Q}\vec{\omega}^{B} = u\vec{b}_1 + (\dot{r}/2)\vec{n} \tag{3-88} $$

这里 u 和 v 是标量，是 A 和 B 在 Q 中的“转动速率”。

只要令：

$$ v = u \tag{3-89} $$

就可以阻止扭转。

为了满足条件（II），我们规定 α 和 β 保持常量。取 $\beta > \alpha$，让 B_1 绕 K，或者等价地，把 θ 从 0 单独变化到 2π，最后让 A_1 和 B_1 的角平分线的惯性方向在运动过程中保持不变。Q 的惯性角速度可以表示为：

$$ \vec{\omega}^{Q} = \dot{\psi} M(\vec{a}_1 + \vec{b}_1) \tag{3-90} $$

这里 $\dot{\psi}$ 是角 ψ 的时间导数，M 代表矢量 $\vec{a}_1 + \vec{b}_1$ 模的倒数。进而，假设初始时 $\psi = 0$，翻转发生在 $\psi = \pm\pi$。

A 和 B 在两个时刻内有相同的方向。当 $\psi = 0$ 时，$\theta = 0$，当 $\psi = \pm\pi$，$\theta = 2\pi$。

经过一系列的运算推出 ψ 的下面微分方程：

$$ \frac{d\psi}{d\theta} = \frac{(J/I)S}{(T-1)[1-T+(J/I)(1+T)](1+T)^{\frac{1}{2}}} \tag{3-91} $$

这里 I 和 J 代表每个物体的横向和轴向的惯性矩。S 和 T 被给为：

$$ S = -\sqrt{2}(\cos\alpha \sin\beta + \sin\alpha \cos\beta \cos\theta)\sin\beta \tag{3-92} $$

$$ T = \cos\alpha\, \mathrm{si}\cos\beta - \sin\alpha \sin\beta \cos\theta \tag{3-93} $$

我们现在来证明此方程，我们首先定义：

$$ \omega_i^{A} = \vec{\omega}^{A} \cdot \vec{a}_i \tag{3-94} $$

$$\omega_i^B = \vec{\omega}^B \cdot \vec{b}_i \tag{3-95}$$

这里$\vec{\omega}^A$和$\vec{\omega}^B$表示A和B的惯性角速度。事实上系统相对于质心的角动量必须一直等于0。(因为按照假设它一开始等于0)，它可以表示为

$$J\omega_1^A\vec{a}_1 + I\omega_2^A\vec{a}_2 + I\omega_3^A\vec{a}_3 + J\omega_1^B\vec{b}_1 + I\omega_2^B\vec{b}_2 + I\omega_3^B\vec{b}_3 = 0 \tag{3-96}$$

用$\vec{a}_1 + \vec{b}_1$点乘此方程得到：

$$(J/I)(\omega_1^A + \omega_1^B)(1+T_{11}) + \omega_2^A T_{21} + \omega_3^A T_{31} + \omega_2^B T_{12} + \omega_3^B T_{13} = 0 \tag{3-97}$$

这里T_{ij}定义为：

$$T_{ij} = \vec{a}_i \cdot \vec{b}_j \tag{3-98}$$

下一步，我们寻找$\vec{\omega}_i^A$和$\vec{\omega}_i^B$关于$\alpha,\beta,\theta,\dot{\theta}$和$\dot{\psi}$的函数。

注意到如下关系式

$$\vec{\omega}^A = \vec{\omega}^Q + {}^Q\vec{\omega}^A = \dot{\psi}M(\vec{a}_1 + \vec{b}_1) + u\vec{a}_1 - (\dot{r}/2)\vec{n} \tag{3-99}$$

$$\vec{\omega}^B = \vec{\omega}^Q + {}^Q\vec{\omega}^B = \dot{\psi}M(\vec{a}_1 + \vec{b}_1) + u\vec{b}_1 + (\dot{r}/2)\vec{n} \tag{3-100}$$

我们得到：

$$\omega_1^A = \dot{\psi}M(1+T_{11}) + u \tag{3-101}$$

$$\omega_2^A = \dot{\psi}MT_{21} - (\dot{r}/2)\vec{n}\cdot\vec{a}_2 \tag{3-102}$$

$$\omega_3^A = \dot{\psi}MT_{31} - (\dot{r}/2)\vec{n}\cdot\vec{a}_3 \tag{3-103}$$

$$\omega_1^B = \dot{\psi}M(1+T_{11}) + u \tag{3-104}$$

$$\omega_2^B = \dot{\psi}MT_{12} + (\dot{r}/2)\vec{n}\cdot\vec{b}_2 \tag{3-105}$$

$$\omega_3^B = \dot{\psi} M T_{13} + (\dot{r}/2)\vec{n}\cdot\vec{b}_3 \tag{3-106}$$

单位矢量$\vec{n}$表示为：

$$\vec{n} = \vec{a}_1 \times \vec{b}_1 / \sin\gamma \tag{3-107}$$

因此

$$\vec{n}\cdot\vec{a}_2 = -\vec{a}_3\cdot\vec{b}_1/\sin\gamma = -T_{31}/\sin\gamma \tag{3-108}$$

$$\vec{n}\cdot\vec{a}_3 = T_{21}/\sin\gamma \tag{3-109}$$

$$\vec{n}\cdot\vec{b}_2 = T_{13}/\sin\gamma \tag{3-110}$$

$$\vec{n}\cdot\vec{b}_3 = -T_{12}/\sin\gamma \tag{3-111}$$

在九个量T_{ij}中，只有三个后文需要，参考方程（3-98）和图 3.3 可以表示为：

$$T_{11} = \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\theta \tag{3-112}$$

$$T_{12} = -\cos\alpha\cos\beta - \sin\alpha\cos\beta\cos\theta \tag{3-113}$$

$$T_{13} = \sin\alpha\sin\theta \tag{3-114}$$

下文将使用的γ被构造为：

$$\cos\gamma = \vec{a}_1\cdot\vec{b}_1 = T_{11}$$

因此关于时间求导得：

$$-\sin\gamma\dot{\gamma} = \dot{T}_{11} = \dot{\theta}T_{13}\sin\beta$$

或者

$$\dot{\gamma} = -\dot{\theta}T_{13}\sin\beta/\sin\gamma \tag{3-115}$$

出现在方程（3-101）和（3-104）中的量u可以表示为：

$$u = \dot{\theta}T_{12}\sin\beta/(1-T_{11}^2) \tag{3-116}$$

B相对A的角速度为

$$ {}^{A}\vec{\omega}^{B} = {}^{Q}\vec{\omega}^{B} - {}^{Q}\vec{\omega}^{A} = \dot{\gamma}\vec{n} + \dot{u}(\vec{b}_1 - \vec{a}_1) \tag{3-117} $$

或者
$$ {}^{A}\vec{\omega}^{B} = {}^{A}\vec{\omega}^{P} + {}^{P}\vec{\omega}^{B} \tag{3-118} $$

这里 P 为由 B_1 和 K 规定的参考面，进一步规定：

$$ {}^{A}\omega^{P} = \dot{\theta}\vec{k} $$

${}^{A}\vec{\omega}^{P}$ 必须平行于 $\vec{b}_1$，因此

$$ {}^{A}\vec{\omega}^{B} = \dot{\theta}\vec{k} + s\vec{b}_1 \tag{3-119} $$

让方程（3-117）和（3-119）的右边相等，所得的方程点乘 $\vec{b}_2$，得方程 u

$$ u = \frac{\dot{r}\vec{n}\cdot\vec{b}_2 - \dot{\theta}\vec{k}\cdot\vec{b}_2}{\vec{a}_1\cdot\vec{b}_2} $$

使用方程（3-110）、（3-115）和（3-98）和关系 $\vec{k}\cdot\vec{b}_2 = -\sin\beta$ 就会导致方程（3-116）。把方程（3-108）—（3-111），（3-115）和（3-116）代入方程（3-101）—（3-106），得到：

$$ \omega_1^A = \dot{\psi}M(1+T_{11}) + \dot{\theta}T_{12}\sin\beta/(1-T_{11}^2) $$

$$ \omega_2^A = \dot{\psi}MT_{21} - \dot{\theta}T_{13}T_{31}\sin\beta/2(1-T_{11}^2) $$

$$ \omega_3^A = \dot{\psi}MT_{31} + \dot{\theta}T_{13}T_{21}\sin\beta/2(1-T_{11}^2) $$

$$ \omega_1^B = \dot{\psi}M(1+T_{11}) + \dot{\theta}T_{12}\sin\beta/(1-T_{11}^2) $$

$$ \omega_2^B = \dot{\psi}MT_{12} - \dot{\theta}T_{13}T_{13}\sin\beta/2(1-T_{11}^2) $$

$$ \omega_3^B = \dot{\psi}MT_{13} + \dot{\theta}T_{13}T_{12}\sin\beta/2(1-T_{11}^2) $$

$$ M = [(\vec{a}_1 + \vec{b}_1)^2]^{-\frac{1}{2}} = [2(1+T_{11})]^{-\frac{1}{2}} $$

用后七个方程代入方程（3-97），得到：

$$\frac{d\psi}{d\theta}=\frac{\sqrt{2}(J/I)T_{12}\sin\beta}{(T_{11}-1)[1-T_{11}+(J/I)(1+T_{11})](1+T_{11})^{\frac{1}{2}}}$$

S 和 T 定义为

$$S=\sqrt{2}T_{12}\sin\beta$$

$$T=T_{11}$$

通过上一段文章，可以看出牛顿—欧拉法是一种矢量方法，有着很强的几何性，列式时物理意义明显，所用的分析方法是整体分析法和隔离分析法，隔离时候要对约束力和力矩作出判断。约束方程数目较多，但是同拉格朗日方法得出的控制方程相比，每一个方程简单多了，虽然拉格朗日型方程组无约束方程只有 8 个方程。

3.3 用凯恩方程解落猫问题

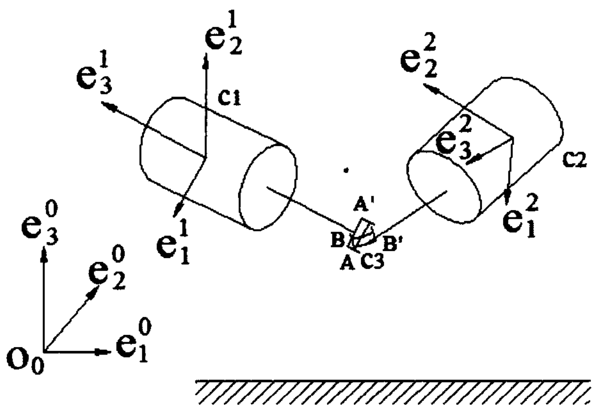

图 3.5 猫体模型的简化图

猫体模型的简化图见图 3.5，坐标系的定义与前一小节描述的一样，系统具有 8 个自由度。因为凯恩方法中需要用矢量计算，故要给出各坐标系的矢量基，固定矢量基为$[\vec{e}_1^0,\vec{e}_2^0,\vec{e}_3^0]$，1 物体的体矢量基为$[\vec{e}_1^2,\vec{e}_2^1,\vec{e}_3^1]$，2 物体的体

矢量基为$[\overrightarrow{e_1^2},\overrightarrow{e_2^2},\overrightarrow{e_3^2}]$。$C_1,C_2$分别为1物体和2物体的质心，$\overrightarrow{e_3^1}$，$\overrightarrow{e_3^2}$，分别平行于 1 物体、2 物体的纵向对称轴。$\overrightarrow{e_2^1}$，$\overrightarrow{e_1^2}$分别平行于$\overline{AA'}$和$\overline{BB'}$。$C_1C_3=s_1,C_3C_2=s_2$，$C_2e_1^1e_2^1e_3^1$坐标系依次绕$\overrightarrow{e_2^1}$轴$\overrightarrow{e_1^1}$轴转角$\beta$，$\alpha$角可得到$C_2e_1^2e_2^2e_3^2$坐标系。1物体相对固定地面的欧拉角为$(\psi,\theta,\varphi)$，$C_1$点在地面坐标系的位置为$x_1,y_1,z_1$，取广义坐标为$x,y,z,\alpha,\beta,\psi,\theta,\varphi$。

物体1的质心速度为

$$\overrightarrow{oc_1}=x_1\overrightarrow{e_1^0}+y_1\overrightarrow{e_2^0}+z_1\overrightarrow{e_3^0} \tag{3-120}$$

$$\vec{v}_1=\dot{x}_1\overrightarrow{e_1^0}+\dot{y}_1\overrightarrow{e_2^0}+\dot{z}_1\overrightarrow{e_3^0} \tag{3-121}$$

物体2 的质心速度

$$\overrightarrow{oc_2}=\overrightarrow{oc_1}-s_1\overrightarrow{e_3^1}-s_2\overrightarrow{e_3^2}$$

$$=x_1\overrightarrow{e_1^0}+y_1\overrightarrow{e_2^0}+z_1\overrightarrow{e_3^0}-s_1\overrightarrow{e_3^1}-s_2\overrightarrow{e_3^2}$$

$$\begin{aligned}\overrightarrow{v_2}&=\dot{\overrightarrow{oc_2}}\\&=\dot{x}_1\overrightarrow{e_1^0}+\dot{y}_1\overrightarrow{e_2^0}+\dot{z}_1\overrightarrow{e_3^0}-s_1\overrightarrow{\omega_1}\times\overrightarrow{e_3^1}-s_2\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\\&=\dot{x}_1\overrightarrow{e_1^0}+\dot{y}_1\overrightarrow{e_2^0}+\dot{z}_1\overrightarrow{e_3^0}+s_1(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)\overrightarrow{e_2^1}-\\&\quad s_1(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)\overrightarrow{e_1^1}-s_2(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)\overrightarrow{e_1^1}\times\overrightarrow{e_3^2}-\\&\quad s_2(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)\overrightarrow{e_2^1}\times\overrightarrow{e_3^2}-s_2(\dot{\psi}\cos\theta+\dot{\varphi})\overrightarrow{e_3^1}\times\overrightarrow{e_3^2}-\\&\quad s_2\dot{\beta}\overrightarrow{e_2^1}\times\overrightarrow{e_3^2}-s_2\dot{\alpha}\overrightarrow{e_1^2}\times\overrightarrow{e_3^2}\end{aligned} \tag{3-122}$$

由前文可知：

$$\begin{Bmatrix}\vec{e}_1^1\\\vec{e}_2^1\\\vec{e}_3^1\end{Bmatrix}=\begin{bmatrix}\cos\psi\cos\varphi-\sin\psi\cos\theta\sin\varphi & \sin\psi\cos\varphi+\cos\psi\cos\theta\sin\varphi & \sin\theta\sin\varphi\\-\cos\psi\sin\varphi-\sin\psi\cos\theta\cos\varphi & -\sin\psi\sin\varphi+\cos\psi\cos\theta\cos\varphi & \sin\theta\cos\varphi\\\sin\psi\sin\theta & -\cos\psi\sin\theta & \cos\theta\end{bmatrix}\begin{Bmatrix}\vec{e}_1^0\\\vec{e}_2^0\\\vec{e}_3^0\end{Bmatrix}$$

简记为

$$\begin{Bmatrix}\vec{e}_1^1\\\vec{e}_2^1\\\vec{e}_3^1\end{Bmatrix}=\begin{bmatrix}l_{11}^{10} & l_{12}^{10} & l_{13}^{10}\\l_{21}^{10} & l_{22}^{10} & l_{23}^{10}\\l_{31}^{10} & l_{32}^{10} & l_{33}^{10}\end{bmatrix}\begin{Bmatrix}\vec{e}_1^0\\\vec{e}_2^0\\\vec{e}_3^0\end{Bmatrix} \quad (3\text{-}123)$$

又有

$$\begin{Bmatrix}\vec{e}_1^2\\\vec{e}_2^2\\\vec{e}_3^2\end{Bmatrix}=\begin{bmatrix}\cos\beta & 0 & -\sin\beta\\-\sin\alpha\sin\beta & \cos\alpha & \sin\alpha\cos\beta\\-\sin\beta\cos\alpha & -\sin\alpha & \cos\alpha\cos\beta\end{bmatrix}\begin{Bmatrix}\vec{e}_1^1\\\vec{e}_2^1\\\vec{e}_3^1\end{Bmatrix}$$

简记为

$$\begin{Bmatrix}\vec{e}_1^2\\\vec{e}_2^2\\\vec{e}_3^2\end{Bmatrix}=\begin{bmatrix}l_{11}^{21} & l_{12}^{21} & l_{13}^{21}\\l_{21}^{21} & l_{22}^{21} & l_{23}^{21}\\l_{31}^{21} & l_{32}^{21} & l_{33}^{21}\end{bmatrix}\begin{Bmatrix}\vec{e}_1^1\\\vec{e}_2^1\\\vec{e}_3^1\end{Bmatrix} \quad (3\text{-}124)$$

所以式（3-122）中的矢量运算为

$$\vec{e}_1^1\times\vec{e}_3^2=\vec{e}_1^1\times(-\sin\beta\cos\alpha\vec{e}_1^1-\sin\alpha\vec{e}_2^1+\cos\alpha\cos\beta\vec{e}_3^1)$$
$$=-\sin\alpha\vec{e}_3^1-\cos\alpha\cos\beta\vec{e}_2^1$$
$$\vec{e}_1^1\times\vec{e}_3^2=\vec{e}_2^1\times(-\sin\beta\cos\alpha\vec{e}_1^1-\sin\alpha\vec{e}_2^1+\cos\alpha\cos\beta\vec{e}_3^1)$$
$$=\sin\beta\cos\alpha\vec{e}_3^1+\cos\beta\cos\alpha\vec{e}_1^1$$
$$\vec{e}_3^1\times\vec{e}_3^2=\vec{e}_3^1\times(-\sin\beta\cos\alpha\vec{e}_1^1-\sin\alpha\vec{e}_2^1+\cos\alpha\cos\beta\vec{e}_3^1)$$
$$=-\sin\beta\cos\alpha\vec{e}_2^1+\sin\alpha\vec{e}_1^1$$
$$\vec{e}_2^1\times\vec{e}_3^2=\vec{e}_2^1\times(-\sin\beta\cos\alpha\vec{e}_1^1-\sin\alpha\vec{e}_2^1+\cos\alpha\cos\beta\vec{e}_3^1)$$
$$=\sin\beta\cos\alpha\vec{e}_3^1+\cos\beta\cos\alpha\vec{e}_1^1$$
$$\vec{e}_1^2\times\vec{e}_3^2=\sin\beta\sin\alpha\vec{e}_1^1-\cos\alpha\vec{e}_2^1-\sin\alpha\cos\beta\vec{e}_3^1$$

（3-125）

代入$\overrightarrow{v_2}$的表达式（3-122），得：

$$\overrightarrow{v_2}=\dot{x}_1\overrightarrow{e_1^0}+\dot{y}_1\overrightarrow{e_2^0}+\dot{z}_1\overrightarrow{e_3^0}+s_1(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)\overrightarrow{e_2^1}-s_1(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)\overrightarrow{e_1^1}-$$
$$s_2[(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)\cos\alpha\cos\beta+(\dot{\psi}\cos\theta+\dot{\varphi})\sin\alpha+\dot{\beta}\cos\alpha\cos\beta+$$
$$\dot{\alpha}\sin\alpha\sin\beta]\overrightarrow{e_1^1}+s_2[(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)\cos\alpha\cos\beta+(\dot{\psi}\cos\theta+$$
$$\dot{\varphi})\sin\beta\cos\alpha+\dot{\alpha}\cos\alpha]\overrightarrow{e_2^1}-s_2[-\sin\alpha(\dot{\psi}\cos\theta+\dot{\varphi})+\sin\beta\cos\alpha(\dot{\psi}\sin\theta\cos\varphi-$$
$$\dot{\theta}\sin\varphi)+\dot{\beta}\sin\beta\cos\alpha-\dot{\alpha}\sin\alpha\cos\beta]\overrightarrow{e_3^1}$$

（3-126）

由本章第一节拉格朗日法中已经得出结果知 1 物体与 2 物体的角速度：

$$\overrightarrow{\omega_1}=(\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi)\overrightarrow{e_1^1}+(\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi)\overrightarrow{e_2^1}+(\dot{\psi}\cos\theta+\dot{\varphi})\overrightarrow{e_3^1}$$

（3-127）

$$\overrightarrow{\omega_2}=\overrightarrow{\omega_1}+\dot{\beta}\overrightarrow{e_2^1}+\dot{\alpha}\overrightarrow{e_1^2} \qquad (3\text{-}128)$$

现在取广义速率：

$$u_1=\dot{x}_1,u_2=\dot{y}_1,u_3=\dot{z}_1$$

$$\begin{aligned}u_4&=\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi\\u_5&=\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi\\u_6&=\dot{\psi}\cos\theta+\dot{\varphi}\\u_7&=\dot{\beta}\\u_8&=\dot{\alpha}\end{aligned} \qquad (3\text{-}129)$$

所以上面诸式可以写成：

$$\begin{aligned}&\overrightarrow{v_1}=u_1\overrightarrow{e_1^0}+u_2\overrightarrow{e_2^0}+u_3\overrightarrow{e_3^0}\\&\overrightarrow{v_2}=u_1\overrightarrow{e_1^0}+u_2\overrightarrow{e_2^0}+u_3\overrightarrow{e_3^0}-\\&\quad[s_1u_5+s_2(u_5\cos\alpha\cos\beta+u_4\sin\alpha+u_7\cos\alpha\cos\beta+\\&\quad u_8\sin\alpha\sin\beta)]\overrightarrow{e_1^1}+[s_1u_4+s_2(u_4\cos\alpha\cos\beta+\\&\quad u_6\sin\beta\cos\alpha+u_8\cos\alpha)]\overrightarrow{e_2^1}-s_2(-u_6\sin\alpha+\\&\quad u_5\sin\beta\cos\alpha+u_7\sin\beta\cos\alpha-u_6\sin\alpha\cos\beta)\overrightarrow{e_3^1}\end{aligned} \qquad (3\text{-}130)$$

$$\overrightarrow{\omega_1}=u_4\overrightarrow{e_1^1}+u_5\overrightarrow{e_2^1}+u_6\overrightarrow{e_3^1} \tag{3-131}$$

$$\overrightarrow{\omega_2}=u_4\overrightarrow{e_1^1}+u_5\overrightarrow{e_2^1}+u_6\overrightarrow{e_3^1}+u_7\overrightarrow{e_2^1}+u_8\overrightarrow{e_1^2} \tag{3-132}$$

由于计算铰的控制力主矢和主矩的需要，1、2 两刚体间的相对速度和相对角速度为：

$$\begin{aligned}\overrightarrow{V_2}&=\overrightarrow{v_2}-\overrightarrow{v_1}\\&=-[s_1u_5+s_2(u_5\cos\alpha\cos\beta+u_4\sin\alpha+u_7\cos\alpha\cos\beta+\\&u_8\sin\alpha\sin\beta)]\overrightarrow{e_1^1}+[s_1u_4+s_2(u_4\cos\alpha\cos\beta+\\&u_6\sin\beta\cos\alpha+u_8\cos\alpha)]\overrightarrow{e_2^1}-s_2(-u_6\sin\alpha+\\&u_5\sin\beta\cos\alpha+u_7\sin\beta\cos\alpha-u_8\sin\alpha\cos\beta)\overrightarrow{e_3^1}\end{aligned} \tag{3-133}$$

$$\overrightarrow{\Omega_2}=\overrightarrow{\omega_2}-\overrightarrow{\omega_1}=u_7\overrightarrow{e_2^1}+u_8\overrightarrow{e_1^2} \tag{3-134}$$

根据偏速度和偏角度定义，以上诸式的矢量系数为相应的广义速率 u_r 的偏速度、偏角速度，列写出来如下：

$$\overrightarrow{v_1}^1=\overrightarrow{e_1^0},\overrightarrow{v_1}^2=\overrightarrow{e_2^0},\overrightarrow{v_1}^3=\overrightarrow{e_3^0},\qquad \text{其余}\ \overrightarrow{v_1}^r=0 \tag{3-135}$$

$$\begin{aligned}&\overrightarrow{v_2}^1=\overrightarrow{e_1^0},\overrightarrow{v_2}^2=\overrightarrow{e_2^0},\overrightarrow{v_2}^3=\overrightarrow{e_3^0},\\&\overrightarrow{v_2}^4=-s_2\sin\alpha\overrightarrow{e_1^1}+(s_1+s_2\cos\alpha\cos\beta)\overrightarrow{e_2^1}\\&\overrightarrow{v_2}^5=-[s_1+s_2\cos\alpha\cos\beta)\overrightarrow{e_1^1}-s_2\sin\beta\cos\alpha\overrightarrow{e_3^1}\\&\overrightarrow{v_2}^6=s_2\sin\beta\cos\alpha\overrightarrow{e_2^1}+s_2\sin\alpha\overrightarrow{e_3^1}\\&\overrightarrow{v_2}^7=-s_2\cos\alpha\cos\beta\overrightarrow{e_1^1}-s_2\sin\beta\cos\alpha\overrightarrow{e_3^1}\\&\overrightarrow{v_2}^8=-s_2\sin\alpha\cos\beta\overrightarrow{e_1^1}+s_2\cos\alpha\overrightarrow{e_2^1}+s_2\sin\alpha\cos\beta\overrightarrow{e_3^1}\end{aligned} \tag{3-136}$$

$$\overrightarrow{\omega_1}^4=\overrightarrow{e_1^1},\ \overrightarrow{\omega_1}^5=\overrightarrow{e_2^1},\ \overrightarrow{\omega_1}^6=\overrightarrow{e_3^1},\quad \text{其 余}\ \overrightarrow{\omega_1}^r=0 \tag{3-137}$$

$$\overrightarrow{\omega_2}^4=\overrightarrow{e_1^1},\ \overrightarrow{\omega_2}^5=\overrightarrow{e_2^1},\ \overrightarrow{\omega_2}^6=\overrightarrow{e_3^1},\ \overrightarrow{\omega_2}^7=\overrightarrow{e_2^1},\ \overrightarrow{\omega_2}^8=\overrightarrow{e_1^2}\ \text{其余}\ \overrightarrow{\omega_2}^r=0 \tag{3-138}$$

$$\overrightarrow{V_2^1}=\overrightarrow{V_2^2}=\overrightarrow{V_2^3}=0 \tag{3-139}$$

$$\begin{aligned}
\overrightarrow{V_2^4}&=-s_2\sin\alpha\overrightarrow{e_1^1}+s_2\cos\alpha\cos\beta\overrightarrow{e_2^1}\\
\overrightarrow{V_2^5}&=-(s_1+s_2\cos\alpha\cos\beta)\overrightarrow{e_1^1}-\sin\beta\cos\alpha\overrightarrow{e_3^1}\\
\overrightarrow{V_2^6}&=s_2\cos\alpha\sin\beta\overrightarrow{e_2^1}+s_2\sin\alpha\overrightarrow{e_3^1}\\
\overrightarrow{V_2^7}&=-s_2\cos\alpha\cos\beta\overrightarrow{e_1^1}-s_2\sin\beta\cos\alpha\overrightarrow{e_3^1}\\
\overrightarrow{V_2^8}&=-s_2\sin\alpha\sin\beta\overrightarrow{e_1^1}+s_2\cos\alpha\overrightarrow{e_2^1}+s_2\sin\alpha\cos\beta\overrightarrow{e_3^1}
\end{aligned} \tag{3-140}$$

$$\overrightarrow{\Omega_2^7}=\overrightarrow{e_2^1},\overrightarrow{\Omega_2^8}=\overrightarrow{e_1^2},\ \text{其余}\ \overrightarrow{\Omega_2^r}=0 \tag{3-141}$$

各刚体的质心加速度和角加速度为：

$$\begin{aligned}
\dot{\vec{v}}_1&=\dot{u}_1\overrightarrow{e_1^0}+\dot{u}_2\overrightarrow{e_2^0}+\dot{u}_3\overrightarrow{e_3^0}\\
\dot{\vec{v}}_2&=\dot{u}_1\overrightarrow{e_1^0}+\dot{u}_2\overrightarrow{e_2^0}+\dot{u}_3\overrightarrow{e_3^0}-[s_1\dot{u}_5+s_2(\dot{u}_5\cos\alpha\cos\beta-\dot{\alpha}u_5\sin\alpha\cos\beta-\\
&\quad\dot{\beta}u_5\cos\alpha\sin\beta+\dot{u}_4\sin\alpha+\dot{\alpha}\cos\alpha u_4+\dot{u}_7\cos\alpha\cos\beta-\\
&\quad\dot{\alpha}u_7\sin\alpha\cos\beta-\dot{\beta}u_7\cos\alpha\sin\beta+\dot{u}_8\sin\alpha\sin\beta+\dot{\alpha}u_8\cos\alpha\sin\beta+\\
&\quad\dot{\beta}u_8\sin\alpha\cos\beta)]\overrightarrow{e_1^1}-[s_1u_5+s_2(u_5\cos\alpha\cos\beta+u_4\sin\alpha+u_7\cos\alpha\cos\beta+\\
&\quad u_8\sin\alpha\sin\beta)]\cdot\overrightarrow{\omega_1}\times\overrightarrow{e_1^1}+[s_1\dot{u}_4+s_2(\dot{u}_4\cos\alpha\cos\beta-\dot{\alpha}u_4\sin\alpha\cos\beta-\\
&\quad\dot{\beta}u_4\cos\alpha\sin\beta+\dot{u}_6\cos\alpha\sin\beta+\dot{\beta}u_6\cos\alpha\cos\beta-\dot{\alpha}u_6\sin\beta\sin\alpha+\\
&\quad\dot{u}_8\cos\alpha-\dot{\alpha}u_8\sin\alpha)]\overrightarrow{e_2^1}+[s_1u_4+s_2(u_4\cos\alpha\cos\beta+u_6\sin\beta\cos\alpha+\\
&\quad u_8\cos\alpha)]\cdot\overrightarrow{\omega_1}\times\overrightarrow{e_2^1}-s_2(-\dot{u}_6\sin\alpha-\dot{\alpha}u_6\cos\alpha+\dot{u}_5\sin\beta\cos\alpha+\\
&\quad\dot{\beta}u_5\cos\beta\cos\alpha-\dot{\alpha}u_5\sin\beta\sin\alpha+\dot{u}_7\sin\beta\cos\alpha+\dot{\beta}u_7\cos\beta\cos\alpha-\\
&\quad\dot{\alpha}u_7\sin\beta\sin\alpha-\dot{u}_8\cos\beta\sin\alpha-\dot{\alpha}u_8\cos\alpha\cos\beta+\dot{\beta}u_8\sin\alpha\sin\beta)\overrightarrow{e_3^1}-\\
&\quad s_2(-u_6\sin\alpha+u_5\cos\alpha\sin\beta+u_7\sin\beta\cos\alpha-u_8\sin\alpha\cos\beta)\cdot\overrightarrow{\omega_1}\times\overrightarrow{e_3^1}
\end{aligned} \tag{3-142}$$

式（3-142）中：

$$
\begin{aligned}
&\overrightarrow{\omega_1}\times\overrightarrow{e_1^1}=(u_4\overrightarrow{e_1^1}+u_5\overrightarrow{e_2^1}+u_6\overrightarrow{e_3^1})\times\overrightarrow{e_1^1}=-u_5\overrightarrow{e_3^1}+u_6\overrightarrow{e_2^1}\\
&\overrightarrow{\omega_1}\times\overrightarrow{e_2^1}=(u_4\overrightarrow{e_1^1}+u_5\overrightarrow{e_2^1}+u_6\overrightarrow{e_3^1})\times\overrightarrow{e_2^1}=u_4\overrightarrow{e_3^1}-u_6\overrightarrow{e_1^1}\\
&\overrightarrow{\omega_1}\times\overrightarrow{e_3^1}=(u_4\overrightarrow{e_1^1}+u_5\overrightarrow{e_2^1}+u_6\overrightarrow{e_3^1})\times\overrightarrow{e_3^1}=-u_4\overrightarrow{e_2^1}+u_5\overrightarrow{e_1^1}\\
&\dot{\overrightarrow{\omega_1}}=u_4\overrightarrow{e_1^1}+\dot{u}_4\overrightarrow{\omega_1}\times\overrightarrow{e_1^1}+\dot{u}_5\overrightarrow{e_2^1}+u_5\overrightarrow{\omega_1}\times\overrightarrow{e_2^1}+\dot{u}_6\overrightarrow{e_3^1}+u_6\overrightarrow{\omega_1}\times\overrightarrow{e_3^1}\\
&\quad=\dot{u}_4\overrightarrow{e_1^1}+u_4(-u_5\overrightarrow{e_3^1}+u_6\overrightarrow{e_2^1})+\dot{u}_5\overrightarrow{e_2^1}+u_5(u_4\overrightarrow{e_3^1}-u_6\overrightarrow{e_1^1})+\\
&\qquad\dot{u}_6\overrightarrow{e_3^1}+u_6(-u_4\overrightarrow{e_2^1}+u_5\overrightarrow{e_1^1})\\
&\quad=\dot{u}_4\overrightarrow{e_1^1}+\dot{u}_5\overrightarrow{e_2^1}+\dot{u}_6\overrightarrow{e_3^1}\\
&\dot{\overrightarrow{\omega_2}}=\dot{\overrightarrow{\omega_1}}+\dot{u}_7\overrightarrow{e_2^1}+u_7\overrightarrow{\omega_1}\times\overrightarrow{e_2^1}+\dot{u}_8\overrightarrow{e_1^2}+u_8\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}
\end{aligned}
\tag{3-143}
$$

式（3-143）中

$$
\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}=u_4\overrightarrow{e_1^1}\times\overrightarrow{e_1^2}+u_5\overrightarrow{e_2^1}\times\overrightarrow{e_1^2}+u_6\overrightarrow{e_3^1}\times\overrightarrow{e_1^2}+u_7\overrightarrow{e_2^1}\times\overrightarrow{e_1^2}
\tag{3-144}
$$

式（3-144）中

$$
\begin{aligned}
&\overrightarrow{e_1^1}\times\overrightarrow{e_1^2}=\overrightarrow{e_1^1}\times(\cos\beta\overrightarrow{e_1^1}-\sin\beta\overrightarrow{e_3^1})=\sin\beta\overrightarrow{e_2^1}\\
&\overrightarrow{e_2^1}\times\overrightarrow{e_1^2}=\overrightarrow{e_2^1}\times(\cos\beta\overrightarrow{e_1^1}-\sin\beta\overrightarrow{e_3^1})=-\cos\beta\overrightarrow{e_3^1}-\sin\beta\overrightarrow{e_1^1}
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ &\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}=u_4\sin\beta\overrightarrow{e_2^1}+(u_5+u_7)(-\cos\beta\overrightarrow{e_3^1}-\sin\beta\overrightarrow{e_1^1})+u_6\cos\beta\overrightarrow{e_2^1}\\
\therefore\ &\dot{\overrightarrow{\omega_2}}=\dot{\overrightarrow{\omega_1}}+\dot{u}_7\overrightarrow{e_2^1}+u_7(u_4\overrightarrow{e_3^1}-u_6\overrightarrow{e_1^1})+\dot{u}_8\overrightarrow{e_1^2}+\\
&\qquad u_8[u_4\sin\beta\overrightarrow{e_2^1}+(u_5+u_7)(-\cos\beta\overrightarrow{e_3^1}-\sin\beta\overrightarrow{e_1^1})+u_6\cos\beta\overrightarrow{e_2^1}]\\
&\quad=[\dot{u}_4-u_6u_7-u_8(u_5+u_7)\sin\beta]\overrightarrow{e_1^1}+[\dot{u}_7+u_4u_8\sin\beta+u_6u_8\cos\beta]\overrightarrow{e_2^1}+\\
&\qquad[u_4u_7-u_8(u_5+u_7)\cos\beta]\overrightarrow{e_3^1}+\dot{u}_8\overrightarrow{e_1^2}
\end{aligned}
\tag{3-145}
$$

各刚体的重力为 $m_i g(i=1,2)$，则重力的主矢相对质心 c_i 的主矩为：

$$
\overrightarrow{G_i}=-m_i g\overrightarrow{e_3^0},\qquad \overrightarrow{M_i}=0
\tag{3-146}
$$

各铰的控制力及相对铰的主矩为：

$$\overrightarrow{F^a}=0,\qquad \overrightarrow{M^a}=M_\alpha\overrightarrow{e_2^1}+M_\beta\overrightarrow{e_1^2} \tag{3-147}$$

下面计算对应于广义速率u_r的广义主动力和广义惯性力。

对应于速率u_1，系统的广义主动力为：

$$F_1'=(-m_1g\overrightarrow{e_3^0})\cdot\overrightarrow{e_1^0}+(-m_2g\overrightarrow{e_3^0})\cdot\overrightarrow{e_1^0}+(M_\beta\overrightarrow{e_2^1}+M_\alpha\overrightarrow{e_1^2})\cdot 0=0 \tag{3-148}$$

为了下文行文简洁之便，我们在这里把$\dot{\overrightarrow{v_2}}$，$\dot{\overrightarrow{\omega_2}}$写成如下的表达形式：

$$\dot{\vec{\omega}}_2=F_1\vec{e}_1^1+F_2\vec{e}_2^1+F_3\vec{e}_3^1+\dot{u}_8\vec{e}_1^2 \tag{3-149}$$

$$\dot{\overrightarrow{v_2}}=\dot{u}_1\overrightarrow{e_1^0}+\dot{u}_2\overrightarrow{e_2^0}+\dot{u}_3\overrightarrow{e_3^0}+E_1\overrightarrow{e_1^1}+E_2\overrightarrow{\omega_1}\times\overrightarrow{e_1^1}+E_3\overrightarrow{e_2^1}+E_4\overrightarrow{\omega_1}\times\overrightarrow{e_2^1}+E_5\overrightarrow{e_3^1}+E_6\overrightarrow{\omega_1}\times\overrightarrow{e_3^1} \tag{3-150}$$

对应于广义速率u_1，系统的广义惯性力为：

$$\begin{aligned}F_1^*&=-m_1\dot{\overrightarrow{v_1}}\overrightarrow{v_1^1}-\overline{J_1}\dot{\overrightarrow{\omega_1}}\cdot\overrightarrow{\omega_1^1}-m_2\dot{\overrightarrow{v_2}}\cdot\overrightarrow{v_2^1}-\overline{J_2}\dot{\overrightarrow{\omega_2}}\cdot\overrightarrow{\omega_2^1}\\&=-m_1(\dot{u}_1\overrightarrow{e_1^0}+\dot{u}_2\overrightarrow{e_2^0}+\dot{u}_3\overrightarrow{e_3^0})\cdot\overrightarrow{e_1^0}-m_2(\dot{u}_1\overrightarrow{e_1^0}+\dot{u}_2\overrightarrow{e_2^0}+\dot{u}_3\overrightarrow{e_3^0}+E_1\overrightarrow{e_1^1}+E_2\overrightarrow{\omega_1}\times\overrightarrow{e_1^1}+\\&\quad E_3\overrightarrow{e_2^1}+E_4\overrightarrow{\omega_1}\times\overrightarrow{e_2^1}+E_5\overrightarrow{e_3^1}+E_6\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})\cdot\overrightarrow{e_1^0}\\&=-m_1\dot{u}_1-m_2[\dot{u}_1+E_1l_{11}^{10}+E_2(-u_5l_{31}^{10}+u_6l_{21}^{10})+E_3l_{21}^{10}+E_4(u_4l_{31}^{10}-u_6l_{11}^{10})+\\&\quad E_5l_{31}^{10}+E_6(-u_4l_{21}^{10}+u_5l_{11}^{10})]\end{aligned} \tag{3-151}$$

上文中出现的$\overline{J_1},\overline{J_2}$为刚体 1、刚体 2 的转动张量

$$\begin{aligned}\overline{J_1}&=J_{11}\overrightarrow{e_1^1}\overrightarrow{e_1^1}+J_{12}\overrightarrow{e_2^1}\overrightarrow{e_2^1}+J_{13}\overrightarrow{e_3^1}\overrightarrow{e_3^1}\\\overline{J_2}&=J_{21}\overrightarrow{e_1^2}\overrightarrow{e_1^2}+J_{22}\overrightarrow{e_2^2}\overrightarrow{e_2^2}+J_{23}\overrightarrow{e_3^2}\overrightarrow{e_3^2}\end{aligned} \tag{3-152}$$

对广义速率u_2系统的广义主动力为：

$$\begin{aligned}F_2'&=(-m_1g\overrightarrow{e_3^0})\cdot\overrightarrow{e_2^0}+(-m_2g\overrightarrow{e_3^0})\cdot\overrightarrow{e_2^0}+M_\alpha\cdot 0\\&=0\end{aligned} \tag{3-153}$$

对应于广义速率u_2，系统的广义惯性力为：

$$F_2^* = -m_1 \overrightarrow{\dot{v}_1}\cdot\overrightarrow{v_1}^2 - \overline{J_1}\,\overrightarrow{\dot{\omega}_1}\cdot\overrightarrow{\omega_1}^2 - m_2 \overrightarrow{\dot{v}_2}\cdot\overrightarrow{v_2}^2 - \overline{J_2}\,\overrightarrow{\dot{\omega}_2}\cdot\overrightarrow{\omega_2}^2$$

$$\begin{aligned} &= -m_1\dot{u}_2 - m_2(\dot{u}_1\overrightarrow{e_1^0} + \dot{u}_2\overrightarrow{e_2^0} + \dot{u}_3\overrightarrow{e_3^0} + E_1\overrightarrow{e_1^1} + E_2\overrightarrow{\omega_1}\times\overrightarrow{e_1^1} + \\ &E_3\overrightarrow{e_2^1} + E_4\overrightarrow{\omega_1}\times\overrightarrow{e_2^1} + E_5\overrightarrow{e_3^1} + E_6\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})\cdot\overrightarrow{e_2^0} \\ &= -m_1\dot{u}_2 - m_2[\dot{u}_2 + E_1 l_{12}^{10} + E_2(-u_5 l_{32}^{10} + u_6 l_{22}^{10}) + E_3 l_{22}^{10} + E_4(u_4 l_{32}^{10} - u_6 l_{12}^{10}) + \\ &E_5 l_{32}^{10} + E_6(-u_4 l_{22}^{10} + u_5 l_{12}^{10})] \end{aligned} \tag{3-154}$$

对于广义速率u_3，系统的广义主动力为：

$$F_3' = (-m_1 g\overrightarrow{e_3^0})\cdot\overrightarrow{e_3^0} + (-m_2 g\overrightarrow{e_3^0})\cdot\overrightarrow{e_3^0} + \overrightarrow{M^a}\cdot 0 = -m_1 g - m_2 g \tag{3-155}$$

对于广义速率u_3，系统的广义惯性力为：

$$\begin{aligned} F_3^* &= -m_1 \overrightarrow{\dot{v}_1}\cdot\overrightarrow{v_1}^3 - \overline{J_1}\,\overrightarrow{\dot{\omega}_1}\cdot\overrightarrow{\omega_1}^3 - m_2 \overrightarrow{\dot{v}_2}\cdot\overrightarrow{v_2}^3 - \overline{J_2}\,\overrightarrow{\dot{\omega}_2}\cdot\overrightarrow{\omega_2}^3 \\ &= -m_1\dot{u}_3 - m_2[\dot{u}_3 + E_1 l_{13}^{10} + E_2(-u_5 l_{33}^{10} + u_6 l_{23}^{10}) + E_3 l_{23}^{10} + \\ &\quad E_4(u_4 l_{33}^{10} - u_6 l_{13}^{10}) + E_5 l_{33}^{10} + E_6(-u_4 l_{23}^{10} + u_5 l_{13}^{10})] \end{aligned} \tag{3-156}$$

对于广义速率u_4，系统的广义主动力为：

$$\begin{aligned} F_4' &= (-m_1 g\overrightarrow{e_3^0})\cdot\vec{0} + (-m_2 g\overrightarrow{e_3^0})\cdot[-s_2\sin\alpha\overrightarrow{e_1^1} + (s_1 + s_2\cos\alpha\cos\beta)\overrightarrow{e_2^1}] + \overrightarrow{M^a}\cdot\vec{0} \\ &= m_2 g s_2\sin\alpha l_{13}^{10} + (-m_2 g\cos\alpha\cos\beta) l_{23}^{10} \end{aligned} \tag{3-157}$$

对于广义速率u_4系统的广义惯性力为：

$$\begin{aligned} F_4^* &= -m_1 \overrightarrow{\dot{v}_1}\cdot\overrightarrow{v_1}^4 - \overline{J_1}\,\overrightarrow{\dot{\omega}_1}\cdot\overrightarrow{\omega_1}^4 - m_2 \overrightarrow{\dot{v}_2}\cdot\overrightarrow{v_2}^4 - \overline{J_2}\,\overrightarrow{\dot{\omega}_2}\cdot\overrightarrow{\omega_2}^4\,] \\ &= -m_1\overrightarrow{\dot{v}_1}\cdot\vec{0} - (J_{11}\overrightarrow{e_1^1}\overrightarrow{e_1^1} + J_{12}\overrightarrow{e_2^1}\overrightarrow{e_2^1} + J_{13}\overrightarrow{e_3^1}\overrightarrow{e_3^1})\cdot(u_4\overrightarrow{e_1^1} + \dot{u}_5\overrightarrow{e_2^1} + \dot{u}_6\overrightarrow{e_3^1})\cdot\overrightarrow{e_1^1} - \\ &\quad m_2(\dot{u}_1\overrightarrow{e_1^0} + \dot{u}_2\overrightarrow{e_2^0} + \dot{u}_3\overrightarrow{e_3^0} + E_1\overrightarrow{e_1^1} + E_2\overrightarrow{\omega_1}\times\overrightarrow{e_1^1} + E_3\overrightarrow{e_2^1} + E_2\overrightarrow{\omega_1}\times\overrightarrow{e_2^1} + E_5\overrightarrow{e_3^1} + \\ &\quad E_6\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})\cdot[-s_2\sin\alpha\overrightarrow{e_1^1} + (s_1 + s_2\cos\alpha\cos\beta)\overrightarrow{e_2^1}] - (J_{21}\overrightarrow{e_1^2}\overrightarrow{e_1^2} + J_{22}\overrightarrow{e_2^2}\overrightarrow{e_2^2} + \\ &\quad J_{23}\overrightarrow{e_3^2}\overrightarrow{e_3^2})\cdot(F_1\overrightarrow{e_1^1} + F_2\overrightarrow{e_2^1} + F_3\overrightarrow{e_3^1} + \dot{u}_8 e_1^2)\cdot\overrightarrow{e_1^1} \end{aligned}$$

$$= -J_{11}\dot{u}_4 - m_2\{-s_2 \sin\alpha(\dot{u}_1 l_{11}^{10} + \dot{u}_2 l_{12}^{10} + \dot{u}_3 l_{13}^{10} + E_1 - E_4 u_6 + E_6 u_5) + (s_1 + s_2\cos\alpha\cos\beta)(\dot{u}_1 l_{21}^{10} + \dot{u}_2 l_{22}^{10} + \dot{u}_3 l_{23}^{10} + E_2 u_6 + E_3 - E_6 u_4) - [J_{21}F_1 l_{11}^{21} l_{11}^{21} + J_{21}F_2 l_{12}^{21} l_{11}^{21} + J_{21}F_3 l_{13}^{21} l_{11}^{21} + \dot{u}_8 J_{21} l_{11}^{21} + J_{22}F_1 l_{21}^{21} l_{21}^{21} + J_{22}F_2 l_{22}^{21} l_{21}^{21} + J_{22}F_3 l_{23}^{21} l_{21}^{21} + J_{23}F_1 l_{31}^{21} l_{31}^{21} + J_{23}F_2 l_{32}^{21} l_{31}^{21} + J_{23}F_3 l_{33}^{21} l_{33}^{21} - l_{21}^{21}]\} \quad (3\text{-}158)$$

对于广义速率u_5的广义主动力为：

$$F_5' = (-m_1 g\overrightarrow{e_3^0})\cdot 0 + (-m_2 g\overrightarrow{e_3^0})\cdot[-(s_1 + s_2\cos\alpha\cos\beta)\overrightarrow{e_1^1} - s_2\sin\beta\cos\alpha\overrightarrow{e_3^1}] + \overrightarrow{M^a}\cdot\vec{0} = m_2 g[(s_1 + s_2\cos\alpha\cos\beta)l_{13}^{10} + s_2\sin\beta\cos\alpha l_{30}^{10}] \quad (3\text{-}159)$$

对于广义速率u_5的广义惯性力为：

$$F_5^* = -m_1\dot{\vec{v}}_1\vec{v}_1^5 - \overline{J_1}\dot{\vec{\omega}}_1\cdot\vec{\omega}_1^5 - m_2\dot{\vec{v}}_2\vec{v}_2^5 - \overline{J_2}\dot{\vec{\omega}}_2\cdot\vec{\omega}_2^5$$

$$= -(J_{11}\dot{u}_4\overrightarrow{e_1^1} + J_{12}\dot{u}_5\overrightarrow{e_2^1} + J_{13}\dot{u}_6\overrightarrow{e_3^1})\cdot\overrightarrow{e_2^1} - m_2(\dot{u}_1\overrightarrow{e_1^0} + \dot{u}_2\overrightarrow{e_2^0} + \dot{u}_3\overrightarrow{e_3^0} + E_1\overrightarrow{e_1^1} + E_2\overrightarrow{\omega_1}\times\overrightarrow{e_1^1} + E_3\overrightarrow{e_2^1} + E_4\overrightarrow{\omega_1}\times\overrightarrow{e_2^1} + E_5\overrightarrow{e_3^1} + E_6\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})\cdot[-(s_1 + s_2\cos\alpha\cos\beta)\overrightarrow{e_1^1} - s_2\sin\beta\cos\alpha\overrightarrow{e_3^1}] - (J_{21}\overrightarrow{e_1^2}\overrightarrow{e_1^2} + J_{22}\overrightarrow{e_2^2}\overrightarrow{e_2^2} + J_{23}\overrightarrow{e_3^2}\overrightarrow{e_3^2})\cdot(F_1\overrightarrow{e_1^1} + F_2\overrightarrow{e_2^1} + F_3\overrightarrow{e_3^1} + \dot{u}_8\overrightarrow{e_1^2})\cdot\overrightarrow{e_2^1}$$

$$= -J_{12}\dot{u}_5 + m_2[(s_1 + s_2\cos\alpha\cos\beta)(\dot{u}_1 l_{11}^{10} + \dot{u}_2 l_{12}^{10} + \dot{u}_3 l_{13}^{10} + E_1 - E_4 u_6 + E_6 u_5) + s_2\sin\beta\cos\alpha(\dot{u}_1 l_{31}^{10} + \dot{u}_2 l_{32}^{10} + \dot{u}_3 l_{33}^{10} - E_2 u_5 + E_4 u_4 + E_5)] - [J_{21}F_1 l_{11}^{21} l_{12}^{21} + J_{21}F_2 l_{12}^{21} l_{12}^{21} + J_{21}F_3 l_{13}^{21} l_{12}^{21} + \dot{u}_8 J_{21} l_{12}^{21} + J_{22}F_1 l_{21}^{21} l_{22}^{21} + J_{22}F_2 l_{22}^{21} l_{22}^{21} + J_{22}F_3 l_{23}^{21} l_{22}^{21} + J_{23}F_1 l_{31}^{21} l_{32}^{21} + J_{23}F_2 l_{32}^{21} l_{32}^{21} + J_{23}F_3 l_{33}^{21} l_{32}^{21}] \quad (3\text{-}160)$$

对于广义速率u_6的广义主动力为：

$$F_6' = (-m_1 g\overrightarrow{e_3^0})\cdot\vec{0} + (-m_2 g\overrightarrow{e_3^0})\cdot(s_2\cos\alpha\sin\beta\overrightarrow{e_2^1} + s_2\sin\alpha\overrightarrow{e_3^1}) + \overrightarrow{M^a}\cdot\vec{0} = -m_2 g(s_2\cos\alpha\sin\beta l_{23}^{10} + s_2\sin\alpha l_{33}^{10}) \quad (3\text{-}161)$$

对于广义速率u_6的广义惯性力为：

$$F_6^* = -m_1 \dot{\vec{v}}_1 \cdot \vec{v}_1^6 - \overline{J_1}\dot{\vec{\omega}}_1 \cdot \vec{\omega}_1^6 - m_2 \dot{\vec{v}}_2 \cdot \vec{v}_2^6 - \overline{J_2}\dot{\vec{\omega}}_2 \cdot \vec{\omega}_2^6$$

$$= -(J_{11}\dot{u}_4\vec{e_1^1} + J_{12}\dot{u}_5\vec{e_2^1} + J_{13}\dot{u}_6\vec{e_3^1}) \cdot \vec{e_3^1} - m_2(\dot{u}_1\vec{e_1^0} + \dot{u}_2\vec{e_2^0} + \dot{u}_3\vec{e_3^0} +$$

$$E_1\vec{e_1^1} + E_2\vec{\omega_1} \times \vec{e_1^1} + E_3\vec{e_2^1} + E_4\vec{\omega_1} \times \vec{e_2^1} + E_5\vec{e_3^1} + E_6\vec{\omega_1} \times \vec{e_3^1}) \cdot [s_2\cos\alpha\sin\beta\vec{e_2^1} +$$

$$s_2\sin\alpha\vec{e_3^1}] - (J_{21}\vec{e_1^2}\vec{e_1^2} + J_{22}\vec{e_2^2}\vec{e_2^2} + J_{23}\vec{e_3^2}\vec{e_3^2}) \cdot (F_1\vec{e_1^1} + F_2\vec{e_2^1} + F_3\vec{e_3^1} + \dot{u}_8\vec{e_1^2}) \cdot \vec{e_3^1}$$

$$= -J_{13}\dot{u}_6 - m_2[s_2\cos\alpha\sin\beta(\dot{u}_1 l_{21}^{10} + \dot{u}_2 l_{22}^{10} + \dot{u}_3 l_{23}^{10} + E_2 u_6 + E_3 - E_6 u_4) +$$

$$s_2\sin\alpha(\dot{u}_1 l_{31}^{10} + \dot{u}_2 l_{32}^{10} + \dot{u}_3 l_{33}^{10} - E_2 u_5 + E_4 u_4 + E_5)] - (J_{21}F_1 l_{11}^{21} l_{13}^{21} + J_{21}F_2 l_{12}^{21} l_{13}^{21}) -$$

$$J_{22} l_{23}^{21}(F_1 l_{21}^{21} + F_2 l_{22}^{21} + F_3 l_{23}^{21}) - J_{23} l_{33}^{21}(F_1 l_{31}^{21} + F_2 l_{32}^{21} + F_3 l_{33}^{21}) - \dot{u}_8 J_{21} l_{13}^{21}$$

（3-162）

对于广义速率u_7的广义主动力为：

$$F_7' = (-m_1 g\vec{e_3^0}) \cdot \vec{0} + (-m_2 g\vec{e_3^0}) \cdot (-s_2\cos\alpha\cos\beta\vec{e_1^1} - s_2\cos\alpha\sin\beta\vec{e_3^1}) +$$

$$(M_\beta\vec{e_2^1} + M_\alpha\vec{e_1^2}) \cdot \vec{e_2^1} \tag{3-163}$$

$$= -m_2 g(s_2\cos\alpha\cos\beta l_{13}^{10} + s_2\sin\beta\cos\alpha l_{33}^{10}) + M_\beta + M_\alpha l_{12}^{21}$$

对于广义速率u_7的广义惯性力为：

$$F_7^* = -m_1 \dot{\vec{v}}_1 \cdot \vec{v}_1^7 - \overline{J_1}\dot{\vec{\omega}}_1 \cdot \vec{\omega}_1^7 - m_2 \dot{\vec{v}}_2 \cdot \vec{v}_2^7 - \overline{J_2}\dot{\vec{\omega}}_2 \cdot \vec{\omega}_2^7$$

$$= -m_2(\dot{u}_1\vec{e_1^0} + \dot{u}_2\vec{e_2^0} + \dot{u}_3\vec{e_3^0} + E_1\vec{e_1^1} + E_2\vec{\omega_1} \times \vec{e_1^1} + E_3\vec{e_2^1} + E_4\vec{\omega_1} \times \vec{e_2^1} + E_5\vec{e_3^1} +$$

$$E_6\vec{\omega_1} \times \vec{e_3^1}) \cdot [-s_2\cos\alpha\cos\beta\vec{e_1^1} - s_2\sin\beta\cos\alpha\vec{e_3^1}] -$$

$$(J_{21}\vec{e_1^2}\vec{e_1^2} + J_{22}\vec{e_2^2}\vec{e_2^2} + J_{23}\vec{e_3^2}\vec{e_3^2}) \cdot (F_1\vec{e_1^1} + F_2\vec{e_2^1} + F_3\vec{e_3^1} + \dot{u}_8\vec{e_1^2}) \cdot \vec{e_2^1}$$

$$= m_2[s_2\cos\alpha\cos\beta(\dot{u}_1 l_{11}^{10} + \dot{u}_2 l_{12}^{10} + \dot{u}_3 l_{13}^{10} + E_1 - E_4 u_6 + E_6 u_5) +$$

$$s_2\sin\beta\sin\alpha(\dot{u}_1 l_{31}^{10} + \dot{u}_2 l_{32}^{10} + \dot{u}_3 l_{33}^{10} - E_2 u_5 + E_4 u_4 + E_5)] -$$

$$J_{21} l_{12}^{21}(F_1 l_{11}^{21} + F_2 l_{12}^{21} + F_3 l_{13}^{21}) - \dot{u}_8 J_{21} l_{12}^{21} - J_{22} l_{22}^{21}(F_1 l_{21}^{21} + F_2 l_{22}^{21} + F_3 l_{23}^{21}) -$$

$$J_{23} l_{32}^{21}(F_1 l_{31}^{21} + F_2 l_{32}^{21} + F_3 l_{33}^{21})$$

（3-164）

对于广义速率u_8的广义主动力为：

$$F_8' = (-m_1 g\overrightarrow{e_3^0})\cdot\vec{0} + (-m_2 g\overrightarrow{e_3^0})\cdot(-s_2\sin\alpha\cos\beta\overrightarrow{e_1^1} + s_2\cos\alpha\overrightarrow{e_2^1} + s_2\cos\beta\sin\alpha\overrightarrow{e_3^1}) + (M_\beta\overrightarrow{e_2^1} + M_\alpha\overrightarrow{e_1^2})\cdot\overrightarrow{e_1^2}$$
$$= -m_2 g(-s_2\sin\alpha\cos\beta l_{13}^{10} + s_2\cos\alpha l_{23}^{10} + s_2\sin\alpha\cos\beta l_{33}^{10}) + M_\beta l_{12}^{21} + M_\alpha \tag{3-165}$$

对于广义速率u_8的广义惯性力为：

$$F_8^* = -m_1\dot{\vec{v}}_1\vec{v}_1^{\,8} - \overline{J_1}\dot{\vec{\omega}}_1\cdot\vec{\omega}_1^{\,8} - m_2\dot{\vec{v}}_2\vec{v}_2^{\,8} - \overline{J_2}\dot{\vec{\omega}}_2\cdot\vec{\omega}_2^{\,8}$$

$$= -m_2(\dot{u}_1\overrightarrow{e_1^0} + \dot{u}_2\overrightarrow{e_2^0} + \dot{u}_3\overrightarrow{e_3^0} + E_1\overrightarrow{e_1^1} + E_2\overrightarrow{\omega_1}\times\overrightarrow{e_1^1} + E_3\overrightarrow{e_2^1} + E_4\overrightarrow{\omega_1}\times\overrightarrow{e_2^1} + E_5\overrightarrow{e_3^1} + E_6\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})\cdot (-s_2\sin\alpha\cos\beta\overrightarrow{e_1^1} + s_2\cos\alpha\overrightarrow{e_2^1} + s_2\sin\alpha\cos\beta\overrightarrow{e_3^1}) - (J_{21}\overrightarrow{e_1^2}\overrightarrow{e_1^2} + J_{22}\overrightarrow{e_2^2}\overrightarrow{e_2^2} + J_{23}\overrightarrow{e_3^2}\overrightarrow{e_3^2})\cdot (F_1\overrightarrow{e_1^1} + F_2\overrightarrow{e_2^1} + F_3\overrightarrow{e_3^1} + \dot{u}_8\overrightarrow{e_1^2})\cdot\overrightarrow{e_1^2}$$
$$= -m_2[-s_2\sin\alpha\cos\beta(\dot{u}_1 l_{11}^{10} + \dot{u}_2 l_{12}^{10} + \dot{u}_3 l_{13}^{10} + E_1 - E_4 u_6 + E_6 u_5) + s_2\cos\alpha(\dot{u}_1 l_{21}^{10} + \dot{u}_2 l_{22}^{10} + \dot{u}_3 l_{23}^{10} + E_2 u_6 + E_3 - E_6 u_4) + s_2\sin\alpha\cos\beta(\dot{u}_1 l_{31}^{10} + \dot{u}_2 l_{32}^{10} + \dot{u}_3 l_{33}^{10} - E_2 + E_4 u_4 + E_5)] - (J_{21}F_1 l_{11}^{21} + J_{21}F_2 l_{12}^{21} + J_{21}F_3 l_{13}^{21} + \dot{u}_8 J_{21}) \tag{3-166}$$

把上述式子代入凯恩方程：

$$F_k' + F_k^* = 0 \qquad (k = 1\sim 8) \tag{3-167}$$

即得运动方程。

3.4 本章小结

本章对落猫问题进行研究，把猫体简化成两个圆柱体用万向接头相连的多刚体模型。分别用拉格朗日方程、牛顿—欧拉法、凯恩方程对模型进行分析与详细的理论推导得到动力学方程。本章采用矩阵推导，这使全文在形式上比较简洁，计算上更加方便。

（1）本章采用 1 物体的质心作为基点，若采用系统的质心作为基点如本章第一节末尾所示需要增加约束方程。通过 1 物体与地面的方向余弦矩阵，1 物体与 2 物体的方向余弦矩阵，得到 2 物体与地面的方向余弦矩阵，最终可以确定整个系统的位形。再通过有关的公式与求导运算得出质心的速度与加速度，物体 1、2 的角速度与角加速度。

（2）在第二节中牛顿—欧拉法解题，从文中可以看出系统的约束方程其实质就是系统的质心运动方程与角动量守恒方程。凯恩在《A DYNAMICAL EXPLANATION OF THE FALLING CAT PHENOMENON》一文中运用角动量守恒得出控制方程（3-91），其实它仅是约束方程里的一类约束方程。为了与本文对照，对其进行局部摘录，本文第五章要给出其方程（3-91）的级数解。

（3）本章第三节采用凯恩方程解题，虽然该方法需要矢量运算，但与前文一样也是采用了较多的矩阵运算。

第 4 章　落猫的多柔体模型

4.1 用 Kane 方程求解落猫的多柔体模型

柔性多体系统，又称多柔体系统。它相对于多刚体而言的，其特点是柔体而不是刚体。这实质上是模型的建立问题。对同一个多体系统，当系统的组成部分在运动过程中不发生显著的弹性变形时，可以把它看成刚体模型。当发生显著的弹性变形时，我们不能不考虑变形时就取柔体模型。总之多刚体与多柔性是两种模型。若模型建立过程中考虑单元的弹性变形，就称为多柔体模型。我们可以看出多柔体模型更加逼近系统的固有性质，但是它的控制方程也是繁的多的。故不能一概说，多柔体模型优于多刚体模型。选择哪种模型要看问题的本身以及我们要求的精度。

针对柔性，我们如何去描述它，即建立什么样的模型，各国的学者提出不同的看法。本文采用一种“有限段”建模法[4]，此法与结构分析的有限元法类似。我们把柔性体离散化成有限个刚体；刚体与刚体之间用弹性元件连接。并且假设弹性元件都处于小变形，线弹性的状态下。

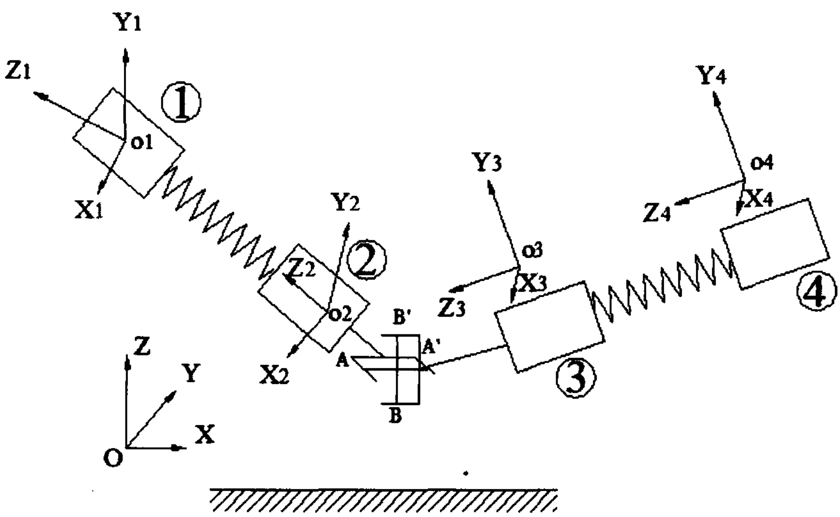

图 4.1 多柔体模型

如图 4. 1 把猫体的前躯看成 1, 2 两圆柱体模型，中间用一弯曲弹簧连接，刚度系数为 k_1(N/rad)；猫体后躯看成 3, 4 两圆柱体，中间也用一弯曲弹簧连接，刚度系数为 k_2（N/rad）可以假设 $k_1 = k_2 = k$ (N/rad)，再进一步假设 $m_1 = m_2 = m_3 = m_4$，$[J_1]=[J_2]=[J_3]=[J_4]$，$[J_i]$，i=1, 2, 3, 4 代表物体 i 的主转动惯量矩阵。在初步模型选取时，可以认为 1 与 2，3 与 4 之间无线位移变形。猫的躯体不能扭转的假设依旧成立。显然可以看出系统的自由度增加至 8+4=12。建立惯性坐标系（地面坐标系）$oxyz$；固定于 2 物体的坐标系 $o_2x_2y_2z_2$，其中 oz_2 平行于 2 柱体的纵向对称轴，$o_2y_2 // AA'$，绕 AA' 轴的相对转角为 β；固定于 3 物体的坐标系 $o_3x_3y_3z_3$，其中 o_3z_3 平行 3 物体的纵向对称轴，$o_3x_3 // BB'$，绕 BB' 轴的相对转角为 α。由于假设猫体只弯不扭，故 1 物体相对于 2 物体的定位只需要两个变量 θ_1, ψ_1，4 物体相对于 3 物体的定位，也只要两个变量 θ_4, ψ_4。2 物体相对于地面坐标系的三个欧拉角为 $\psi_2, \theta_2, \varphi_2$，用 $\overrightarrow{e^i}$ 代表 i 坐标系的矢量基。

由前文章节知

$$\{\overrightarrow{e^2}\} = [L]_{20}\{\overrightarrow{e^0}\} \tag{4-1}$$

式中

$$[L]_{20} = \begin{bmatrix} \cos\psi_2\cos\varphi_2 - \sin\psi_2\cos\theta_2\sin\varphi_2 & \sin\psi_2\cos\varphi_2 + \cos\psi_2\cos\theta_2\sin\varphi_2 & \sin\theta_2\sin\varphi_2 \\ -\cos\psi_2\sin\varphi_2 - \sin\psi_2\cos\theta_2\cos\varphi_2 & -\sin\psi_2\sin\varphi_2 + \cos\psi_2\cos\theta_2\cos\varphi_2 & \sin\theta_2\cos\varphi_2 \\ \sin\psi_2\sin\theta_2 & -\cos\psi_2\sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$\{\overrightarrow{e^3}\} = [L]_{32}\{\overrightarrow{e^2}\} \tag{4-2}$$

$$[L]_{32} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ -\sin\alpha\sin\beta & \cos\alpha & \sin\alpha\cos\beta \\ -\sin\beta\cos\alpha & -\sin\alpha & \cos\alpha\cos\beta \end{bmatrix}$$

$$\{\overrightarrow{e^4}\}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_4 & \sin\theta_4\\ 0 & -\sin\theta_4 & \cos\theta_4\end{bmatrix}\begin{bmatrix}\cos\psi_4 & \sin\psi_4 & 0\\ -\sin\psi_4 & \cos\psi_4 & 0\\ 0 & 0 & 1\end{bmatrix}\{\overrightarrow{e^3}\}$$

$$=\begin{bmatrix}\cos\psi_4 & \sin\psi_4 & 0\\ -\sin\psi_4\cos\theta_4 & \cos\theta_4\cos\psi_4 & \sin\theta_4\\ \sin\theta_4\sin\psi_4 & -\sin\theta_4\cos\psi_4 & \cos\theta_4\end{bmatrix}\{\overrightarrow{e^3}\} \tag{4-3}$$

$$\{\overrightarrow{e^1}\}=\begin{bmatrix}\cos\psi_1 & \sin\psi_1 & 0\\ -\sin\psi_1\cos\theta_1 & \cos\theta_1\cos\psi_1 & \sin\theta_1\\ \sin\theta_1\sin\psi_1 & -\sin\theta_1\cos\psi_1 & \cos\theta_1\end{bmatrix}\{\overrightarrow{e^2}\} \tag{4-4}$$

设各柱体的轴向长均为 s，o_2 点在地面坐标系上的坐标为 x_2, y_2, z_2，下面求各柱体质心的速度：

求 $\overrightarrow{v_2}$：

$$\overrightarrow{oo_2}=x_2\overrightarrow{e_1^0}+y_2\overrightarrow{e_2^0}+z_2\overrightarrow{e_3^0} \tag{4-5}$$

$$\overrightarrow{v_2}=\dot{\overrightarrow{oo_2}}=\dot{x}_2\overrightarrow{e_1^0}+\dot{y}_2\overrightarrow{e_2^0}+\dot{z}_2\overrightarrow{e_3^0} \tag{4-6}$$

求 $\overrightarrow{v_3}$：

$$\overrightarrow{oo_3}=\overrightarrow{oo_2}+(-\frac{s}{2})\overrightarrow{e_3^2}+(-\frac{s}{2})\overrightarrow{e_3^3} \tag{4-7}$$

$$\overrightarrow{v_3}=\dot{\overrightarrow{oo_3}}=\dot{\overrightarrow{oo_2}}-\frac{s}{2}\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}-\frac{s}{2}\overrightarrow{\omega_3}\times\overrightarrow{e_3^3} \tag{4-8}$$

式中

$$\overrightarrow{\omega_2}=(\dot{\psi}_2\sin\theta_2\sin\varphi_2+\dot{\theta}_2+\cos\varphi_2)\overrightarrow{e_1^2}+$$

$$(\dot{\psi}_2\sin\theta_2\cos\varphi_2-\dot{\theta}_2\sin\varphi_2)\overrightarrow{e_2^2}+(\dot{\psi}_2\cos\theta_2+\dot{\varphi}_2)\overrightarrow{e_3^2}$$

(4-9)

$$\overrightarrow{\omega_3} = \overrightarrow{\omega_2} + \dot{\beta}\overrightarrow{e_2^2} + \dot{\alpha}\overrightarrow{e_1^3} \tag{4-10}$$

$$\begin{aligned}
\overrightarrow{v_3} &= \dot{\overrightarrow{oo_3}} \\
&= \dot{x}_2\overrightarrow{e_1^0} + \dot{y}_2\overrightarrow{e_2^0} + \dot{z}_2\overrightarrow{e_3^0} + \frac{s}{2}(\dot{\psi}\sin\theta_2\sin\varphi_2 + \dot{\theta}_2\cos\varphi_2)\overrightarrow{e_2^2} - \\
&= \frac{s}{2}(\dot{\psi}\sin\theta_2\cos\varphi_2 - \dot{\theta}_2\sin\varphi_2)\overrightarrow{e_1^2} - \frac{s}{2}(\dot{\psi}_2\sin\theta_2\sin\varphi_2 + \dot{\theta}_2\cos\varphi_2)\overrightarrow{e_1^2}\times\overrightarrow{e_3^3} - \\
&\quad \frac{s}{2}(\dot{\psi}_2\sin\theta_2\cos\varphi_2 - \dot{\theta}_2\sin\varphi_2)\overrightarrow{e_3^2}\times\overrightarrow{e_3^3} - \frac{s}{2}(\dot{\psi}_2\cos\theta_2 + \dot{\varphi}_2)\overrightarrow{e_3^2} - \frac{s}{2}\dot{\beta}\overrightarrow{e_2^2}\times\overrightarrow{e_3^3} - \\
&\quad \frac{s}{2}\dot{\alpha}\overrightarrow{e_1^3}\times\overrightarrow{e_3^3}
\end{aligned} \tag{4-11}$$

式中

$$\begin{aligned}
&\overrightarrow{e_1^2}\times\overrightarrow{e_3^3} = -\sin\alpha\overrightarrow{e_3^2} - \cos\alpha\cos\beta\overrightarrow{e_2^2} \\
&\overrightarrow{e_2^2}\times\overrightarrow{e_3^3} = \sin\beta\cos\alpha\overrightarrow{e_3^2} + \cos\alpha\cos\beta\overrightarrow{e_1^2} \\
&\overrightarrow{e_3^2}\times\overrightarrow{e_3^3} = -\sin\beta\cos\alpha\overrightarrow{e_2^2} + \sin\alpha\overrightarrow{e_1^2} \\
&\overrightarrow{e_2^2}\times\overrightarrow{e_3^3} = \sin\beta\cos\alpha\overrightarrow{e_3^2} + \cos\alpha\cos\beta\overrightarrow{e_1^2} \\
&\overrightarrow{e_1^3}\times\overrightarrow{e_3^3} = -\overrightarrow{e_3^3} = \sin\alpha\sin\beta\overrightarrow{e_1^2} - \cos\alpha\overrightarrow{e_2^2} - \sin\alpha\cos\beta\overrightarrow{e_3^2}
\end{aligned} \tag{4-12}$$

$$\begin{aligned}
\therefore \overrightarrow{v_3} &= \dot{x}_2\overrightarrow{e_1^0} + \dot{y}_2\overrightarrow{e_2^0} + \dot{z}_2\overrightarrow{e_3^0} + \frac{s}{2}(\dot{\psi}_2\sin\theta_2\sin\varphi_2 + \dot{\theta}_2\cos\varphi_2)\overrightarrow{e_2^2} - \frac{s}{2}(\dot{\psi}_2\sin\theta_2\cos\varphi_2 - \\
&\quad \dot{\theta}_2\sin\varphi_2)\overrightarrow{e_1^2} - \frac{s}{2}[(\dot{\psi}_2\sin\theta_2\cos\varphi_2 - \dot{\theta}_2\sin\varphi_2)\cos\alpha\cos\beta + (\dot{\psi}_2\cos\theta_2 + \dot{\varphi})\sin\alpha + \\
&\quad \dot{\beta}\cos\alpha\cos\beta + \dot{\alpha}\sin\alpha\sin\beta]\overrightarrow{e_1^2} - \frac{s}{2}[(\dot{\psi}_2\sin\theta_2\cos\varphi_2 - \dot{\theta}_2\sin\varphi_2)\cos\alpha\cos\beta + \\
&\quad (\dot{\psi}_2\cos\theta_2 + \dot{\varphi})\sin\alpha + \dot{\beta}\cos\alpha\cos\beta + \dot{\alpha}\sin\alpha\sin\beta]\overrightarrow{e_1^2} + \frac{s}{2}[(\dot{\psi}_2\sin\theta_2\sin\varphi_2 +
\end{aligned}$$

$$\dot{\theta}_2\cos\varphi_2)\cos\alpha\cos\beta+(\dot{\psi}_2\cos\theta_2+\dot{\varphi}_2)\sin\beta\cos\alpha+\dot{\alpha}\cos\alpha]\overrightarrow{e_2^2}-$$

$$\frac{s}{2}[-\sin\alpha(\dot{\psi}_2\cos\theta_2+\dot{\varphi}_2)+\sin\beta\cos\alpha(\dot{\psi}_2\sin\theta_2\cos\varphi_2-\dot{\theta}_2\sin\varphi_2)+$$

$$\dot{\beta}\sin\beta\cos\alpha-\dot{\alpha}\sin\alpha\cos\beta]\overrightarrow{e_3^2}$$

（4-13）

求 $\overrightarrow{v_4}$：

$$\overrightarrow{oo_4}=\overrightarrow{oo_3}-\frac{s}{2}\overrightarrow{e_3^3}-\frac{s}{2}\overrightarrow{e_3^4} \quad (4\text{-}14)$$

$$\overrightarrow{v_4}=\dot{\overrightarrow{oo_3}}-\frac{s}{2}\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}-\frac{s}{2}\overrightarrow{\omega_4}\times\overrightarrow{e_3^4} \quad (4\text{-}15)$$

式（4-15）中

$$\overrightarrow{\omega_4}=\overrightarrow{\omega_3}+\dot{\psi}_4\overrightarrow{e_3^3}+\dot{\theta}_4\overrightarrow{e_1^4} \quad (4\text{-}16)$$

$$\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}=(\overrightarrow{\omega_2}+\dot{\beta}\overrightarrow{e_2^2}+\dot{\alpha}\overrightarrow{e_1^3})\times\overrightarrow{e_1^3} \quad (4\text{-}17)$$

因为

$$\overrightarrow{e_1^2}\times\overrightarrow{e_3^3}=\overrightarrow{e_1^2}\times(-\sin\beta\cos\alpha\overrightarrow{e_1^2}-\sin\alpha\overrightarrow{e_2^2}+\cos\alpha\cos\beta\overrightarrow{e_3^2})$$

$$=-\sin\alpha\overrightarrow{e_3^2}-\cos\alpha\cos\beta\overrightarrow{e_2^2}$$

$$\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}=\overrightarrow{e_2^2}\times(-\sin\beta\cos\alpha\overrightarrow{e_1^2}-\sin\alpha\overrightarrow{e_2^2}+\cos\alpha\cos\beta\overrightarrow{e_3^2})$$

$$=\sin\beta\cos\alpha\overrightarrow{e_3^2}+\cos\alpha\cos\beta\overrightarrow{e_1^2}$$

$$\overrightarrow{e_3^2}\times\overrightarrow{e_3^3}=\overrightarrow{e_3^2}\times(-\sin\beta\cos\alpha\overrightarrow{e_1^2}-\sin\alpha\overrightarrow{e_2^2}+\cos\alpha\cos\beta\overrightarrow{e_3^2})=-\sin\beta\cos\alpha\overrightarrow{e_2^2}+\sin\alpha\overrightarrow{e_1^2}$$

$$\overrightarrow{e_1^3}\times\overrightarrow{e_3^3}=-\overrightarrow{e_2^3}$$

（4-18）

式（4-15）中

$$\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}=(\dot{\psi}_2\sin\theta_2\sin\varphi_2+\dot{\theta}_2+\cos\varphi_2)(-\sin\alpha\overrightarrow{e_3^2}-\cos\beta\cos\alpha)\overrightarrow{e_2^2}+$$

$$(\dot{\psi}_2\sin\theta_2\cos\varphi_2-\dot{\theta}_2\sin\varphi_2)(\sin\beta\cos\alpha\overrightarrow{e_3^2}+\cos\alpha\cos\beta\overrightarrow{e_1^2})+$$

$$(\dot{\psi}_2\cos\theta_2+\dot{\varphi}_2)(-\sin\beta\cos\alpha\overrightarrow{e_2^2}+\sin\alpha\overrightarrow{e_1^2})+$$

$$\dot{\beta}(\sin\beta\cos\alpha\overrightarrow{e_3^2}+\cos\alpha\cos\beta)\overrightarrow{e_1^2}+\dot{\alpha}(-\overrightarrow{e_2^3})$$

（4-19）

式（4-15）中

$$\overrightarrow{\omega_4}\times\overrightarrow{e_3^4}=(\overrightarrow{\omega_3}+\dot{\psi}_4\overrightarrow{e_3^3}+\dot{\theta}_4\overrightarrow{e_1^4})\times\overrightarrow{e_3^4}$$

$$=(\dot{\psi}_2\sin\theta_2\sin\varphi_2+\dot{\theta}_2+\cos\varphi_2)\overrightarrow{e_1^2}\times\overrightarrow{e_3^4}+(\dot{\psi}_2\sin\theta_2\cos\varphi_2-\dot{\theta}_2\sin\varphi_2)\overrightarrow{e_2^2}\times\overrightarrow{e_3^4}+$$

$$(\dot{\psi}_2\cos\theta_2+\dot{\varphi}_2)\overrightarrow{e_3^2}\times\overrightarrow{e_3^4}+\dot{\beta}\overrightarrow{e_2^2}\times\overrightarrow{e_3^4}+\dot{\alpha}\overrightarrow{e_1^3}\times\overrightarrow{e_3^4}+\dot{\psi}_4\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}+\dot{\theta}_4\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}$$

（4-20）

求 $\overrightarrow{v_1}$：

$$\overrightarrow{oo_1}=\overrightarrow{oo_2}+\frac{s}{2}\overrightarrow{e_3^2}+\frac{s}{2}\overrightarrow{e_3^1}$$

$$\overrightarrow{v_1}=\dot{\overrightarrow{oo_1}}=\dot{\overrightarrow{oo_2}}+\frac{s}{2}\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\frac{s}{2}\overrightarrow{\omega_1}\times\overrightarrow{e_3^1}\quad（4-21）$$

式中

$$\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}=-(\dot{\psi}_2\sin\theta_2\sin\varphi_2+\dot{\theta}_2+\cos\varphi_2)\overrightarrow{e_2^2}+(\dot{\psi}_2\sin\theta_2\cos\varphi_2-\dot{\theta}_2\sin\varphi_2)\overrightarrow{e_1^2}$$

（4-22）

$$\overrightarrow{\omega_1}=\overrightarrow{\omega_2}+\dot{\psi}_1\overrightarrow{e_3^2}+\dot{\theta}_1\overrightarrow{e_1^1}\quad（4-23）$$

由上面的过程，我们可以选择广义速率 ·

$$u_1 = \dot{x}_2, u_2 = \dot{y}_2, u_3 = \dot{z}_2$$

$$u_4 = \dot{\psi}_2 \sin\theta_2 \sin\varphi_2 + \dot{\theta}_2 + \cos\varphi_2, u_5 = \dot{\psi}_2 \sin\theta_2 \cos\varphi_2 - \dot{\theta}_2 \sin\varphi_2$$

$$u_6 = \dot{\psi}_2 \cos\theta_2 + \dot{\varphi}_2$$

$$u_7 = \dot{\beta}, u_8 = \dot{\alpha}, u_9 = \dot{\psi}_1, u_{10} = \dot{\theta}_1, u_{11} = \dot{\psi}_4, u_{12} = \dot{\theta}_4$$

这样 $\overrightarrow{v_2}$, $\overrightarrow{\omega_2}$, $\overrightarrow{v_3}$, $\overrightarrow{\omega_3}$, $\overrightarrow{v_1}$, $\overrightarrow{\omega_1}$, $\overrightarrow{v_4}$, $\overrightarrow{\omega_4}$ 可以写成如下形式

$$\overrightarrow{v_2} = u_1 \overrightarrow{e_1^0} + u_2 \overrightarrow{e_2^0} + u_3 \overrightarrow{e_3^0} \tag{4-24}$$

$$\overrightarrow{\omega_2} = u_4 \overrightarrow{e_1^2} + u_5 \overrightarrow{e_2^2} + u_6 \overrightarrow{e_3^2} \tag{4-25}$$

$$\begin{aligned}
\overrightarrow{v_3} = {} & u_1 \overrightarrow{e_1^0} + u_2 \overrightarrow{e_2^0} + u_3 \overrightarrow{e_3^0} + \frac{s}{2}(\dot{\psi}_2 \sin\theta_2 \sin\varphi_2 + \dot{\theta}_2 \cos\varphi_2) \overrightarrow{e_2^2} - \\
& \frac{s}{2}(\dot{\psi}_2 \sin\theta_2 \cos\varphi_2 - \dot{\theta}_2 \sin\varphi_2) \overrightarrow{e_1^2} - \frac{s}{2}[(\dot{\psi}_2 \sin\theta_2 \sin\varphi_2 + \dot{\theta}_2 \cos\varphi_2) \cos\alpha \cos\beta + \\
& u_6 \sin\alpha + u_7 \cos\alpha \cos\beta + u_8 \sin\alpha \sin\beta] \overrightarrow{e_1^2} + \frac{s}{2}[(\dot{\psi}_2 \sin\theta_2 \sin\varphi_2 + \\
& \dot{\theta}_2 \cos\varphi_2) \cos\alpha \cos\beta + u_6 \sin\beta \cos\alpha + u_8 \cos\alpha] \overrightarrow{e_2^2} - \\
& \frac{s}{2}[-u_6 \sin\alpha + \sin\beta \cos\alpha (\dot{\psi}_2 \sin\theta_2 \cos\varphi_2 - \dot{\theta}_2 \sin\varphi_2) + \\
& u_7 \sin\beta \cos\alpha - u_8 \sin\alpha \cos\beta] \overrightarrow{e_3^2}
\end{aligned} \tag{4-26}$$

$$\overrightarrow{\omega_3} = u_4 \overrightarrow{e_1^2} + u_5 \overrightarrow{e_2^2} + u_6 \overrightarrow{e_3^2} + u_7 \overrightarrow{e_2^2} + u_8 \overrightarrow{e_1^3} \tag{4-27}$$

$$\begin{aligned}
\overrightarrow{v_1} = {} & u_1 \overrightarrow{e_1^2} + u_2 \overrightarrow{e_2^2} + u_3 \overrightarrow{e_3^2} + \frac{s}{2}(-u_4 \overrightarrow{e_2^2} + u_5 \overrightarrow{e_1^2}) + \frac{s}{2}(u_4 \overrightarrow{e_1^2} + u_5 \overrightarrow{e_2^2} + u_6 \overrightarrow{e_3^2} + u_9 \overrightarrow{e_3^2} + u_{10} \overrightarrow{e_1^1}) \times \overrightarrow{e_3^1} \\
= {} & u_1 \overrightarrow{e_1^2} + u_2 \overrightarrow{e_2^2} + u_3 \overrightarrow{e_3^2} + u_4 (-\frac{s}{2} \overrightarrow{e_2^2} + \frac{s}{2} \overrightarrow{e_1^2} \times \overrightarrow{e_3^1}) + u_5 (\frac{s}{2} \overrightarrow{e_1^2} + \frac{s}{2} \overrightarrow{e_2^2} \times \overrightarrow{e_3^1}) + u_6 \frac{s}{2} \overrightarrow{e_3^2} \times \overrightarrow{e_3^1} +
\end{aligned}$$

$$\frac{s}{2}u_9\overrightarrow{e_3^2}\times\overrightarrow{e_3^1}+\frac{s}{2}u_{10}\overrightarrow{e_1^1}\times\overrightarrow{e_3^1} \tag{4-28}$$

$$\overrightarrow{\omega_1}=u_4\overrightarrow{e_1^2}+u_5\overrightarrow{e_2^2}+u_6\overrightarrow{e_3^2}+u_9\overrightarrow{e_3^2}+u_{10}\overrightarrow{e_1^1} \tag{4-29}$$

$$\overrightarrow{\omega_4}=u_4\overrightarrow{e_1^2}+u_5\overrightarrow{e_2^2}+u_6\overrightarrow{e_3^2}+u_7\overrightarrow{e_2^2}+u_8\overrightarrow{e_1^3}+u_{11}\overrightarrow{e_3^3}+u_{12}\overrightarrow{e_1^4} \tag{4-30}$$

$$\overrightarrow{v_4}=\overrightarrow{v_3}-\frac{s}{2}(u_4\overrightarrow{e_1^2}\times\overrightarrow{e_3^3}+u_5\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}+u_6\overrightarrow{e_3^2}\times\overrightarrow{e_3^3}+u_7\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}-u_8\overrightarrow{e_2^3})-$$

$$\frac{s}{2}(u_4\overrightarrow{e_1^2}\times\overrightarrow{e_3^4}+u_5\overrightarrow{e_2^2}\times\overrightarrow{e_3^4}+u_6\overrightarrow{e_3^2}\times\overrightarrow{e_3^4}+u_7\overrightarrow{e_2^2}\times\overrightarrow{e_3^4}+u_8\overrightarrow{e_1^3}\times\overrightarrow{e_3^4}+u_{11}\overrightarrow{e_3^3}\times\overrightarrow{e_3^4}+u_{12}\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}) \tag{4-31}$$

此外还有

$$\overrightarrow{\Omega_{2,3}}=\dot{\alpha}\overrightarrow{e_2^2}+\dot{\beta}\overrightarrow{e_1^3}=u_7\overrightarrow{e_2^2}+u_8\overrightarrow{e_1^3} \tag{4-32}$$

$$\overrightarrow{\Omega_{3,4}}=u_{11}\overrightarrow{e_3^3}+u_{12}\overrightarrow{e_1^4} \tag{4-33}$$

$$\overrightarrow{\Omega_{1,2}}=u_9\overrightarrow{e_3^2}+u_{10}\overrightarrow{e_1^1} \tag{4-34}$$

下面列出偏速度与偏角速度：

$\overrightarrow{v_1}$ 的偏速度

$$\overrightarrow{v_1}^1=\overrightarrow{e_1^2},\overrightarrow{v_1}^2=\overrightarrow{e_2^2},\overrightarrow{v_1}^3=\overrightarrow{e_3^2}$$

$$\overrightarrow{v_1}^4=\frac{s}{2}(-\overrightarrow{e_2^2}+\overrightarrow{e_1^2}\times\overrightarrow{e_3^1}),\overrightarrow{v_1}^5=\frac{s}{2}(\overrightarrow{e_1^2}+\overrightarrow{e_2^2}\times\overrightarrow{e_3^1}),\overrightarrow{v_1}^6=\frac{s}{2}\overrightarrow{e_3^2}\times\overrightarrow{e_3^1}$$

$$\overrightarrow{v_1}^7=0=\overrightarrow{v_1}^8,\overrightarrow{v_1}^9=\frac{s}{2}\overrightarrow{e_3^2}\times\overrightarrow{e_3^1},\overrightarrow{v_1}^{10}=\frac{s}{2}\overrightarrow{e_1^1}\times\overrightarrow{e_3^1} \tag{4-35}$$

$\overrightarrow{\omega_1}$ 的偏速度

$$\overrightarrow{\omega_1}^1=\overrightarrow{\omega_1}^2=\overrightarrow{\omega_1}^3=0$$

$$\overrightarrow{\omega_1}^4=\overrightarrow{e_1^2},\overrightarrow{\omega_1}^5=\overrightarrow{e_2^2},\overrightarrow{\omega_1}^6=\overrightarrow{e_3^2}$$

$$\overrightarrow{\omega_1}^7=\overrightarrow{\omega_1}^8=0$$

$$\overrightarrow{\omega_1}^9=\overrightarrow{e_3^2},\overrightarrow{\omega_1}^{10}=\overrightarrow{e_1^1},\overrightarrow{\omega_1}^{11}=\overrightarrow{\omega_1}^{12}=0$$

(4-36)

$\overrightarrow{v_2}$ 的偏速度

$$\overrightarrow{v_2}^1=\overrightarrow{e_1^0},\overrightarrow{v_2}^2=\overrightarrow{e_2^0},\overrightarrow{v_2}^3=\overrightarrow{e_3^0}，其它\overrightarrow{v_2}^r=0 \tag{4-37}$$

$\overrightarrow{\omega_2}$ 的偏速度

$$\overrightarrow{\omega_2}^1=\overrightarrow{\omega_2}^2=\overrightarrow{\omega_2}^3=0$$

$$\overrightarrow{\omega_2}^4=\overrightarrow{e_1^2},\overrightarrow{\omega_2}^5=\overrightarrow{e_2^2},\overrightarrow{\omega_2}^6=\overrightarrow{e_3^2}$$

$$\overrightarrow{\omega_2}^7=\overrightarrow{\omega_2}^8=\overrightarrow{\omega_2}^9=\overrightarrow{\omega_2}^{10}=\overrightarrow{\omega_2}^{11}=\overrightarrow{\omega_2}^{12}=0$$

(4-38)

$\overrightarrow{v_3}$ 的偏速度

$$\overrightarrow{v_3}^1=\overrightarrow{e_1^0},\overrightarrow{v_3}^2=\overrightarrow{e_2^0},\overrightarrow{v_3}^3=\overrightarrow{e_3^0}，\overrightarrow{v_3}^4=\overrightarrow{v_3}^5=\overrightarrow{v_3}^9=\overrightarrow{v_3}^{10}=\overrightarrow{v_3}^{11}=\overrightarrow{v_3}^{12}=0$$

$$\overrightarrow{v_3}^6=-\frac{s}{2}\sin\alpha\overrightarrow{e_1^2}+\frac{s}{2}\sin\beta\cos\alpha\overrightarrow{e_2^2}+\frac{s}{2}\sin\alpha\overrightarrow{e_3^2}$$

$$\overrightarrow{v_3}^7=-\frac{s}{2}\cos\alpha\cos\beta\overrightarrow{e_1^2}-\frac{s}{2}\sin\beta\cos\alpha\overrightarrow{e_3^2}$$

$$\overrightarrow{v_3}^8=-\frac{s}{2}\sin\alpha\sin\beta\overrightarrow{e_1^2}+\frac{s}{2}\cos\alpha\overrightarrow{e_2^2}+\frac{s}{2}\sin\alpha\cos\beta\overrightarrow{e_3^2}$$

(4-39)

$\overrightarrow{\omega_3}$ 的偏速度

$$\overrightarrow{\omega_3}^4 = \overrightarrow{e_1^2}, \overrightarrow{\omega_3}^5 = \overrightarrow{e_2^2}, \overrightarrow{\omega_3}^6 = \overrightarrow{e_3^2}, \tag{4-40}$$

$$\overrightarrow{\omega_3}^7 = \overrightarrow{e_2^2}, \overrightarrow{\omega_3}^8 = \overrightarrow{e_1^3}, \text{其余} \overrightarrow{\omega_3}^\gamma \text{为零}$$

$\overrightarrow{\omega_4}$ 的偏速度

$$\overrightarrow{\omega_4}^1 = \overrightarrow{\omega_4}^2 = \overrightarrow{\omega_4}^3 = 0$$

$$\overrightarrow{\omega_4}^4 = \overrightarrow{e_1^2}, \overrightarrow{\omega_4}^5 = \overrightarrow{e_2^2}, \overrightarrow{\omega_4}^6 = \overrightarrow{e_3^2}, \overrightarrow{\omega_4}^7 = \overrightarrow{e_2^2}, \overrightarrow{\omega_4}^8 = \overrightarrow{e_1^3}$$

$$\overrightarrow{\omega_4}^9 = \overrightarrow{\omega_4}^{10} = 0, \overrightarrow{\omega_4}^{11} = \overrightarrow{e_3^3}, \overrightarrow{\omega_4}^{12} = \overrightarrow{e_1^4}$$

（4-41）

$\overrightarrow{v_4}$ 的偏速度

$$\overrightarrow{v_4}^1 = \overrightarrow{e_1^0}, \overrightarrow{v_4}^2 = \overrightarrow{e_2^0}, \overrightarrow{v_4}^3 = \overrightarrow{e_3^0}$$

$$\overrightarrow{v_4}^4 = -\frac{s}{2}(\overrightarrow{e_1^2} \times \overrightarrow{e_3^3} + \overrightarrow{e_1^2} \times \overrightarrow{e_3^4}), \overrightarrow{v_4}^5 = -\frac{s}{2}(\overrightarrow{e_2^2} \times \overrightarrow{e_3^3} + \overrightarrow{e_2^2} \times \overrightarrow{e_3^4})$$

$$\overrightarrow{v_4}^6 = -\frac{s}{2}\sin\alpha \overrightarrow{e_1^2} + \frac{s}{2}\sin\beta\cos\alpha \overrightarrow{e_2^2} + \frac{s}{2}\sin\alpha \overrightarrow{e_3^2} - \frac{s}{2}(\overrightarrow{e_3^2} \times \overrightarrow{e_3^3} + \overrightarrow{e_3^2} \times \overrightarrow{e_3^4})$$

$$\overrightarrow{v_4}^7 = -\frac{s}{2}\cos\alpha\cos\beta \overrightarrow{e_1^2} - \frac{s}{2}\sin\beta\cos\alpha \overrightarrow{e_3^2} - \frac{s}{2}(\overrightarrow{e_2^2} \times \overrightarrow{e_3^3} + \overrightarrow{e_2^2} \times \overrightarrow{e_3^4})$$

$$\overrightarrow{v_4}^8 = -\frac{s}{2}\sin\alpha\sin\beta \overrightarrow{e_1^2} + \frac{s}{2}\cos\alpha \overrightarrow{e_2^2} + \frac{s}{2}\sin\alpha\sin\beta \overrightarrow{e_3^2} - \frac{s}{2}(-\overrightarrow{e_2^3} + \overrightarrow{e_1^3} \times \overrightarrow{e_3^4})$$

$$\overrightarrow{v_4}^9 = 0, \overrightarrow{v_4}^{10} = 0, \overrightarrow{v_4}^{11} = -\frac{s}{2}\overrightarrow{e_3^3} \times \overrightarrow{e_3^4}, \overrightarrow{v_4}^{12} = -\frac{s}{2}\overrightarrow{e_1^4} \times \overrightarrow{e_3^4}$$

（4-42）

求各 $\dot{\overrightarrow{v_i}}$, $\dot{\overrightarrow{\omega_i}}, i = 1 \sim 4$

$$\dot{\vec{v_1}} = \dot{u}_1\vec{e_1^2} + u_1\vec{\omega_2}\times\vec{e_2^2} + \dot{u}_2\vec{e_2^2} + u_2\vec{\omega_2}\times\vec{e_3^2} + \dot{u}_4(-\frac{s}{2}\vec{e_2^2} + \frac{s}{2}\vec{e_1^2}\times\vec{e_3^1}) +$$
$$u_4\frac{s}{2}[-\vec{\omega_2}\times\vec{e_2^2} + \vec{\omega_2}\times\vec{e_1^2}\times\vec{e_3^1} + \vec{e_1^2}\times(\vec{\omega_1}\times\vec{e_3^1})] + \frac{s}{2}\dot{u}_5(\vec{e_1^2} + \vec{e_2^2}\times\vec{e_3^1}) +$$
$$\frac{s}{2}u_5[\vec{\omega_2}\times\vec{e_1^2} + \vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^1} + \vec{e_2^2}\times(\vec{\omega_1}\times\vec{e_3^1})] + \dot{u}_6\frac{s}{2}\vec{e_3^2}\times\vec{e_3^1} +$$
$$u_6\frac{s}{2}[\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^1} + \vec{e_3^2}\times(\vec{\omega_1}\times\vec{e_3^1})] + \frac{s}{2}\dot{u}_9\vec{e_3^2}\times\vec{e_3^1} +$$
$$\frac{s}{2}u_9[\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^1} + \vec{e_3^2}\times(\vec{\omega_1}\times\vec{e_3^1})] + \frac{s}{2}\dot{u}_{10}\vec{e_1^1}\times\vec{e_3^1} +$$
$$\frac{s}{2}u_{10}[\vec{\omega_1}\times\vec{e_1^1}\times\vec{e_3^1} + \vec{e_1^1}\times(\vec{\omega_1}\times\vec{e_3^1})]$$

（4-43）

$$\dot{\vec{\omega_1}} = \dot{u}_4\vec{e_1^2} + u_4\vec{\omega_2}\times\vec{e_1^2} + \dot{u}_5\vec{e_2^2} + u_5\vec{\omega_2}\times\vec{e_2^2} + \dot{u}_6\vec{e_3^2} + u_6\vec{\omega_2}\times\vec{e_3^2} +$$
$$\dot{u}_9\vec{e_3^2} + u_9\vec{\omega_2}\times\vec{e_3^2} + \dot{u}_{10}\vec{e_1^1} + u_{10}\vec{\omega_1}\times\vec{e_1^1}$$

（4-44）

$$\dot{\vec{v_2}} = \dot{u}_1\vec{e_1^0} + \dot{u}_2\vec{e_2^0} + \dot{u}_3\vec{e_3^0} \qquad (4\text{-}45)$$

$$\dot{\vec{\omega_2}} = \dot{u}_4\vec{e_1^2} + u_4\vec{\omega_2}\times\vec{e_1^2} + \dot{u}_5\vec{e_2^2} + u_5\vec{\omega_2}\times\vec{e_2^2} + \dot{u}_6\vec{e_3^2} + u_6\vec{\omega_2}\times\vec{e_3^2} \qquad (4\text{-}46)$$

$$\dot{\vec{v_3}} = \dot{u}_1\vec{e_1^0} + \dot{u}_2\vec{e_2^0} + \dot{u}_3\vec{e_3^0} + \frac{s}{2}(\ddot{\psi}\sin\theta_2\sin\varphi_2 + \dot{\theta}_2\dot{\psi}_2\cos\theta_2\sin\varphi_2 + \dot{\varphi}_2\dot{\psi}_2\sin\theta_2\cos\varphi_2 +$$
$$\ddot{\theta}_2\cos\varphi_2 - \dot{\theta}_2\dot{\psi}_2\sin\varphi_2)\vec{e_2^2} + \frac{s}{2}(\dot{\psi}_2\sin\theta_2\sin\varphi_2 + \dot{\theta}_2\cos\varphi_2)\vec{\omega_2}\times\vec{e_2^2} -$$
$$\frac{s}{2}(\ddot{\psi}_2\sin\theta_2\cos\varphi_2 + \dot{\theta}_2\dot{\psi}_2\cos\theta_2\cos\varphi_2 - \dot{\psi}_2\dot{\varphi}_2\sin\theta_2\sin\varphi_2 - \ddot{\theta}_2\sin\varphi_2 -$$
$$\dot{\theta}_2\dot{\varphi}_2\cos\varphi_2)\vec{e_1^2} - \frac{s}{2}(\dot{\psi}_2\sin\theta_2\cos\varphi_2 - \dot{\theta}_2\sin\varphi_2)\vec{\omega_2}\times\vec{e_1^2} - \frac{s}{2}[(\ddot{\psi}_2\sin\theta_2\sin\varphi_2 +$$
$$\dot{\theta}_2\dot{\varphi}_2\cos\theta_2\sin\varphi_2 + \dot{\varphi}_2\dot{\psi}_2 + \dot{\theta}_2\cos\varphi_2 - \dot{\theta}_2\dot{\varphi}_2\sin\varphi_2)\cos\alpha\cos\beta +$$

$$(\dot{\psi}_2 \sin\theta_2 \sin\varphi_2 + \dot{\theta}_2 \cos\varphi_2)(-\dot{\alpha}\sin\alpha\cos\beta - \dot{\beta}\cos\alpha\sin\beta) +$$

$$\dot{u}_6 \sin\alpha + u_6 u_8 \cos\alpha - u_7 u_8 \sin\alpha\cos\beta - u_7^2 \cos\alpha\sin\beta + u_8{}^2 \cos\alpha\sin\beta +$$

$$u_8^2 \sin\alpha\cos\beta]\overrightarrow{e_1^2} - \frac{s}{2}[(\dot{\psi}_2 \sin\theta_2 \sin\varphi_2 + \dot{\theta}_2 \cos\varphi_2)\cos\alpha\cos\beta + u_6 \sin\alpha +$$

$$u_7 \cos\alpha\cos\beta + u_8 \sin\alpha\sin\beta]\overrightarrow{\omega_2} \times \overrightarrow{e_1^2} +$$

$$\frac{s}{2}[(\ddot{\psi}_2 \sin\theta_2 \sin\varphi_2 + \dot{\theta}_2\dot{\psi}_2 \cos\theta_2 \sin\varphi_2 + \dot{\varphi}_2\dot{\psi}_2 \sin\theta_2 \cos\varphi_2 + \ddot{\theta}_2 \cos\varphi_2 -$$

$$\dot{\varphi}_2\dot{\theta}_2 \sin\varphi_2)\cos\alpha\cos\beta + (\dot{\psi}_2 \sin\theta_2 \sin\varphi_2 + \dot{\theta}_2 \cos\varphi_2)(-u_8 \sin\alpha\cos\beta -$$

$$u_7 \cos\alpha\sin\beta) + \dot{u}_6 \cos\alpha\sin\beta + u_6 u_7 \cos\alpha\cos\beta - u_6 u_8 \sin\alpha\sin\beta +$$

$$\dot{u}_8 \cos\alpha - u_8{}^2 \sin\alpha]\overrightarrow{e_2^2} + \frac{s}{2}[(\dot{\psi}_2 \sin\theta_2 \sin\varphi_2 + \dot{\theta}_2 \cos\varphi_2)\cos\alpha\cos\beta + u_6 \cos\alpha\sin\beta +$$

$$u_8 \cos\alpha]\overrightarrow{\omega_2} \times \overrightarrow{e_2^2} - \frac{s}{2}[-u_6 u_8 \cos\alpha - \dot{u}_6 \sin\alpha + (u_7 \cos\alpha\cos\beta - u_8 \sin\alpha\sin\beta)$$

$$(\dot{\psi}_2 \sin\theta_2 \sin\varphi_2 - \dot{\theta}_2 \sin\varphi_2) + \cos\alpha\sin\beta(\ddot{\psi}_2 \sin\theta_2 \cos\varphi_2 + \dot{\psi}_2\dot{\theta}_2 \cos\theta_2 \cos\varphi_2 -$$

$$\dot{\psi}_2\dot{\varphi}_2 \sin\theta_2 \sin\varphi_2 - \ddot{\theta}_2 \sin\varphi_2 - \dot{\theta}_2\dot{\varphi}_2 \cos\varphi_2) + \dot{u}_7 \cos\alpha\sin\beta - u_7 u_8 \sin\alpha\sin\beta +$$

$$u_7^2 \cos\alpha\cos\beta - \dot{u}_8 \sin\alpha\cos\beta - u_7 u_8 \cos\alpha\cos\beta + u_8^2 \sin\alpha\sin\beta]\overrightarrow{e_3^2} - \frac{s}{2}[-\sin\alpha u_6 +$$

$$\cos\alpha\sin\beta(\dot{\psi}_2 \sin\theta_2 \sin\varphi_2 - \dot{\theta}_2 \sin\varphi_2) + u_7 \cos\alpha\sin\beta - u_8 \sin\alpha\cos\beta]\overrightarrow{\omega_2} \times \overrightarrow{e_3^2}$$

（4-47）

$$\dot{\overrightarrow{\omega_3}} = \dot{u}_4 \overrightarrow{e_1^2} + u_4 \overrightarrow{\omega_2} \times \overrightarrow{e_1^2} + \dot{u}_5 \overrightarrow{e_2^2} + u_5 \overrightarrow{\omega_2} \times \overrightarrow{e_2^2} + \dot{u}_6 \overrightarrow{e_3^2} + u_6 \overrightarrow{\omega_2} \times \overrightarrow{e_3^2}$$

$$+\dot{u}_7 \overrightarrow{e_2^2} + u_7 \overrightarrow{\omega_2} \times \overrightarrow{e_2^2} + \dot{u}_8 \overrightarrow{e_1^3} + u_8 \overrightarrow{\omega_3} \times \overrightarrow{e_1^3}$$

（4-48）

$$\dot{\vec{v}}_4=\dot{\vec{v}}_3-\frac{s}{2}[\dot{u}_4\vec{e}_1^2\times\vec{e}_3^3+u_4\vec{\omega}_2\times\vec{e}_1^2\times\vec{e}_3^3+u_4\vec{e}_1^2\times(\vec{\omega}_3\times\vec{e}_3^3)+$$

$$\dot{u}_5\vec{e}_2^2\times\vec{e}_3^3+u_5\vec{\omega}_2\times\vec{e}_2^2\times\vec{e}_3^3+u_5\vec{e}_2^2\times(\vec{\omega}_3\times\vec{e}_3^3)+\dot{u}_6\vec{e}_3^2\times\vec{e}_3^3+u_6\vec{\omega}_2\times\vec{e}_3^2\times\vec{e}_3^3+$$

$$u_6\vec{e}_3^2\times(\vec{\omega}_3\times\vec{e}_3^3)+\dot{u}_7\vec{e}_2^2\times\vec{e}_3^3+u_7\vec{\omega}_2\times\vec{e}_2^2\times\vec{e}_3^3+u_7\vec{e}_2^2\times(\vec{\omega}_3\times\vec{e}_3^3)-\dot{u}_8\vec{e}_2^3-u_8\vec{\omega}_3\times\vec{e}_2^3]-$$

$$\frac{s}{2}(\dot{u}_4\vec{e}_1^2\times\vec{e}_3^4+u_4\vec{\omega}_2\times\vec{e}_1^2\times\vec{e}_3^4+u_4\vec{e}_1^2\times(\vec{\omega}_4\times\vec{e}_3^4)+$$

$$\dot{u}_5\vec{e}_2^2\times\vec{e}_3^4+u_5\vec{\omega}_2\times\vec{e}_2^2\times\vec{e}_3^4+u_5\vec{e}_2^2\times(\vec{\omega}_4\times\vec{e}_3^4)+\dot{u}_6\vec{e}_3^2\times\vec{e}_3^4+u_6\vec{\omega}_2\times\vec{e}_3^2\times\vec{e}_3^4+$$

$$u_6\vec{e}_3^2\times(\vec{\omega}_4\times\vec{e}_3^4)+\dot{u}_8\vec{e}_1^3\times\vec{e}_3^4+u_8\vec{\omega}_3\times\vec{e}_1^3\times\vec{e}_3^4+u_8\vec{e}_1^3\times(\vec{\omega}_4\times\vec{e}_3^4)+\dot{u}_{11}\vec{e}_3^3\times\vec{e}_3^4+$$

$$u_{11}\vec{\omega}_3\times\vec{e}_3^3\times\vec{e}_3^4+u_{11}\vec{e}_3^3\times(\vec{\omega}_4\times\vec{e}_3^4)+\dot{u}_{12}\vec{e}_1^4\times\vec{e}_3^4+u_{12}\vec{\omega}_4\times\vec{e}_1^4\times\vec{e}_3^4+u_{12}\vec{e}_1^4\times(\vec{\omega}_4\times\vec{e}_3^4)$$

(4-49)

$$\begin{aligned}\dot{\vec{\omega}}_4=&\dot{u}_4\vec{e}_1^2+\dot{u}_5\vec{e}_2^2+\dot{u}_6\vec{e}_3^2+\dot{u}_7\vec{e}_2^2+\dot{u}_8\vec{e}_1^3+\dot{u}_{11}\vec{e}_3^3+\dot{u}_{12}\vec{e}_1^4\\&u_4\vec{\omega}_2\times\vec{e}_1^2+u_5\vec{\omega}_2\times\vec{e}_2^2+u_6\vec{\omega}_2\times\vec{e}_3^2+u_7\vec{\omega}_2\times\vec{e}_2^2+u_8\vec{\omega}_3\times\vec{e}_1^3+\\&u_{11}\vec{\omega}_3\times\vec{e}_3^3+u_{12}\vec{\omega}_4\times\vec{e}_1^4\end{aligned}$$

(4-50)

主动力有，各刚体的重力

$$\vec{G}_i=-m_ig\vec{e}_3^0 \tag{4-51}$$

重力主矢对质心（i）的主矩为

$$\vec{M}_i=0 \tag{4-52}$$

铰的控制力及相对铰的主矩为

$$\vec{F^a}=0,\vec{M^a}=M_\alpha\vec{e}_2^2+M_\beta\vec{e}_1^3 \tag{4-53}$$

对于反映柔性作用的弯曲弹其作用力矩相当于相对铰的主矩，但要注意力矩的方向。

$$\overrightarrow{M_{1,2}}=-k\theta_1\overrightarrow{e_1^1},\overrightarrow{M_{3,4}}=-k\theta_4\overrightarrow{e_1^4} \qquad (4-54)$$

对于广义速率u_1，系统的广义主动力为

$$F_1'=\sum_{i=1}^{4}[\overrightarrow{G_i}\bullet\overrightarrow{v_i}^1+\overrightarrow{M_i}\bullet\overrightarrow{\omega_i}^1]+\overrightarrow{M^a}\bullet\overrightarrow{\Omega_{2,3}}^1+\overrightarrow{M_{1,2}}\bullet\overrightarrow{\Omega_{1,2}}^1+\overrightarrow{M_{3,4}}\bullet\overrightarrow{\Omega_{3,4}}^1$$

$$=-mg\overrightarrow{e_3^0}\bullet\overrightarrow{e_1^2}-mg\overrightarrow{e_3^0}\bullet\overrightarrow{e_1^0}-mg\overrightarrow{e_3^0}\bullet\overrightarrow{e_1^0}-mg\overrightarrow{e_3^0}\bullet\overrightarrow{e_1^0}+(M_\beta\overrightarrow{e_2^2}+M_\alpha\overrightarrow{e_1^3})\bullet\vec{0}-$$

$$k\theta_1\overrightarrow{e_1^1}\bullet\vec{0}-k\theta_4\overrightarrow{e_1^4}\bullet\vec{0}$$

$$=-mg\overrightarrow{e_3^0}\bullet\overrightarrow{e_1^2}$$

$$=-mg\overrightarrow{e_3^0}\bullet[(\cos\psi_2\cos\varphi_2-\sin\psi_2\cos\theta_2\sin\varphi_2)\overrightarrow{e_1^0}+(\sin\psi_2\cos\varphi_2+$$

$$\cos\psi_2\cos\theta_2\sin\varphi_2)\overrightarrow{e_2^0}+\sin\theta_2\sin\varphi_2\overrightarrow{e_3^0}]$$

$$=-mg\sin\theta_2\sin\varphi_2 \qquad (4-55a)$$

对于u_1的广义惯性力为

$$F_1^*=-\sum_{i=1}^{4}m_i\dot{\overrightarrow{v_i}}\bullet\overrightarrow{v_i}^1-\sum_{i=1}^{4}\overline{J_i}\bullet\dot{\overrightarrow{\omega_i}}\bullet\overrightarrow{\omega_i}^1$$

$$=-\sum_{i=1}^{4}m_i\dot{\overrightarrow{v_i}}\bullet\overrightarrow{v_i}^1-0$$

(4-56a)

式中：$\overline{J_i}$——为物体i的惯性张量

求出式（4-55）中的有关分量

$$\dot{\overrightarrow{v_1}}\bullet\overrightarrow{v_1}^1=\dot{u}_1\overrightarrow{e_1^2}\bullet\overrightarrow{e_1^2}+u_1\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\bullet\overrightarrow{e_1^2}+\dot{u}_2\overrightarrow{e_2^2}\bullet\overrightarrow{e_1^2}+u_2\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\bullet\overrightarrow{e_1^2}+\frac{s}{2}\dot{u}_4(-\overrightarrow{e_2^2}\bullet\overrightarrow{e_1^2}+\overrightarrow{e_1^2}\times\overrightarrow{e_3^1}\bullet\overrightarrow{e_1^2})+$$

$$\overrightarrow{e_3^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})\bullet\overrightarrow{e_1^2}]+\frac{s}{2}\dot{u}_{10}\overrightarrow{e_1^1}\times\overrightarrow{e_3^1}\bullet\overrightarrow{e_1^2}+\frac{s}{2}u_{10}[\overrightarrow{\omega_1}\times\overrightarrow{e_1^1}\times\overrightarrow{e_3^1}\bullet\overrightarrow{e_1^2}+\overrightarrow{e_1^1}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})\bullet\overrightarrow{e_1^2}]+$$

$$\frac{s}{2}\dot{u}_{10}\overrightarrow{e_1^1}\times\overrightarrow{e_3^1}\bullet\overrightarrow{e_1^2}+\frac{s}{2}u_{10}[\overrightarrow{\omega_1}\times\overrightarrow{e_1^1}\times\overrightarrow{e_3^1}\bullet\overrightarrow{e_1^2}+\overrightarrow{e_1^1}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})\bullet\overrightarrow{e_1^2}]$$

(4-57)

$$\dot{\vec{v}}_2 \bullet \vec{v}_2^{\,1} = \dot{u}_1 \qquad (4\text{-}58)$$

为书写方便把$\dot{\vec{v}}_3$简记为

$$\dot{\vec{v}}_3 = \dot{u}_1\vec{e}_1^{\,0} + \dot{u}_2\vec{e}_2^{\,0} + \dot{u}_3\vec{e}_3^{\,0} + E_1\vec{e}_2^{\,2} + E_2\vec{\omega}_2\times\vec{e}_2^{\,2} + E_3\vec{e}_1^{\,2} + E_4\vec{\omega}_2\times\vec{e}_1^{\,2} + E_5\vec{e}_1^{\,2} + E_6\vec{\omega}_2\times\vec{e}_1^{\,2} \qquad (4\text{-}59)$$

$$\begin{aligned}\dot{\vec{v}}_3\bullet\vec{v}_3^{\,1} &= \dot{u}_1 + E_1\vec{e}_2^{\,2}\bullet\vec{e}_1^{\,0} + E_2\vec{\omega}_2\times\vec{e}_2^{\,2}\bullet\vec{e}_1^{\,0} + E_3\vec{e}_1^{\,2}\bullet\vec{e}_1^{\,0} + E_4\vec{\omega}_2\times\vec{e}_1^{\,2}\bullet\vec{e}_1^{\,0} + E_5\vec{e}_1^{\,2}\bullet\vec{e}_1^{\,0} + E_6\vec{\omega}_2\times\vec{e}_1^{\,2}\bullet\vec{e}_1^{\,0}\\
&= \dot{u}_1 + E_1(-\cos\psi_2\sin\varphi_2 - \sin\psi_2\cos\theta_2\varphi_2) +\\
&\quad (E_4+E_6)[-\sin\psi_2\sin\theta_2(\dot{\psi}_2\sin\theta_2\cos\varphi_2 - \dot{\theta}_2\sin\varphi_2) +\\
&\quad (\dot{\psi}_2\cos\theta_2 + \dot{\varphi}_2)(-\cos\psi_2\sin\varphi_2 - \sin\psi_2\cos\theta_2\cos\varphi_2)] +\\
&\quad (E_3+E_5)(\cos\psi_2\cos\varphi_2 - \sin\psi_2\cos\theta_2\sin\varphi_2) +\\
&\quad E_2[(\dot{\psi}_2\sin\theta_2\sin\varphi_2 + \dot{\theta}_2 + \cos\varphi_2)\sin\psi_2\sin\theta_2 -\\
&\quad (\dot{\psi}_2\cos\theta_2 + \dot{\varphi}_2)\bullet(\cos\psi_2\cos\varphi_2 - \sin\psi_2\cos\theta_2\sin\varphi_2)]\end{aligned} \qquad (4\text{-}60)$$

$$\begin{aligned}\dot{\vec{v}}_4\bullet\vec{v}_4^{\,1} &= \dot{\vec{v}}_3\bullet\vec{e}_1^{\,0} - \frac{s}{2}[\dot{u}_4\vec{e}_1^{\,2}\times\vec{e}_3^{\,3}\bullet\vec{e}_1^{\,0} + u_4\vec{\omega}_2\times\vec{e}_1^{\,2}\times\vec{e}_3^{\,3} + u_4\vec{e}_1^{\,2}\times(\vec{\omega}_3\times\vec{e}_3^{\,3})\bullet\vec{e}_1^{\,0} +\\
&\quad \dot{u}_5\vec{e}_2^{\,2}\times\vec{e}_3^{\,3}\bullet\vec{e}_1^{\,0} + u_5\vec{\omega}_2\times\vec{e}_2^{\,2}\times\vec{e}_3^{\,3}\bullet\vec{e}_1^{\,0} + u_5\vec{e}_2^{\,2}\times(\vec{\omega}_3\times\vec{e}_3^{\,3})\bullet\vec{e}_1^{\,0} + \dot{u}_6\vec{e}_3^{\,2}\times\vec{e}_3^{\,3}\bullet\vec{e}_1^{\,0} +\\
&\quad u_6\vec{\omega}_2\times\vec{e}_3^{\,2}\times\vec{e}_3^{\,3} + u_6\vec{e}_3^{\,2}\times(\vec{\omega}_3\times\vec{e}_3^{\,3})\bullet\vec{e}_1^{\,0} + \dot{u}_7\vec{e}_2^{\,2}\times\vec{e}_3^{\,3}\bullet\vec{e}_1^{\,0} + u_7\vec{\omega}_2\times\vec{e}_2^{\,2}\times\vec{e}_3^{\,3}\bullet\vec{e}_1^{\,0} +\\
&\quad u_7\vec{e}_2^{\,2}\times(\vec{\omega}_3\times\vec{e}_3^{\,3})\bullet\vec{e}_1^{\,0} - \dot{u}_8\vec{e}_2^{\,3}\bullet\vec{e}_1^{\,0} - u_8\vec{\omega}_3\times\vec{e}_2^{\,3}\bullet\vec{e}_1^{\,0}] - \frac{s}{2}[\dot{u}_4\vec{e}_1^{\,2}\times\vec{e}_3^{\,4} + u_4\vec{\omega}_2\times\vec{e}_1^{\,2}\times\vec{e}_3^{\,4}\bullet\vec{e}_1^{\,0} +\\
&\quad u_4\vec{e}_1^{\,2}\times(\vec{\omega}_4\times\vec{e}_3^{\,4})\bullet\vec{e}_1^{\,0} + \dot{u}_5\vec{e}_2^{\,2}\times\vec{e}_3^{\,4}\bullet\vec{e}_1^{\,0} + u_5\vec{\omega}_2\times\vec{e}_2^{\,2}\times\vec{e}_3^{\,4}\bullet\vec{e}_1^{\,0} + u_5\vec{e}_2^{\,2}\times(\vec{\omega}_4\times\vec{e}_3^{\,4})\bullet\vec{e}_1^{\,0} +\\
&\quad \dot{u}_6\vec{e}_3^{\,2}\times\vec{e}_3^{\,4}\bullet\vec{e}_1^{\,0} + u_6\vec{\omega}_2\times\vec{e}_3^{\,2}\times\vec{e}_3^{\,4}\bullet\vec{e}_1^{\,0} + u_6\vec{e}_3^{\,2}\times(\vec{\omega}_4\times\vec{e}_3^{\,4})\bullet\vec{e}_1^{\,0} + \dot{u}_8\vec{e}_1^{\,3}\times\vec{e}_3^{\,4}\bullet\vec{e}_1^{\,0} +\\
&\quad u_8\vec{\omega}_3\times\vec{e}_1^{\,3}\times\vec{e}_3^{\,4}\bullet\vec{e}_1^{\,0} + u_8\vec{e}_1^{\,3}\times(\vec{\omega}_4\times\vec{e}_3^{\,4})\bullet\vec{e}_1^{\,0} + \dot{u}_{11}\vec{e}_3^{\,3}\times\vec{e}_3^{\,4}\bullet\vec{e}_1^{\,0} + u_{11}\vec{\omega}_3\times\vec{e}_3^{\,3}\times\vec{e}_3^{\,4}\bullet\vec{e}_1^{\,0} +\end{aligned}$$

$$u_{11}\overrightarrow{e_3^3}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})\bullet\overrightarrow{e_1^0}+\dot{u}_{12}\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}\bullet\overrightarrow{e_1^0}+u_{12}\overrightarrow{\omega_4}\times\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}\bullet\overrightarrow{e_1^0}+$$

$$u_{12}\overrightarrow{e_1^4}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})\bullet\overrightarrow{e_1^0}]$$

(4-61)

把(4-57),(4-58),(4-59),(4-60),(4-61)代入(4-56a),最后把(4-55a),(4-56a)代入凯恩方程

$$F_1'+F_1^*=0$$

对于广义速率u_2,系统的广义主动力为

$$F_2'=\sum_{i=1}^{4}[\overrightarrow{G_i}\bullet\overrightarrow{v_i}^2+\overrightarrow{M_i}\bullet\overrightarrow{\omega_i}^2]+\overrightarrow{M^a}\bullet\overrightarrow{\Omega_{2,3}}^2+\overrightarrow{M_{1,2}}\bullet\overrightarrow{\Omega_{1,2}}^2+\overrightarrow{M_{3,4}}\bullet\overrightarrow{\Omega_{3,4}}^2$$

$$=-mg\overrightarrow{e_3^0}\bullet\overrightarrow{e_2^2}-3mg\overrightarrow{e_3^0}\bullet\overrightarrow{e_2^0} \qquad (4\text{-}55b)$$

对于u_2的广义惯性力为

$$F_2^*=-\sum_{i=1}^{4}m_i\dot{\overrightarrow{v_i}}\bullet\overrightarrow{v_i}^2-\sum_{i=1}^{4}\overline{J_i}\bullet\dot{\overrightarrow{\omega_i}}\bullet\overrightarrow{\omega_i}^2=-\sum_{i=1}^{4}m_i\dot{\overrightarrow{v_i}}\bullet\overrightarrow{v_i}^2 \qquad (4\text{-}56b)$$

式中

$$\dot{\overrightarrow{v_1}}\bullet\overrightarrow{v_1}^2=\{\dot{u}_1\overrightarrow{e_1^2}+u_1\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+\dot{u}_2\overrightarrow{e_2^2}+u_2\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\dot{u}_4(-\frac{s}{2}\overrightarrow{e_2^2}+\frac{s}{2}\overrightarrow{e_1^2}\times\overrightarrow{e_3^1})+$$

$$u_4\frac{s}{2}[-\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}\times\overrightarrow{e_3^1}+\overrightarrow{e_1^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})]+\frac{s}{2}\dot{u}_5(\overrightarrow{e_1^2}+\overrightarrow{e_2^2}\times\overrightarrow{e_3^1})+$$

$$\frac{s}{2}u_5[\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\times\overrightarrow{e_3^1}+\overrightarrow{e_2^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})]+\dot{u}_6\frac{s}{2}\overrightarrow{e_3^2}\times\overrightarrow{e_3^1}+$$

$$u_6\frac{s}{2}[\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\times\overrightarrow{e_3^1}+\overrightarrow{e_3^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})]+\frac{s}{2}\dot{u}_9\overrightarrow{e_3^2}\times\overrightarrow{e_3^1}+$$

$$\frac{s}{2}u_9[\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\times\overrightarrow{e_3^1}+\overrightarrow{e_3^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})]+\frac{s}{2}\dot{u}_{10}\overrightarrow{e_1^1}\times\overrightarrow{e_3^1}+$$

$$\frac{s}{2}u_{10}[\overrightarrow{\omega_1}\times\overrightarrow{e_1^1}\times\overrightarrow{e_3^1}+\overrightarrow{e_1^1}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})]\}\bullet\overrightarrow{e_2^2}$$

$$\dot{\overrightarrow{v_2}}\bullet\overrightarrow{v_2}^2=(\dot{u}_1\overrightarrow{e_1^0}+\dot{u}_2\overrightarrow{e_2^0}+\dot{u}_3\overrightarrow{e_3^0})\bullet\overrightarrow{e_2^0}$$

$$\dot{\vec{v}}_3 \bullet \vec{v}_3^{\,2} = (\dot{u}_1\vec{e_1^0} + \dot{u}_2\vec{e_2^0} + \dot{u}_3\vec{e_3^0} + E_1\vec{e_2^2} + E_2\vec{\omega_2}\times\vec{e_2^2} + E_3\vec{e_1^2} + E_4\vec{\omega_2}\times\vec{e_1^2} + E_5\vec{e_1^2} + E_6\vec{\omega_2}\times\vec{e_1^2})\bullet\vec{e_2^0}$$

$$\dot{\vec{v}}_4 \bullet \vec{v}_4^{\,2} = \{\dot{u}_4\vec{e_1^2} + u_4\vec{\omega_2}\times\vec{e_1^2} + \dot{u}_5\vec{e_2^2} + u_5\vec{\omega_2}\times\vec{e_2^2} + \dot{u}_6\vec{e_3^2} + u_6\vec{\omega_2}\times\vec{e_3^2} + \dot{u}_7\vec{e_2^2} + u_7\vec{\omega_2}\times\vec{e_2^2} + \dot{u}_8\vec{e_1^3} + u_8\vec{\omega_3}\times\vec{e_1^3} - \frac{s}{2}[\dot{u}_4\vec{e_1^2}\times\vec{e_3^3} + u_4\vec{\omega_2}\times\vec{e_1^2}\times\vec{e_3^3} + u_4\vec{e_1^2}\times(\vec{\omega_3}\times\vec{e_3^3}) + \dot{u}_5\vec{e_2^2}\times\vec{e_3^3} + u_5\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^3} + u_5\vec{e_2^2}\times(\vec{\omega_3}\times\vec{e_3^3}) + \dot{u}_6\vec{e_3^2}\times\vec{e_3^3} + u_6\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^3} + u_6\vec{e_3^2}\times(\vec{\omega_3}\times\vec{e_3^3}) + \dot{u}_7\vec{e_2^2}\times\vec{e_3^3} + u_7\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^3} + u_7\vec{e_2^2}\times(\vec{\omega_3}\times\vec{e_3^3}) - \dot{u}_8\vec{e_2^3} - u_8\vec{\omega_3}\times\vec{e_2^3}] - \frac{s}{2}(\dot{u}_4\vec{e_1^2}\times\vec{e_3^4} + u_4\vec{\omega_2}\times\vec{e_1^2}\times\vec{e_3^4} + u_4\vec{e_1^2}\times(\vec{\omega_4}\times\vec{e_3^4}) + \dot{u}_5\vec{e_2^2}\times\vec{e_3^4} + u_5\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^4} + u_5\vec{e_2^2}\times(\vec{\omega_4}\times\vec{e_3^4}) + \dot{u}_6\vec{e_3^2}\times\vec{e_3^4} + u_6\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^4} + u_6\vec{e_3^2}\times(\vec{\omega_4}\times\vec{e_3^4}) + \dot{u}_8\vec{e_1^3}\times\vec{e_3^4} + u_8\vec{\omega_3}\times\vec{e_1^3}\times\vec{e_3^4} + u_8\vec{e_1^3}\times(\vec{\omega_4}\times\vec{e_3^4}) + \dot{u}_{11}\vec{e_3^3}\times\vec{e_3^4} + u_{11}\vec{\omega_3}\times\vec{e_3^3}\times\vec{e_3^4} + u_{11}\vec{e_3^3}\times(\vec{\omega_4}\times\vec{e_3^4}) + \dot{u}_{12}\vec{e_1^4}\times\vec{e_3^4} + u_{12}\vec{\omega_4}\times\vec{e_1^4}\times\vec{e_3^4} + u_{12}\vec{e_1^4}\times(\vec{\omega_4}\times\vec{e_3^4})\}\bullet\vec{e_2^0}$$

代入凯恩方程

$$F_2' + F_2^* = 0$$

对于广义速率u_3，系统的广义主动力为

$$F_3' = \sum_{i=1}^{4}[\vec{G_i}\bullet\vec{v}_i^{\,3} + \vec{M_i}\bullet\vec{\omega}_i^{\,3}] + \vec{M^a}\bullet\vec{\Omega}_{2,3}^{\,3} + \vec{M_{1,2}}\bullet\vec{\Omega}_{1,2}^{\,3} + \vec{M_{3,4}}\bullet\vec{\Omega}_{3,4}^{\,3}$$

$$= -mg\vec{e_3^0}\bullet\vec{e_3^2} - 3mg$$

（4-55c）

对于u_3的广义惯性力为

$$F_3^* = -\sum_{i=1}^{4} m_i\dot{\vec{v}}_i\bullet\vec{v}_i^{\,3} - \sum_{i=1}^{4}\overline{J_i}\bullet\dot{\vec{\omega}}_i\bullet\vec{\omega}_i^{\,3}$$

$$= -\sum_{i=1}^{4} m_i\dot{\vec{v}}_i\bullet\vec{v}_i^{\,3}$$

（4-56c）

式中

$$\dot{\vec{v}_1}\cdot\vec{v}_1^{\,3}=\{\dot{u}_1\vec{e_1^2}+u_1\vec{\omega_2}\times\vec{e_2^2}+\dot{u}_2\vec{e_2^2}+u_2\vec{\omega_2}\times\vec{e_3^2}+\dot{u}_4(-\frac{s}{2}\vec{e_2^2}+\frac{s}{2}\vec{e_1^2}\times\vec{e_3^1})+$$
$$u_4\frac{s}{2}[-\vec{\omega_2}\times\vec{e_2^2}+\vec{\omega_2}\times\vec{e_1^2}\times\vec{e_3^1}+\vec{e_1^2}\times(\vec{\omega_1}\times\vec{e_3^1})]+\frac{s}{2}\dot{u}_5(\vec{e_1^2}+\vec{e_2^2}\times\vec{e_3^1})+$$
$$\frac{s}{2}u_5[\vec{\omega_2}\times\vec{e_1^2}+\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^1}+\vec{e_2^2}\times(\vec{\omega_1}\times\vec{e_3^1})]+\dot{u}_6\frac{s}{2}\vec{e_3^2}\times\vec{e_3^1}+$$
$$u_6\frac{s}{2}[\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^1}+\vec{e_3^2}\times(\vec{\omega_1}\times\vec{e_3^1})]+\frac{s}{2}\dot{u}_9\vec{e_3^2}\times\vec{e_3^1}+$$
$$\frac{s}{2}u_9[\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^1}+\vec{e_3^2}\times(\vec{\omega_1}\times\vec{e_3^1})]+\frac{s}{2}\dot{u}_{10}\vec{e_1^1}\times\vec{e_3^1}+$$
$$\frac{s}{2}u_{10}[\vec{\omega_1}\times\vec{e_1^1}\times\vec{e_3^1}+\vec{e_1^1}\times(\vec{\omega_1}\times\vec{e_3^1})]\}\cdot\vec{e_3^2}$$

$$\dot{\vec{v}_2}\cdot\vec{v}_2^{\,3}=(\dot{u}_1\vec{e_1^0}+\dot{u}_2\vec{e_2^0}+\dot{u}_3\vec{e_3^0})\cdot\vec{e_3^0}$$

$$\dot{\vec{v}_3}\cdot\vec{v}_3^{\,3}=(\dot{u}_1\vec{e_1^0}+\dot{u}_2\vec{e_2^0}+\dot{u}_3\vec{e_3^0}+E_1\vec{e_2^2}+E_2\vec{\omega_2}\times\vec{e_2^2}+E_3\vec{e_1^2}+E_4\vec{\omega_2}\times\vec{e_1^2}+E_5\vec{e_1^2}+$$
$$E_6\vec{\omega_2}\times\vec{e_1^2})\cdot\vec{e_3^0}$$

$$\dot{\vec{v}_4}\cdot\vec{v}_4^{\,3}=\{\dot{u}_1\vec{e_1^0}+\dot{u}_2\vec{e_2^0}+\dot{u}_3\vec{e_3^0}+E_1\vec{e_2^2}+E_2\vec{\omega_2}\times\vec{e_2^2}+E_3\vec{e_1^2}+E_4\vec{\omega_2}\times\vec{e_1^2}+E_5\vec{e_1^2}+$$
$$E_6\vec{\omega_2}\times\vec{e_1^2}-\frac{s}{2}[\dot{u}_4\vec{e_1^2}\times\vec{e_3^3}+u_4\vec{\omega_2}\times\vec{e_1^2}\times\vec{e_3^3}+u_4\vec{e_1^2}\times(\vec{\omega_3}\times\vec{e_3^3})+\dot{u}_5\vec{e_2^2}\times\vec{e_3^3}+$$
$$u_5\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^3}+u_5\vec{e_2^2}\times(\vec{\omega_3}\times\vec{e_3^3})+\dot{u}_6\vec{e_3^2}\times\vec{e_3^3}+u_6\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^3}+u_6\vec{e_3^2}\times$$
$$(\vec{\omega_3}\times\vec{e_3^3})+\dot{u}_7\vec{e_2^2}\times\vec{e_3^3}+u_7\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^3}+u_7\vec{e_2^2}\times(\vec{\omega_3}\times\vec{e_3^3})-\dot{u}_8\vec{e_2^3}-u_8\vec{\omega_3}\times\vec{e_2^3}]-$$
$$\frac{s}{2}(\dot{u}_4\vec{e_1^2}\times\vec{e_3^4}+u_4\vec{\omega_2}\times\vec{e_1^2}\times\vec{e_3^4}+u_4\vec{e_1^2}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_5\vec{e_2^2}\times\vec{e_3^4}+u_5\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^4}+$$
$$u_5\vec{e_2^2}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_6\vec{e_3^2}\times\vec{e_3^4}+u_6\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^4}+u_6\vec{e_3^2}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_8\vec{e_1^3}\times\vec{e_3^4}+$$
$$u_8\vec{\omega_3}\times\vec{e_1^3}\times\vec{e_3^4}+u_8\vec{e_1^3}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_{11}\vec{e_3^3}\times\vec{e_3^4}+u_{11}\vec{\omega_3}\times\vec{e_3^3}\times\vec{e_3^4}+u_{11}\vec{e_3^3}\times$$
$$(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_{12}\vec{e_1^4}\times\vec{e_3^4}+u_{12}\vec{\omega_4}\times\vec{e_1^4}\times\vec{e_3^4}+u_{12}\vec{e_1^4}\times(\vec{\omega_4}\times\vec{e_3^4})\}\cdot\vec{e_3^0}$$

代入凯恩方程

$$F_3'+F_3^*=0$$

对于广义速率u_4，系统的广义主动力为

$$F_4' = \sum_{i=1}^{4}[\overrightarrow{G_i}\cdot\overrightarrow{v_i}^4 + \overrightarrow{M_i}\cdot\overrightarrow{\omega_i}^4] + \overrightarrow{M^a}\cdot\overrightarrow{\Omega_{2,3}}^4 + \overrightarrow{M_{1,2}}\cdot\overrightarrow{\Omega_{1,2}}^4 + \overrightarrow{M_{3,4}}\cdot\overrightarrow{\Omega_{3,4}}^4$$

$$= -mg\overrightarrow{e_3^0}\cdot[\frac{s}{2}(-\overrightarrow{e_2^2} + \overrightarrow{e_1^2}\times\overrightarrow{e_3^1})] - mg\overrightarrow{e_3^0}\cdot[-\frac{s}{2}(\overrightarrow{e_2^2}\times\overrightarrow{e_3^3} + \overrightarrow{e_2^2}\times\overrightarrow{e_3^4})]$$

（4-55d）

对于u_4的广义惯性力为

$$F_4^* = -\sum_{i=1}^{4}m_i\dot{\overrightarrow{v_i}}\cdot\overrightarrow{v_i}^4 - \sum_{i=1}^{4}\overline{J_i}\cdot\dot{\overrightarrow{\omega_i}}\cdot\overrightarrow{\omega_i}^4 \qquad (4\text{-}56d)$$

式中

$$\dot{\overrightarrow{v_1}}\cdot\overrightarrow{v_1}^4 = \{\dot{u}_1\overrightarrow{e_1^2} + u_1\overrightarrow{\omega_2}\times\overrightarrow{e_2^2} + \dot{u}_2\overrightarrow{e_2^2} + u_2\overrightarrow{\omega_2}\times\overrightarrow{e_3^2} + \dot{u}_4(-\frac{s}{2}\overrightarrow{e_2^2} + \frac{s}{2}\overrightarrow{e_1^2}\times\overrightarrow{e_3^1}) +$$

$$u_4\frac{s}{2}[-\overrightarrow{\omega_2}\times\overrightarrow{e_2^2} + \overrightarrow{\omega_2}\times\overrightarrow{e_1^2}\times\overrightarrow{e_3^1} + \overrightarrow{e_1^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})] + \frac{s}{2}\dot{u}_5(\overrightarrow{e_1^2} + \overrightarrow{e_2^2}\times\overrightarrow{e_3^1}) +$$

$$\frac{s}{2}u_5[\overrightarrow{\omega_2}\times\overrightarrow{e_1^2} + \overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\times\overrightarrow{e_3^1} + \overrightarrow{e_2^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})] + \dot{u}_6\frac{s}{2}\overrightarrow{e_3^2}\times\overrightarrow{e_3^1} +$$

$$u_6\frac{s}{2}[\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\times\overrightarrow{e_3^1} + \overrightarrow{e_3^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})] + \frac{s}{2}\dot{u}_9\overrightarrow{e_3^2}\times\overrightarrow{e_3^1} +$$

$$\frac{s}{2}u_9[\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\times\overrightarrow{e_3^1} + \overrightarrow{e_3^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})] + \frac{s}{2}\dot{u}_{10}\overrightarrow{e_1^1}\times\overrightarrow{e_3^1} +$$

$$\frac{s}{2}u_{10}[\overrightarrow{\omega_1}\times\overrightarrow{e_1^1}\times\overrightarrow{e_3^1} + \overrightarrow{e_1^1}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})]\}\cdot[\frac{s}{2}(-\overrightarrow{e_2^2} + \overrightarrow{e_1^2}\times\overrightarrow{e_3^1})]$$

$$\dot{\overrightarrow{v_2}}\cdot\overrightarrow{v_2}^4 = 0$$

$$\dot{\overrightarrow{v_3}}\cdot\overrightarrow{v_3}^4 = 0$$

$$\dot{\overrightarrow{v_4}}\cdot\overrightarrow{v_4}^4 = \{\dot{u}_1\overrightarrow{e_1^0} + \dot{u}_2\overrightarrow{e_2^0} + \dot{u}_3\overrightarrow{e_3^0} + E_1\overrightarrow{e_2^2} + E_2\overrightarrow{\omega_2}\times\overrightarrow{e_2^2} + E_3\overrightarrow{e_1^2} + E_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2} +$$

$$E_5\overrightarrow{e_1^2} + E_6\overrightarrow{\omega_2}\times\overrightarrow{e_1^2} - \frac{s}{2}[\dot{u}_4\overrightarrow{e_1^2}\times\overrightarrow{e_3^3} + u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}\times\overrightarrow{e_3^3} + u_4\overrightarrow{e_1^2}\times(\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}) +$$

$$\dot{u}_5\overrightarrow{e_2^2}\times\overrightarrow{e_3^3} + u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\times\overrightarrow{e_3^3} + u_5\overrightarrow{e_2^2}\times(\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}) + \dot{u}_6\overrightarrow{e_3^2}\times\overrightarrow{e_3^3} + u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\times\overrightarrow{e_3^3} +$$

$$u_6\overrightarrow{e_3^2}\times(\overrightarrow{\omega_3}\times\overrightarrow{e_3^3})+\dot{u}_7\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}+u_7\overrightarrow{e_2^2}\times(\overrightarrow{\omega_3}\times\overrightarrow{e_3^3})-\dot{u}_8\overrightarrow{e_2^3}-$$
$$u_8\overrightarrow{\omega_3}\times\overrightarrow{e_2^3}]-\frac{s}{2}(\dot{u}_4\overrightarrow{e_1^2}\times\overrightarrow{e_3^4}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}\times\overrightarrow{e_3^4}+u_4\overrightarrow{e_1^2}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_5\overrightarrow{e_2^2}\times\overrightarrow{e_3^4}+$$
$$u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\times\overrightarrow{e_3^4}+u_5\overrightarrow{e_2^2}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_6\overrightarrow{e_3^2}\times\overrightarrow{e_3^4}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\times\overrightarrow{e_3^4}+u_6\overrightarrow{e_3^2}\times$$
$$(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_8\overrightarrow{e_1^3}\times\overrightarrow{e_3^4}+u_8\overrightarrow{\omega_3}\times\overrightarrow{e_1^3}\times\overrightarrow{e_3^4}+u_8\overrightarrow{e_1^3}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_{11}\overrightarrow{e_3^3}\times\overrightarrow{e_3^4}+$$
$$u_{11}\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}\times\overrightarrow{e_3^4}+u_{11}\overrightarrow{e_3^3}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_{12}\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}+u_{12}\overrightarrow{\omega_4}\times\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}+$$
$$u_{12}\overrightarrow{e_1^4}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})\bullet[-\frac{s}{2}(\overrightarrow{e_1^2}\times\overrightarrow{e_3^3}+\overrightarrow{e_1^2}\times\overrightarrow{e_3^4})]$$

$$\overline{J}_1\bullet\dot{\overrightarrow{\omega_1}}\bullet\overrightarrow{\omega_1}^4=(J_{11}\overrightarrow{e_1^1}\overrightarrow{e_1^1}+J_{12}\overrightarrow{e_2^1}\overrightarrow{e_2^1}+J_{13}\overrightarrow{e_3^1}\overrightarrow{e_3^1})\bullet(\dot{u}_4\overrightarrow{e_1^2}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+$$
$$\dot{u}_6\overrightarrow{e_3^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\dot{u}_9\overrightarrow{e_3^2}+u_9\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\dot{u}_{10}\overrightarrow{e_1^1}+u_{10}\overrightarrow{\omega_1}\times\overrightarrow{e_1^1})\bullet\overrightarrow{e_1^2}$$

$$\overline{J}_2\bullet\dot{\overrightarrow{\omega_2}}\bullet\overrightarrow{\omega_2}^4=(J_{21}\overrightarrow{e_1^2}\overrightarrow{e_1^2}+J_{22}\overrightarrow{e_2^2}\overrightarrow{e_2^2}+J_{23}\overrightarrow{e_3^2}\overrightarrow{e_3^2})\bullet(\dot{u}_4\overrightarrow{e_1^2}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+$$
$$u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+\dot{u}_6\overrightarrow{e_3^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2})\bullet\overrightarrow{e_1^2}$$

$$\overline{J}_3\bullet\dot{\overrightarrow{\omega_3}}\bullet\overrightarrow{\omega_3}^4=(J_{31}\overrightarrow{e_1^3}\overrightarrow{e_1^3}+J_{32}\overrightarrow{e_2^3}\overrightarrow{e_2^3}+J_{33}\overrightarrow{e_3^3}\overrightarrow{e_3^3})\bullet(\dot{u}_4\overrightarrow{e_1^2}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+$$
$$\dot{u}_6\overrightarrow{e_3^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\dot{u}_7\overrightarrow{e_2^2}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+\dot{u}_8\overrightarrow{e_1^3}+u_8\overrightarrow{\omega_3}\times\overrightarrow{e_1^3})\bullet\overrightarrow{e_1^2}$$

$$\overline{J}_4\bullet\dot{\overrightarrow{\omega_4}}\bullet\overrightarrow{\omega_4}^4=(J_{41}\overrightarrow{e_1^4}\overrightarrow{e_1^4}+J_{42}\overrightarrow{e_2^4}\overrightarrow{e_2^4}+J_{43}\overrightarrow{e_3^4}\overrightarrow{e_3^4})\bullet(\dot{u}_4\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+\dot{u}_6\overrightarrow{e_3^2}+\dot{u}_7\overrightarrow{e_2^2}+\dot{u}_8\overrightarrow{e_1^3}+$$
$$\dot{u}_{11}\overrightarrow{e_3^3}+\dot{u}_{12}\overrightarrow{e_1^4}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+$$
$$u_8\overrightarrow{\omega_3}\times\overrightarrow{e_1^3}+u_{11}\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}+u_{12}\overrightarrow{\omega_4}\times\overrightarrow{e_1^4})\bullet\overrightarrow{e_1^2}$$

代入凯恩方程

$$F_4'+F_4^*=0$$

对于广义速率u_5，系统的广义主动力为

$$F_5'=\sum_{i=1}^{4}[\overrightarrow{G_i}\bullet\overrightarrow{v_i}^5+\overrightarrow{M_i}\bullet\overrightarrow{\omega_i}^5]+\overrightarrow{M^a}\bullet\overrightarrow{\Omega_{2,3}}^5+\overrightarrow{M_{1,2}}\bullet\overrightarrow{\Omega_{1,2}}^5+\overrightarrow{M_{3,4}}\bullet\overrightarrow{\Omega_{3,4}}^5$$
$$=-mg\overrightarrow{e_3^0}\bullet[\frac{s}{2}(\overrightarrow{e_1^2}+\overrightarrow{e_2^2}\times\overrightarrow{e_3^1})]-mg\overrightarrow{e_3^0}\bullet[-\frac{s}{2}(\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}+\overrightarrow{e_2^2}\times\overrightarrow{e_3^4})]$$

(4-55e)

对于u_5的广义惯性力为

$$F_5^* = -\sum_{i=1}^{4} m_i \dot{\vec{v}_i} \bullet \vec{v}_i^{\,5} - \sum_{i=1}^{4} \overline{J_i} \bullet \dot{\vec{\omega}_i} \bullet \vec{\omega}_i^{\,5} \tag{4-56e}$$

式中

$$\dot{\vec{v}_1} \bullet \vec{v}_1^{\,5} = \{\dot{u}_1 \vec{e_1^2} + u_1 \vec{\omega_2} \times \vec{e_2^2} + \dot{u}_2 \vec{e_2^2} + u_2 \vec{\omega_2} \times \vec{e_3^2} + \dot{u}_4 (-\frac{s}{2}\vec{e_2^2} + \frac{s}{2}\vec{e_1^2} \times \vec{e_3^1}) +$$

$$u_4 \frac{s}{2}[-\vec{\omega_2} \times \vec{e_2^2} + \vec{\omega_2} \times \vec{e_1^2} \times \vec{e_3^1} + \vec{e_1^2} \times (\vec{\omega_1} \times \vec{e_3^1})] + \frac{s}{2}\dot{u}_5 (\vec{e_1^2} + \vec{e_2^2} \times \vec{e_3^1}) +$$

$$\frac{s}{2} u_5 [\vec{\omega_2} \times \vec{e_1^2} + \vec{\omega_2} \times \vec{e_2^2} \times \vec{e_3^1} + \vec{e_2^2} \times (\vec{\omega_1} \times \vec{e_3^1})] + \dot{u}_6 \frac{s}{2} \vec{e_3^2} \times \vec{e_3^1} +$$

$$u_6 \frac{s}{2}[\vec{\omega_2} \times \vec{e_3^2} \times \vec{e_3^1} + \vec{e_3^2} \times (\vec{\omega_1} \times \vec{e_3^1})] + \frac{s}{2}\dot{u}_9 \vec{e_3^2} \times \vec{e_3^1} +$$

$$\frac{s}{2} u_9 [\vec{\omega_2} \times \vec{e_3^2} \times \vec{e_3^1} + \vec{e_3^2} \times (\vec{\omega_1} \times \vec{e_3^1})] + \frac{s}{2}\dot{u}_{10} \vec{e_1^1} \times \vec{e_3^1} + \} \bullet [\frac{s}{2}(\vec{e_1^2} + \vec{e_2^2} \times \vec{e_3^1})]$$

$$\frac{s}{2} u_{10} [\vec{\omega_1} \times \vec{e_1^1} \times \vec{e_3^1} + \vec{e_1^1} \times (\vec{\omega_1} \times \vec{e_3^1})]\} \bullet [\frac{s}{2}(\vec{e_1^2} + \vec{e_2^2} \times \vec{e_3^1})]$$

$$\dot{\vec{v}_2} \bullet \vec{v}_2^{\,5} = 0$$

$$\dot{\vec{v}_3} \bullet \vec{v}_3^{\,5} = 0$$

$$\dot{\vec{v}_4} \bullet \vec{v}_4^{\,5} = \{\dot{u}_1 \vec{e_1^0} + \dot{u}_2 \vec{e_2^0} + \dot{u}_3 \vec{e_3^0} + E_1 \vec{e_2^2} + E_2 \vec{\omega_2} \times \vec{e_2^2} + E_3 \vec{e_1^2} + E_4 \vec{\omega_2} \times \vec{e_1^2} + E_5 \vec{e_1^2} +$$

$$E_6 \vec{\omega_2} \times \vec{e_1^2} - \frac{s}{2}[\dot{u}_4 \vec{e_1^2} \times \vec{e_3^3} + u_4 \dot{\vec{\omega_2}} \times \vec{e_1^2} \times \vec{e_3^3} + u_4 \vec{e_1^2} \times (\vec{\omega_3} \times \vec{e_3^3}) + \dot{u}_5 \vec{e_2^2} \times$$

$$\vec{e_3^3} + u_5 \vec{\omega_2} \times \vec{e_2^2} \times \vec{e_3^3} + u_5 \vec{e_2^2} \times (\vec{\omega_3} \times \vec{e_3^3}) + \dot{u}_6 \vec{e_3^2} \times \vec{e_3^3} + u_6 \vec{\omega_2} \times \vec{e_3^2} \times \vec{e_3^3} +$$

$$u_6 \vec{e_3^2} \times (\vec{\omega_3} \times \vec{e_3^3}) + \dot{u}_7 \vec{e_2^2} \times \vec{e_3^3} + u_7 \vec{\omega_2} \times \vec{e_2^2} \times \vec{e_3^3} + u_7 \vec{e_2^2} \times (\vec{\omega_3} \times \vec{e_3^3}) -$$

$$\dot{u}_8 \vec{e_2^3} - u_8 \vec{\omega_3} \times \vec{e_2^3}] - \frac{s}{2}(\dot{u}_4 \vec{e_1^2} \times \vec{e_3^4} + u_4 \vec{\omega_2} \times \vec{e_1^2} \times \vec{e_3^4} + u_4 \vec{e_1^2} \times (\vec{\omega_4} \times \vec{e_3^4}) +$$

$$\dot{u}_5 \vec{e_2^2} \times \vec{e_3^4} + u_5 \vec{\omega_2} \times \vec{e_2^2} \times \vec{e_3^4} + u_5 \vec{e_2^2} \times (\vec{\omega_4} \times \vec{e_3^4}) + \dot{u}_6 \vec{e_3^2} \times \vec{e_3^4} + u_6 \vec{\omega_2} \times \vec{e_3^2} \times \vec{e_3^4} +$$

$$u_6 \vec{e_3^2} \times (\vec{\omega_4} \times \vec{e_3^4}) + \dot{u}_8 \vec{e_1^3} \times \vec{e_3^4} + u_8 \vec{\omega_3} \times \vec{e_1^3} \times \vec{e_3^4} + u_8 \vec{e_1^3} \times (\vec{\omega_4} \times \vec{e_3^4}) + \dot{u}_{11} \vec{e_3^3} \times \vec{e_3^4} +$$

$$u_{11}\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}\times\overrightarrow{e_3^4}+u_{11}\overrightarrow{e_3^3}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_{12}\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}+u_{12}\overrightarrow{\omega_4}\times\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}+$$

$$u_{12}\overrightarrow{e_1^4}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})\}\bullet[-\frac{s}{2}(\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}+\overrightarrow{e_2^2}\times\overrightarrow{e_3^4})]$$

$$\overline{J}_1\bullet\dot{\overrightarrow{\omega_1}}\bullet\overrightarrow{\omega_1}^5=(J_{11}\overrightarrow{e_1^1}\overrightarrow{e_1^1}+J_{12}\overrightarrow{e_2^1}\overrightarrow{e_2^1}+J_{13}\overrightarrow{e_3^1}\overrightarrow{e_3^1})\bullet(\dot{u}_4\overrightarrow{e_1^2}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+$$
$$u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+\dot{u}_6\overrightarrow{e_3^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\dot{u}_9\overrightarrow{e_3^2}+u_9\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\dot{u}_{10}\overrightarrow{e_1^1}+$$
$$u_{10}\overrightarrow{\omega_1}\times\overrightarrow{e_1^1})\bullet\overrightarrow{e_2^2}$$

$$\overline{J}_2\bullet\dot{\overrightarrow{\omega_2}}\bullet\overrightarrow{\omega_2}^5=(J_{21}\overrightarrow{e_1^2}\overrightarrow{e_1^2}+J_{22}\overrightarrow{e_2^2}\overrightarrow{e_2^2}+J_{23}\overrightarrow{e_3^2}\overrightarrow{e_3^2})\bullet(\dot{u}_4\overrightarrow{e_1^2}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+$$
$$\dot{u}_5\overrightarrow{e_2^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+\dot{u}_6\overrightarrow{e_3^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2})\bullet\overrightarrow{e_2^2}$$

$$\overline{J}_3\bullet\dot{\overrightarrow{\omega_3}}\bullet\overrightarrow{\omega_3}^5=(J_{31}\overrightarrow{e_1^3}\overrightarrow{e_1^3}+J_{32}\overrightarrow{e_2^3}\overrightarrow{e_2^3}+J_{33}\overrightarrow{e_3^3}\overrightarrow{e_3^3})\bullet(\dot{u}_4\overrightarrow{e_1^2}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+$$
$$\dot{u}_5\overrightarrow{e_2^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+\dot{u}_6\overrightarrow{e_3^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\dot{u}_7\overrightarrow{e_2^2}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+$$
$$\dot{u}_8\overrightarrow{e_1^3}+u_8\overrightarrow{\omega_3}\times\overrightarrow{e_1^3})\bullet\overrightarrow{e_2^2}$$

$$\overline{J}_4\bullet\dot{\overrightarrow{\omega_4}}\bullet\overrightarrow{\omega_4}^5=(J_{41}\overrightarrow{e_1^4}\overrightarrow{e_1^4}+J_{42}\overrightarrow{e_2^4}\overrightarrow{e_2^4}+J_{43}\overrightarrow{e_3^4}\overrightarrow{e_3^4})\bullet(\dot{u}_4\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+\dot{u}_6\overrightarrow{e_3^2}+$$
$$\dot{u}_7\overrightarrow{e_2^2}+\dot{u}_8\overrightarrow{e_1^3}+\dot{u}_{11}\overrightarrow{e_3^3}+\dot{u}_{12}\overrightarrow{e_1^4}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+$$
$$u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+u_8\overrightarrow{\omega_3}\times\overrightarrow{e_1^3}+u_{11}\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}+u_{12}\overrightarrow{\omega_4}\times\overrightarrow{e_1^4})\bullet\overrightarrow{e_2^2}$$

代入凯恩方程 $F_5'+F_5^*=0$

对于广义速率u_6，系统的广义主动力为

$$F_6'=\sum_{i=1}^{4}[\overrightarrow{G_i}\bullet\overrightarrow{v_i}^6+\overrightarrow{M_i}\bullet\overrightarrow{\omega_i}^6]+\overrightarrow{M^a}\bullet\overrightarrow{\Omega_{2,3}}^6+\overrightarrow{M_{1,2}}\bullet\overrightarrow{\Omega_{1,2}}^6+\overrightarrow{M_{3,4}}\bullet\overrightarrow{\Omega_{3,4}}^6$$

$$=-mg\overrightarrow{e_3^0}\bullet[\frac{s}{2}\overrightarrow{e_3^2}\times\overrightarrow{e_3^1}]-mg\overrightarrow{e_3^0}\bullet[-\frac{s}{2}\sin\alpha\overrightarrow{e_1^2}+\frac{s}{2}\sin\beta\cos\alpha\overrightarrow{e_2^2}+\frac{s}{2}\sin\alpha\overrightarrow{e_3^2}]-$$

$$mg\overrightarrow{e_3^0}\bullet[-\frac{s}{2}\sin\alpha\overrightarrow{e_1^2}+\frac{s}{2}\sin\beta\cos\alpha\overrightarrow{e_2^2}+\frac{s}{2}\sin\alpha\overrightarrow{e_3^2}-\frac{s}{2}(\overrightarrow{e_3^2}\times\overrightarrow{e_3^3}+\overrightarrow{e_3^2}\times\overrightarrow{e_3^4})]$$

（4-55f）

对于u_6的广义惯性力为

$$F_6^*=-\sum_{i=1}^{4}m_i\dot{\overrightarrow{v_i}}\bullet\overrightarrow{v_i}^6-\sum_{i=1}^{4}\overline{J_i}\bullet\dot{\overrightarrow{\omega_i}}\bullet\overrightarrow{\omega_i}^6 \qquad (4\text{-}56f)$$

式中

$$\dot{\vec{v}_1}\bullet\vec{v}_1^{\,6}=\{\dot{u}_1\vec{e_1^2}+u_1\vec{\omega_2}\times\vec{e_2^2}+\dot{u}_2\vec{e_2^2}+u_2\vec{\omega_2}\times\vec{e_3^2}+\dot{u}_4(-\frac{s}{2}\vec{e_2^2}+\frac{s}{2}\vec{e_1^2}\times\vec{e_3^1})+$$

$$u_4\frac{s}{2}[-\vec{\omega_2}\times\vec{e_2^2}+\vec{\omega_2}\times\vec{e_1^2}\times\vec{e_3^1}+\vec{e_1^2}\times(\vec{\omega_1}\times\vec{e_3^1})]+\frac{s}{2}\dot{u}_5(\vec{e_1^2}+\vec{e_2^2}\times\vec{e_3^1})+$$

$$\frac{s}{2}u_5[\vec{\omega_2}\times\vec{e_1^2}+\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^1}+\vec{e_2^2}\times(\dot{\vec{\omega_1}}\times\vec{e_3^1})]+\dot{u}_6\frac{s}{2}\vec{e_3^2}\times\vec{e_3^1}+$$

$$u_6\frac{s}{2}[\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^1}+\vec{e_3^2}\times(\vec{\omega_1}\times\vec{e_3^1})]+\frac{s}{2}\dot{u}_9\vec{e_3^2}\times\vec{e_3^1}+$$

$$\frac{s}{2}u_9[\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^1}+\vec{e_3^2}\times(\vec{\omega_1}\times\vec{e_3^1})]+\frac{s}{2}\dot{u}_{10}\vec{e_1^1}\times\vec{e_3^1}+$$

$$\frac{s}{2}u_{10}[\vec{\omega_1}\times\vec{e_1^1}\times\vec{e_3^1}+\vec{e_1^1}\times(\vec{\omega_1}\times\vec{e_3^1})]\}\bullet(\frac{s}{2}\vec{e_3^2}\times\vec{e_3^1})$$

$$\dot{\vec{v}_2}\bullet\vec{v}_2^{\,6}=0$$

$$\dot{\vec{v}_3}\bullet\vec{v}_3^{\,6}=(\dot{u}_1\vec{e_1^0}+\dot{u}_2\vec{e_2^0}+\dot{u}_3\vec{e_3^0}+E_1\vec{e_2^2}+E_2\vec{\omega_2}\times\vec{e_2^2}+E_3\vec{e_1^2}+E_4\vec{\omega_2}\times\vec{e_1^2}+$$

$$E_5\vec{e_1^2}+E_6\vec{\omega_2}\times\vec{e_1^2})\bullet(-\frac{s}{2}\sin\alpha\vec{e_1^2}+\frac{s}{2}\sin\beta\cos\alpha\vec{e_2^2}+\frac{s}{2}\sin\alpha\vec{e_3^2})$$

$$\dot{\vec{v}_4}\bullet\vec{v}_4^{\,6}=\{\dot{u}_1\vec{e_1^0}+\dot{u}_2\vec{e_2^0}+\dot{u}_3\vec{e_3^0}+E_1\vec{e_2^2}+E_2\vec{\omega_2}\times\vec{e_2^2}+E_3\vec{e_1^2}+E_4\vec{\omega_2}\times\vec{e_1^2}+$$

$$E_5\vec{e_1^2}+E_6\vec{\omega_2}\times\vec{e_1^2}-\frac{s}{2}[\dot{u}_4\vec{e_1^2}\times\vec{e_3^3}+u_4\vec{\omega_2}\times\vec{e_1^2}\times\vec{e_3^3}+u_4\vec{e_1^2}\times(\vec{\omega_3}\times\vec{e_3^3})+$$

$$\dot{u}_5\vec{e_2^2}\times\vec{e_3^3}+u_5\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^3}+u_5\vec{e_2^2}\times(\vec{\omega_3}\times\vec{e_3^3})+\dot{u}_6\vec{e_3^2}\times\vec{e_3^3}+u_6\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^3}+$$

$$u_6\vec{e_3^2}\times(\vec{\omega_3}\times\vec{e_3^3})+\dot{u}_7\vec{e_2^2}\times\vec{e_3^3}+u_7\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^3}+u_7\vec{e_2^2}\times(\vec{\omega_3}\times\vec{e_3^3})-\dot{u}_8\vec{e_2^3}-u_8\vec{\omega_3}\times\vec{e_2^3}]-$$

$$\frac{s}{2}(\dot{u}_4\vec{e_1^2}\times\vec{e_3^4}+u_4\vec{\omega_2}\times\vec{e_1^2}\times\vec{e_3^4}+u_4\vec{e_1^2}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_5\vec{e_2^2}\times\vec{e_3^4}+u_5\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^4}+$$

$$u_5\vec{e_2^2}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_6\vec{e_3^2}\times\vec{e_3^4}+u_6\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^4}+u_6\vec{e_3^2}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_8\vec{e_1^3}\times\vec{e_3^4}+$$

$$u_8\vec{\omega_3}\times\vec{e_1^3}\times\vec{e_3^4}+u_8\vec{e_1^3}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_{11}\vec{e_3^3}\times\vec{e_3^4}+u_{11}\vec{\omega_3}\times\vec{e_3^3}\times\vec{e_3^4}+u_{11}\vec{e_3^3}\times(\vec{\omega_4}\times\vec{e_3^4})+$$

$$\dot{u}_{12}\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}+u_{12}\overrightarrow{\omega_4}\times\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}+u_{12}\overrightarrow{e_1^4}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})\}\bullet[-\frac{s}{2}\sin\alpha\overrightarrow{e_1^2}+\frac{s}{2}\sin\beta\cos\alpha\overrightarrow{e_2^2}+$$

$$\frac{s}{2}\sin\alpha\overrightarrow{e_3^2}-\frac{s}{2}(\overrightarrow{e_3^2}\times\overrightarrow{e_3^3}+\overrightarrow{e_3^2}\times\overrightarrow{e_3^4})]$$

$$\overline{J_1}\bullet\dot{\overrightarrow{\omega_1}}\bullet\overrightarrow{\omega_1}^6=(J_{11}\overrightarrow{e_1^1}\overrightarrow{e_1^1}+J_{12}\overrightarrow{e_2^1}\overrightarrow{e_2^1}+J_{13}\overrightarrow{e_3^1}\overrightarrow{e_3^1})\bullet(\dot{u}_4\overrightarrow{e_1^2}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+$$
$$\dot{u}_6\overrightarrow{e_3^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\dot{u}_9\overrightarrow{e_3^2}+u_9\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\dot{u}_{10}\overrightarrow{e_1^1}+u_{10}\overrightarrow{\omega_1}\times\overrightarrow{e_1^1})\bullet\overrightarrow{e_3^2}$$

$$\overline{J_2}\bullet\dot{\overrightarrow{\omega_2}}\bullet\overrightarrow{\omega_2}^6=(J_{21}\overrightarrow{e_1^2}\overrightarrow{e_1^2}+J_{22}\overrightarrow{e_2^2}\overrightarrow{e_2^2}+J_{23}\overrightarrow{e_3^2}\overrightarrow{e_3^2})\bullet(\dot{u}_4\overrightarrow{e_1^2}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+$$
$$\dot{u}_6\overrightarrow{e_3^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2})\bullet\overrightarrow{e_3^2}$$

$$\overline{J_3}\bullet\dot{\overrightarrow{\omega_3}}\bullet\overrightarrow{\omega_3}^6=(J_{31}\overrightarrow{e_1^3}\overrightarrow{e_1^3}+J_{32}\overrightarrow{e_2^3}\overrightarrow{e_2^3}+J_{33}\overrightarrow{e_3^3}\overrightarrow{e_3^3})\bullet(\dot{u}_4\overrightarrow{e_1^2}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+$$
$$\dot{u}_6\overrightarrow{e_3^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\dot{u}_7\overrightarrow{e_2^2}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+\dot{u}_8\overrightarrow{e_1^3}+u_8\overrightarrow{\omega_3}\times\overrightarrow{e_1^3})\bullet\overrightarrow{e_3^2}$$

$$\overline{J_4}\bullet\dot{\overrightarrow{\omega_4}}\bullet\overrightarrow{\omega_4}^6=(J_{41}\overrightarrow{e_1^4}\overrightarrow{e_1^4}+J_{42}\overrightarrow{e_2^4}\overrightarrow{e_2^4}+J_{43}\overrightarrow{e_3^4}\overrightarrow{e_3^4})\bullet(\dot{u}_4\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+\dot{u}_6\overrightarrow{e_3^2}+\dot{u}_7\overrightarrow{e_2^2}+\dot{u}_8\overrightarrow{e_1^3}+$$
$$\dot{u}_{11}\overrightarrow{e_3^3}+\dot{u}_{12}\overrightarrow{e_1^4}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+$$
$$u_8\overrightarrow{\omega_3}\times\overrightarrow{e_1^3}+u_{11}\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}+u_{12}\overrightarrow{\omega_4}\times\overrightarrow{e_1^4})\bullet\overrightarrow{e_3^2}$$

代入凯恩方程 $F_6'+F_6^*=0$

对于广义速率u_7，系统的广义主动力为

$$F_7'=\sum_{i=1}^{4}[\overrightarrow{G_i}\bullet\overrightarrow{v_i}^7+\overrightarrow{M_i}\bullet\overrightarrow{\omega_i}^7]+\overrightarrow{M^a}\bullet\overrightarrow{\Omega_{2,3}}^7+\overrightarrow{M_{1,2}}\bullet\overrightarrow{\Omega_{1,2}}^7+\overrightarrow{M_{3,4}}\bullet\overrightarrow{\Omega_{3,4}}^7$$
$$=mg\overrightarrow{e_3^0}\bullet[-\frac{s}{2}\cos\alpha\cos\beta\overrightarrow{e_1^2}-\frac{s}{2}\sin\beta\cos\alpha\overrightarrow{e_3^2}]-$$
$$mg\overrightarrow{e_3^0}\bullet[-\frac{s}{2}\cos\alpha\cos\beta\overrightarrow{e_1^2}-\frac{s}{2}\sin\beta\cos\alpha\overrightarrow{e_3^2}-\frac{s}{2}(\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}+\overrightarrow{e_2^2}\times\overrightarrow{e_3^4})]+$$
$$(M_\beta\overrightarrow{e_2^2}+M_\alpha\overrightarrow{e_1^3})\bullet\overrightarrow{e_2^2}$$

（4-55g）

对于u_7的广义惯性力为

$$F_7^*=-\sum_{i=1}^{4}m_i\dot{\overrightarrow{v_i}}\bullet\overrightarrow{v_i}^7-\sum_{i=1}^{4}\overline{J_i}\bullet\dot{\overrightarrow{\omega_i}}\bullet\overrightarrow{\omega_i}^7 \qquad (4\text{-}56g)$$

式中

$$\dot{\vec{v_1}}\bullet\vec{v_1}^{\,7}=0$$

$$\dot{\vec{v_2}}\bullet\vec{v_2}^{\,7}=0$$

$$\dot{\vec{v_3}}\bullet\vec{v_3}^{\,7}=(\dot{u}_1\vec{e_1^0}+\dot{u}_2\vec{e_2^0}+\dot{u}_3\vec{e_3^0}+E_1\vec{e_2^2}+E_2\vec{\omega_2}\times\vec{e_2^2}+E_3\vec{e_1^2}+E_4\vec{\omega_2}\times\vec{e_1^2}+E_5\vec{e_1^2}+ E_6\vec{\omega_2}\times\vec{e_1^2})\bullet(-\frac{s}{2}\cos\alpha\cos\beta\vec{e_1^2}-\frac{s}{2}\sin\beta\cos\alpha\vec{e_3^2})$$

$$\dot{\vec{v_4}}\bullet\vec{v_4}^{\,7}=\{\dot{u}_1\vec{e_1^0}+\dot{u}_2\vec{e_2^0}+\dot{u}_3\vec{e_3^0}+E_1\vec{e_2^2}+E_2\vec{\omega_2}\times\vec{e_2^2}+E_3\vec{e_1^2}+E_4\vec{\omega_2}\times\vec{e_1^2}+E_5\vec{e_1^2}+ E_6\vec{\omega_2}\times\vec{e_1^2}-\frac{s}{2}[\dot{u}_4\vec{e_1^2}\times\vec{e_3^3}+u_4\vec{\omega_2}\times\vec{e_1^2}\times\vec{e_3^3}+u_4\vec{e_1^2}\times(\vec{\omega_3}\times\vec{e_3^3})+\dot{u}_5\vec{e_2^2}\times\vec{e_3^3}+ u_5\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^3}+u_5\vec{e_2^2}\times(\vec{\omega_3}\times\vec{e_3^3})+\dot{u}_6\vec{e_3^2}\times\vec{e_3^3}+u_6\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^3}+u_6\vec{e_3^2}\times (\vec{\omega_3}\times\vec{e_3^3})+\dot{u}_7\vec{e_2^2}\times\vec{e_3^3}+u_7\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^3}+u_7\vec{e_2^2}\times(\vec{\omega_3}\times\vec{e_3^3})-\dot{u}_8\vec{e_2^3}-u_8\vec{\omega_3}\times\vec{e_2^3}]- \frac{s}{2}(\dot{u}_4\vec{e_1^2}\times\vec{e_3^4}+u_4\vec{\omega_2}\times\vec{e_1^2}\times\vec{e_3^4}+u_4\vec{e_1^2}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_5\vec{e_2^2}\times\vec{e_3^4}+u_5\vec{\omega_2}\times\vec{e_2^2}\times\vec{e_3^4}+ u_5\vec{e_2^2}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_6\vec{e_3^2}\times\vec{e_3^4}+u_6\vec{\omega_2}\times\vec{e_3^2}\times\vec{e_3^4}+u_6\vec{e_3^2}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_8\vec{e_1^3}\times\vec{e_3^4}+ u_8\vec{\omega_3}\times\vec{e_1^3}\times\vec{e_3^4}+u_8\vec{e_1^3}\times(\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_{11}\vec{e_3^3}\times\vec{e_3^4}+u_{11}\vec{\omega_3}\times\vec{e_3^3}\times\vec{e_3^4}+u_{11}\vec{e_3^3}\times (\vec{\omega_4}\times\vec{e_3^4})+\dot{u}_{12}\vec{e_1^4}\times\vec{e_3^4}+u_{12}\vec{\omega_4}\times\vec{e_1^4}\times\vec{e_3^4}+u_{12}\vec{e_1^4}\times(\vec{\omega_4}\times\vec{e_3^4})\}\bullet[-\frac{s}{2}\cos\alpha\cos\beta\vec{e_1^2}- \frac{s}{2}\sin\beta\cos\alpha\vec{e_3^2}-\frac{s}{2}(\vec{e_2^2}\times\vec{e_3^3}+\vec{e_2^2}\times\vec{e_3^4})]$$

$$\overline{J_1}\bullet\dot{\vec{\omega_1}}\bullet\vec{\omega_1}^{\,7}=0$$

$$\overline{J_2}\bullet\dot{\vec{\omega_2}}\bullet\vec{\omega_2}^{\,7}=0$$

$$\overline{J_3}\bullet\dot{\vec{\omega_3}}\bullet\vec{\omega_3}^{\,7}=(J_{31}\vec{e_1^3}\vec{e_1^3}+J_{32}\vec{e_2^3}\vec{e_2^3}+J_{33}\vec{e_3^3}\vec{e_3^3})\bullet(\dot{u}_4\vec{e_1^2}+u_4\vec{\omega_2}\times\vec{e_1^2}+\dot{u}_5\vec{e_2^2}+u_5\vec{\omega_2}\times\vec{e_2^2}+ \dot{u}_6\vec{e_3^2}+u_6\vec{\omega_2}\times\vec{e_3^2}+\dot{u}_7\vec{e_2^2}+u_7\vec{\omega_2}\times\vec{e_2^2}+\dot{u}_8\vec{e_1^3}+u_8\vec{\omega_3}\times\vec{e_1^3})\bullet\vec{e_2^2}$$

$$\overline{J_4}\bullet\dot{\vec{\omega_4}}\bullet\vec{\omega_4}^{\,7}=(J_{41}\vec{e_1^4}\vec{e_1^4}+J_{42}\vec{e_2^4}\vec{e_2^4}+J_{43}\vec{e_3^4}\vec{e_3^4})\bullet(\dot{u}_4\vec{e_1^2}+\dot{u}_5\vec{e_2^2}+\dot{u}_6\vec{e_3^2}+\dot{u}_7\vec{e_2^2}+\dot{u}_8\vec{e_1^3}+ \dot{u}_{11}\vec{e_3^3}+\dot{u}_{12}\vec{e_1^4}u_4\vec{\omega_2}\times\vec{e_1^2}+u_5\vec{\omega_2}\times\vec{e_2^2}+u_6\vec{\omega_2}\times\vec{e_3^2}+u_7\vec{\omega_2}\times\vec{e_2^2}+u_8\vec{\omega_3}\times$$

$$\overrightarrow{e_1^3} + u_{11}\overrightarrow{\omega_3} \times \overrightarrow{e_3^3} + u_{12}\overrightarrow{\omega_4} \times \overrightarrow{e_1^4})\bullet\overrightarrow{e_2^2}$$

代入凯恩方程

$$F_7' + F_7^* = 0$$

对于广义速率u_8，系统的广义主动力为

$$\begin{aligned}F_8' &= \sum_{i=1}^{4}[\overrightarrow{G_i}\bullet\overrightarrow{v_i}^8 + \overrightarrow{M_i}\bullet\overrightarrow{\omega_i}^8] + \overrightarrow{M^a}\bullet\overrightarrow{\Omega_{2,3}}^8 + \overrightarrow{M_{1,2}}\bullet\overrightarrow{\Omega_{1,2}}^8 + \overrightarrow{M_{3,4}}\bullet\overrightarrow{\Omega_{3,4}}^8 \\ &= mg\overrightarrow{e_3^0}\bullet[-\frac{s}{2}\sin\alpha\sin\beta\overrightarrow{e_1^2} + \frac{s}{2}\cos\alpha\overrightarrow{e_2^2} + \frac{s}{2}\sin\alpha\cos\beta\overrightarrow{e_3^2}] - mg\overrightarrow{e_3^0}\bullet \\ &\quad [-\frac{s}{2}\sin\alpha\sin\beta\overrightarrow{e_1^2} + \frac{s}{2}\cos\alpha\overrightarrow{e_2^2} + \frac{s}{2}\sin\alpha\sin\beta\overrightarrow{e_3^2} - \frac{s}{2}(-\overrightarrow{e_2^3} + \overrightarrow{e_1^3} \times \overrightarrow{e_3^4})] + \\ &\quad (M_\beta\overrightarrow{e_2^2} + M_\alpha\overrightarrow{e_1^3})\bullet\overrightarrow{e_1^3}\end{aligned}$$

（4-55h）

对于u_8的广义惯性力为

$$F_8^* = -\sum_{i=1}^{4} m_i\dot{\overrightarrow{v_i}}\bullet\overrightarrow{v_i}^8 - \sum_{i=1}^{4}\overline{J_i}\bullet\dot{\overrightarrow{\omega_i}}\bullet\overrightarrow{\omega_i}^8 \tag{4-56h}$$

式中

$$\dot{\overrightarrow{v_1}}\bullet\overrightarrow{v_1}^8 = 0$$

$$\dot{\overrightarrow{v_2}}\bullet\overrightarrow{v_2}^8 = 0$$

$$\begin{aligned}\dot{\overrightarrow{v_3}}\bullet\overrightarrow{v_3}^8 &= [\dot{u}_1\overrightarrow{e_1^0} + \dot{u}_2\overrightarrow{e_2^0} + \dot{u}_3\overrightarrow{e_3^0} + E_1\overrightarrow{e_2^2} + E_2\overrightarrow{\omega_2} \times \overrightarrow{e_2^2} + E_3\overrightarrow{e_1^2} + E_4\overrightarrow{\omega_2} \times \overrightarrow{e_1^2} + E_5\overrightarrow{e_1^2} + \\ &\quad E_6\overrightarrow{\omega_2} \times \overrightarrow{e_1^2}]\bullet(-\frac{s}{2}\sin\alpha\sin\beta\overrightarrow{e_1^2} + \frac{s}{2}\cos\alpha\overrightarrow{e_2^2} + \frac{s}{2}\sin\alpha\cos\beta\overrightarrow{e_3^2})\end{aligned}$$

$$\begin{aligned}\dot{\overrightarrow{v_4}}\bullet\overrightarrow{v_4}^8 &= \{\dot{u}_1\overrightarrow{e_1^0} + \dot{u}_2\overrightarrow{e_2^0} + \dot{u}_3\overrightarrow{e_3^0} + E_1\overrightarrow{e_2^2} + E_2\overrightarrow{\omega_2} \times \overrightarrow{e_2^2} + E_3\overrightarrow{e_1^2} + E_4\overrightarrow{\omega_2} \times \overrightarrow{e_1^2} + E_5\overrightarrow{e_1^2} + \\ &\quad E_6\overrightarrow{\omega_2} \times \overrightarrow{e_1^2} - \frac{s}{2}[\dot{u}_4\overrightarrow{e_1^2} \times \overrightarrow{e_3^3} + u_4\overrightarrow{\omega_2} \times \overrightarrow{e_1^2} \times \overrightarrow{e_3^3} + u_4\overrightarrow{e_1^2} \times (\overrightarrow{\omega_3} \times \overrightarrow{e_3^3}) + \dot{u}_5\overrightarrow{e_2^2} \times \overrightarrow{e_3^3} + \\ &\quad u_5\overrightarrow{\omega_2} \times \overrightarrow{e_2^2} \times \overrightarrow{e_3^3} + u_5\overrightarrow{e_2^2} \times (\overrightarrow{\omega_3} \times \overrightarrow{e_3^3}) + \dot{u}_6\overrightarrow{e_3^2} \times \overrightarrow{e_3^3} + u_6\overrightarrow{\omega_2} \times \overrightarrow{e_3^2} \times \overrightarrow{e_3^3} + u_6\overrightarrow{e_3^2} \times (\overrightarrow{\omega_3} \times\end{aligned}$$

$$\overrightarrow{e_3^3})+\dot{u}_7\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}+u_7\overrightarrow{e_2^2}\times(\overrightarrow{\omega_3}\times\overrightarrow{e_3^3})-\dot{u}_8\overrightarrow{e_2^3}-u_8\overrightarrow{\omega_3}\times\overrightarrow{e_2^3}]-$$

$$\frac{s}{2}(\dot{u}_4\overrightarrow{e_1^2}\times\overrightarrow{e_3^4}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}\times\overrightarrow{e_3^4}+u_4\overrightarrow{e_1^2}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_5\overrightarrow{e_2^2}\times\overrightarrow{e_3^4}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\times$$

$$\overrightarrow{e_3^4}+u_5\overrightarrow{e_2^2}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_6\overrightarrow{e_3^2}\times\overrightarrow{e_3^4}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\times\overrightarrow{e_3^4}+u_6\overrightarrow{e_3^2}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+$$

$$\dot{u}_8\overrightarrow{e_1^3}\times\overrightarrow{e_3^4}+u_8\overrightarrow{\omega_3}\times\overrightarrow{e_1^3}\times\overrightarrow{e_3^4}+u_8\overrightarrow{e_1^3}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_{11}\overrightarrow{e_3^3}\times\overrightarrow{e_3^4}+u_{11}\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}\times\overrightarrow{e_3^4}+$$

$$u_{11}\overrightarrow{e_3^3}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_{12}\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}+u_{12}\overrightarrow{\omega_4}\times\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}+u_{12}\overrightarrow{e_1^4}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})\}\bullet$$

$$[-\frac{s}{2}\sin\alpha\sin\beta\overrightarrow{e_1^2}+\frac{s}{2}\cos\alpha\overrightarrow{e_2^2}+\frac{s}{2}\sin\alpha\sin\beta\overrightarrow{e_3^2}-\frac{s}{2}(-\overrightarrow{e_2^3}+\overrightarrow{e_1^3}\times\overrightarrow{e_3^4})]$$

$$\overline{J}_1\bullet\dot{\overrightarrow{\omega_1}}\bullet\overrightarrow{\omega_1}^8=0$$

$$\overline{J}_2\bullet\dot{\overrightarrow{\omega_2}}\bullet\overrightarrow{\omega_2}^8=0$$

$$\overline{J}_3\bullet\dot{\overrightarrow{\omega_3}}\bullet\overrightarrow{\omega_3}^8=(J_{31}\overrightarrow{e_1^3}\overrightarrow{e_1^3}+J_{32}\overrightarrow{e_2^3}\overrightarrow{e_2^3}+J_{33}\overrightarrow{e_3^3}\overrightarrow{e_3^3})\bullet(\dot{u}_4\overrightarrow{e_1^2}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+$$
$$\dot{u}_6\overrightarrow{e_3^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+\dot{u}_7\overrightarrow{e_2^2}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+\dot{u}_8\overrightarrow{e_1^3}+u_8\overrightarrow{\omega_3}\times\overrightarrow{e_1^3})\bullet\overrightarrow{e_1^3}$$

$$\overline{J}_4\bullet\dot{\overrightarrow{\omega_4}}\bullet\overrightarrow{\omega_4}^8=(J_{41}\overrightarrow{e_1^4}\overrightarrow{e_1^4}+J_{42}\overrightarrow{e_2^4}\overrightarrow{e_2^4}+J_{43}\overrightarrow{e_3^4}\overrightarrow{e_3^4})\bullet(\dot{u}_4\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+\dot{u}_6\overrightarrow{e_3^2}+\dot{u}_7\overrightarrow{e_2^2}+\dot{u}_8\overrightarrow{e_1^3}+$$
$$\dot{u}_{11}\overrightarrow{e_3^3}+\dot{u}_{12}\overrightarrow{e_1^4}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+u_8\overrightarrow{\omega_3}\times$$
$$\overrightarrow{e_1^3}+u_{11}\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}+u_{12}\overrightarrow{\omega_4}\times\overrightarrow{e_1^4})\bullet\overrightarrow{e_1^3}$$

代入凯恩方程

$$F_8'+F_8^*=0$$

对于广义速率u_9，系统的广义主动力为

$$F_9'=\sum_{i=1}^{4}[\overrightarrow{G_i}\bullet\overrightarrow{v_i}^9+\overrightarrow{M_i}\bullet\overrightarrow{\omega_i}^9]+\overrightarrow{M^a}\bullet\overrightarrow{\Omega_{2,3}}^9+\overrightarrow{M_{1,2}}\bullet\overrightarrow{\Omega_{1,2}}^9+\overrightarrow{M_{3,4}}\bullet\overrightarrow{\Omega_{3,4}}^9$$

$$=-mg\overrightarrow{e_3^0}\bullet(\frac{s}{2}\overrightarrow{e_3^2}\times\overrightarrow{e_3^1})-k\theta_1\overrightarrow{e_1^1}\bullet\overrightarrow{e_3^2} \tag{4-55i}$$

对于u_9的广义惯性力为

$$F_9^*=-\sum_{i=1}^{4}m_i\dot{\overrightarrow{v_i}}\bullet\overrightarrow{v_i}^9-\sum_{i=1}^{4}\overline{J}_i\bullet\dot{\overrightarrow{\omega_i}}\bullet\overrightarrow{\omega_i}^9 \tag{4-56i}$$

式中

$$\dot{\vec{v}}_1 \bullet \vec{v}_1^{\,9} = \{\dot{u}_1\vec{e}_1^2 + u_1\vec{\omega}_2\times\vec{e}_2^2 + \dot{u}_2\vec{e}_2^2 + u_2\vec{\omega}_2\times\vec{e}_3^2 + \dot{u}_4(-\frac{s}{2}\vec{e}_2^2 + \frac{s}{2}\vec{e}_1^2\times\vec{e}_3^1) +$$

$$u_4\frac{s}{2}[-\vec{\omega}_2\times\vec{e}_2^2 + \vec{\omega}_2\times\vec{e}_1^2\times\vec{e}_3^1 + \vec{e}_1^2\times(\vec{\omega}_1\times\vec{e}_3^1)] + \frac{s}{2}\dot{u}_5(\vec{e}_1^2 + \vec{e}_2^2\times\vec{e}_3^1) +$$

$$\frac{s}{2}u_5[\vec{\omega}_2\times\vec{e}_1^2 + \vec{\omega}_2\times\vec{e}_2^2\times\vec{e}_3^1 + \vec{e}_2^2\times(\vec{\omega}_1\times\vec{e}_3^1)] + \dot{u}_6\frac{s}{2}\vec{e}_3^2\times\vec{e}_3^1 + u_6\frac{s}{2}[\vec{\omega}_2\times$$

$$\vec{e}_3^2\times\vec{e}_3^1 + \vec{e}_3^2\times(\vec{\omega}_1\times\vec{e}_3^1)] + \frac{s}{2}\dot{u}_9\vec{e}_3^2\times\vec{e}_3^1 + \frac{s}{2}u_9[\vec{\omega}_2\times\vec{e}_3^2\times\vec{e}_3^1 + \vec{e}_3^2\times(\vec{\omega}_1\times$$

$$\vec{e}_3^1)] + \frac{s}{2}\dot{u}_{10}\vec{e}_1^1\times\vec{e}_3^1 + \frac{s}{2}u_{10}[\vec{\omega}_1\times\vec{e}_1^1\times\vec{e}_3^1 + \vec{e}_1^1\times(\vec{\omega}_1\times\vec{e}_3^1)]\}\bullet(\frac{s}{2}\vec{e}_3^2\times\vec{e}_3^1)$$

$$\dot{\vec{v}}_2 \bullet \vec{v}_2^{\,9} = 0$$

$$\dot{\vec{v}}_3 \bullet \vec{v}_3^{\,9} = 0$$

$$\dot{\vec{v}}_4 \bullet \vec{v}_4^{\,9} = 0$$

$$\overline{J}_1\bullet\dot{\vec{\omega}}_1\bullet\vec{\omega}_1^{\,9} = (J_{11}\vec{e}_1^1\vec{e}_1^1 + J_{12}\vec{e}_2^1\vec{e}_2^1 + J_{13}\vec{e}_3^1\vec{e}_3^1)\bullet[\dot{u}_4\vec{e}_1^2 + u_4\vec{\omega}_2\times\vec{e}_1^2 + \dot{u}_5\vec{e}_2^2 + u_5\vec{\omega}_2\times$$

$$\vec{e}_2^2 + \dot{u}_6\vec{e}_3^2 + u_6\vec{\omega}_2\times\vec{e}_3^2 + \dot{u}_9\vec{e}_3^2 + u_9\vec{\omega}_2\times\vec{e}_3^2 + \dot{u}_{10}\vec{e}_1^1 + u_{10}\vec{\omega}_1\times\vec{e}_1^1]\bullet\vec{e}_3^2$$

$$\overline{J}_2\bullet\dot{\vec{\omega}}_2\bullet\vec{\omega}_2^{\,9} = 0$$

$$\overline{J}_3\bullet\dot{\vec{\omega}}_3\bullet\vec{\omega}_3^{\,9} = 0$$

$$\overline{J}_4\bullet\dot{\vec{\omega}}_4\bullet\vec{\omega}_4^{\,9} = 0$$

代入凯恩方程

$$F_9' + F_9^* = 0$$

对于广义速率u_{10}，系统的广义主动力为

$$F_{10}{}' = \sum_{i=1}^{4}[\vec{G}_i\bullet\vec{v}_i^{\,10} + \vec{M}_i\bullet\vec{\omega}_i^{\,10}] + \vec{M}^a\bullet\vec{\Omega}_{2,3}^{\,10} + \vec{M}_{1,2}\bullet\vec{\Omega}_{1,2}^{\,10} + \vec{M}_{3,4}\bullet\vec{\Omega}_{3,4}^{\,10}$$

$$= -mg\overrightarrow{e_3^0}\bullet(\frac{s}{2}\overrightarrow{e_1^1}\times\overrightarrow{e_3^1}) - k\theta_1 \tag{4-55j}$$

对于u_{10}的广义惯性力为

$$F_{10}^* = -\sum_{i=1}^{4} m_i \dot{\overrightarrow{v_i}}\bullet\overrightarrow{v_i}^{10} - \sum_{i=1}^{4}\overline{J_i}\bullet\dot{\overrightarrow{\omega_i}}\bullet\overrightarrow{\omega_i}^{10} \tag{4-56j}$$

式中

$$\dot{\overrightarrow{v_1}}\bullet\overrightarrow{v_1}^{10} = \{\dot{u}_1\overrightarrow{e_1^2} + u_1\overrightarrow{\omega_2}\times\overrightarrow{e_2^2} + \dot{u}_2\overrightarrow{e_2^2} + u_2\overrightarrow{\omega_2}\times\overrightarrow{e_3^2} + \dot{u}_4(-\frac{s}{2}\overrightarrow{e_2^2} + \frac{s}{2}\overrightarrow{e_1^2}\times\overrightarrow{e_3^1}) +$$

$$u_4\frac{s}{2}[-\overrightarrow{\omega_2}\times\overrightarrow{e_2^2} + \overrightarrow{\omega_2}\times\overrightarrow{e_1^2}\times\overrightarrow{e_3^1} + \overrightarrow{e_1^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})] + \frac{s}{2}\dot{u}_5(\overrightarrow{e_1^2} + \overrightarrow{e_2^2}\times\overrightarrow{e_3^1}) +$$

$$\frac{s}{2}u_5[\overrightarrow{\omega_2}\times\overrightarrow{e_1^2} + \overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\times\overrightarrow{e_3^1} + \overrightarrow{e_2^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})] + \dot{u}_6\frac{s}{2}\overrightarrow{e_3^2}\times\overrightarrow{e_3^1} +$$

$$u_6\frac{s}{2}[\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\times\overrightarrow{e_3^1} + \overrightarrow{e_3^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})] + \frac{\dot{s}}{2}\dot{u}_9\overrightarrow{e_3^2}\times\overrightarrow{e_3^1} + \frac{s}{2}u_9[\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\times\overrightarrow{e_3^1} +$$

$$\overrightarrow{e_3^2}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})] + \frac{s}{2}\dot{u}_{10}\overrightarrow{e_1^1}\times\overrightarrow{e_3^1} + \frac{s}{2}u_{10}[\overrightarrow{\omega_1}\times\overrightarrow{e_1^1}\times\overrightarrow{e_3^1} + \overrightarrow{e_1^1}\times(\overrightarrow{\omega_1}\times\overrightarrow{e_3^1})]\}\bullet$$

$$(\frac{s}{2}\overrightarrow{e_1^1}\times\overrightarrow{e_3^1})$$

$$\dot{\overrightarrow{v_2}}\bullet\overrightarrow{v_2}^{10} = 0, \dot{\overrightarrow{v_3}}\bullet\overrightarrow{v_3}^{10} = 0, \dot{\overrightarrow{v_4}}\bullet\overrightarrow{v_4}^{10} = 0$$

$$\overline{J_1}\bullet\dot{\overrightarrow{\omega_1}}\bullet\overrightarrow{\omega_1}^{10} = (J_{11}\overrightarrow{e_1^1}\overrightarrow{e_1^1} + J_{12}\overrightarrow{e_2^1}\overrightarrow{e_2^1} + J_{13}\overrightarrow{e_3^1}\overrightarrow{e_3^1})\bullet[\dot{u}_4\overrightarrow{e_1^2} + u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2} + \dot{u}_5\overrightarrow{e_2^2} + u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2} +$$
$$\dot{u}_6\overrightarrow{e_3^2} + u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2} + \dot{u}_9\overrightarrow{e_3^2} + u_9\overrightarrow{\omega_2}\times\overrightarrow{e_3^2} + \dot{u}_{10}\overrightarrow{e_1^1} + u_{10}\overrightarrow{\omega_1}\times\overrightarrow{e_1^1}]\bullet\overrightarrow{e_1^1}$$

$$\overline{J_2}\bullet\dot{\overrightarrow{\omega_2}}\bullet\overrightarrow{\omega_2}^{10} = 0$$

$$\overline{J_3}\bullet\dot{\overrightarrow{\omega_3}}\bullet\overrightarrow{\omega_3}^{10} = 0$$

$$\overline{J_4}\bullet\dot{\overrightarrow{\omega_4}}\bullet\overrightarrow{\omega_4}^{10} = 0$$

代入凯恩方程

$$F_{10}' + F_{10}^* = 0$$

对于广义速率u_{11}，系统的广义主动力为

$$F_{11}'=\sum_{i=1}^{4}[\overrightarrow{G_i}\bullet\overrightarrow{v_i}^{11}+\overrightarrow{M_i}\bullet\overrightarrow{\omega_i}^{11}]+\overrightarrow{M^a}\bullet\overrightarrow{\Omega_{2,3}}^{11}+\overrightarrow{M_{1,2}}\bullet\overrightarrow{\Omega_{1,2}}^{11}+\overrightarrow{M_{3,4}}\bullet\overrightarrow{\Omega_{3,4}}^{11}$$

$$=-mg\overrightarrow{e_3^0}\bullet(-\frac{s}{2}\overrightarrow{e_3^3}\times\overrightarrow{e_3^4})-k\theta_4\overrightarrow{e_1^4}\bullet\overrightarrow{e_3^3}$$

（4-55k）

对于u_{11}的广义惯性力为

$$F_{11}^*=-\sum_{i=1}^{4}m_i\dot{\overrightarrow{v_i}}\bullet\overrightarrow{v_i}^{11}-\sum_{i=1}^{4}\overline{J_i}\bullet\dot{\overrightarrow{\omega_i}}\bullet\overrightarrow{\omega_i}^{11} \quad (4\text{-}56k)$$

式中

$$\dot{\overrightarrow{v_1}}\bullet\overrightarrow{v_1}^{11}=0$$

$$\dot{\overrightarrow{v_2}}\bullet\overrightarrow{v_2}^{11}=0$$

$$\dot{\overrightarrow{v_3}}\bullet\overrightarrow{v_3}^{11}=0$$

$$\dot{\overrightarrow{v_4}}\bullet\overrightarrow{v_4}^{11}=\{\dot{u}_1\overrightarrow{e_1^0}+\dot{u}_2\overrightarrow{e_2^0}+\dot{u}_3\overrightarrow{e_3^0}+E_1\overrightarrow{e_2^2}+E_2\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+E_3\overrightarrow{e_1^2}+E_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+E_5\overrightarrow{e_1^2}+$$
$$E_6\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}-\frac{s}{2}[\dot{u}_4\overrightarrow{e_1^2}\times\overrightarrow{e_3^3}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}\times\overrightarrow{e_3^3}+u_4\overrightarrow{e_1^2}\times(\overrightarrow{\omega_3}\times\overrightarrow{e_3^3})+\dot{u}_5\overrightarrow{e_2^2}\times$$
$$\overrightarrow{e_3^3}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}+u_5\overrightarrow{e_2^2}\times(\overrightarrow{\omega_3}\times\overrightarrow{e_3^3})+\dot{u}_6\overrightarrow{e_3^2}\times\overrightarrow{e_3^3}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\times\overrightarrow{e_3^3}+$$
$$u_6\overrightarrow{e_3^2}\times(\overrightarrow{\omega_3}\times\overrightarrow{e_3^3})+\dot{u}_7\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\times\overrightarrow{e_3^3}+u_7\overrightarrow{e_2^2}\times(\overrightarrow{\omega_3}\times\overrightarrow{e_3^3})-$$
$$\dot{u}_8\overrightarrow{e_2^3}-u_8\overrightarrow{\omega_3}\times\overrightarrow{e_2^3}]-\frac{s}{2}(\dot{u}_4\overrightarrow{e_1^2}\times\overrightarrow{e_3^4}+u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}\times\overrightarrow{e_3^4}+u_4\overrightarrow{e_1^2}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+$$
$$\dot{u}_5\overrightarrow{e_2^2}\times\overrightarrow{e_3^4}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}\times\overrightarrow{e_3^4}+u_5\overrightarrow{e_2^2}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_6\overrightarrow{e_3^2}\times\overrightarrow{e_3^4}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}\times\overrightarrow{e_3^4}+$$
$$u_6\overrightarrow{e_3^2}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_8\overrightarrow{e_1^3}\times\overrightarrow{e_3^4}+u_8\overrightarrow{\omega_3}\times\overrightarrow{e_1^3}\times\overrightarrow{e_3^4}+u_8\overrightarrow{e_1^3}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_{11}\overrightarrow{e_3^3}\times$$
$$\overrightarrow{e_3^4}+u_{11}\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}\times\overrightarrow{e_3^4}+u_{11}\overrightarrow{e_3^3}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})+\dot{u}_{12}\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}+u_{12}\overrightarrow{\omega_4}\times\overrightarrow{e_1^4}\times\overrightarrow{e_3^4}+$$
$$u_{12}\overrightarrow{e_1^4}\times(\overrightarrow{\omega_4}\times\overrightarrow{e_3^4})\}\bullet(-\frac{s}{2}\overrightarrow{e_3^3}\times\overrightarrow{e_3^4})$$

$$\overline{J_1}\bullet\dot{\overrightarrow{\omega_1}}\bullet\overrightarrow{\omega_1}^{11}=0$$

$$\overline{J}_2 \bullet \dot{\overrightarrow{\omega_2}} \bullet \overrightarrow{\omega_2}^{11} = 0 , \overline{J}_3 \bullet \dot{\overrightarrow{\omega_3}} \bullet \overrightarrow{\omega_3}^{11} = 0$$

$$\overline{J}_4 \bullet \dot{\overrightarrow{\omega_4}} \bullet \overrightarrow{\omega_4}^{11} = (J_{41}\overrightarrow{e_1^4}\overrightarrow{e_1^4} + J_{42}\overrightarrow{e_2^4}\overrightarrow{e_2^4} + J_{43}\overrightarrow{e_3^4}\overrightarrow{e_3^4}) \bullet (\dot{u}_4\overrightarrow{e_1^2} + \dot{u}_5\overrightarrow{e_2^2} + \dot{u}_6\overrightarrow{e_3^2} + \dot{u}_7\overrightarrow{e_2^2} + \dot{u}_8\overrightarrow{e_1^3} + \dot{u}_{11}\overrightarrow{e_3^3} + \dot{u}_{12}\overrightarrow{e_1^4} u_4\overrightarrow{\omega_2} \times \overrightarrow{e_1^2} + u_5\overrightarrow{\omega_2} \times \overrightarrow{e_2^2} + u_6\overrightarrow{\omega_2} \times \overrightarrow{e_3^2} + u_7\overrightarrow{\omega_2} \times \overrightarrow{e_2^2} + u_8\overrightarrow{\omega_3} \times \overrightarrow{e_1^3} + u_{11}\overrightarrow{\omega_3} \times \overrightarrow{e_3^3} + u_{12}\overrightarrow{\omega_4} \times \overrightarrow{e_1^4}) \bullet \overrightarrow{e_3^3}$$

代入凯恩方程 $F_{11}' + F_{11}^* = 0$

对于广义速率u_{12}，系统的广义主动力为

$$F_{12}' = \sum_{i=1}^{4}[\overrightarrow{G_i} \bullet \overrightarrow{v_i}^{12} + \overrightarrow{M_i} \bullet \overrightarrow{\omega_i}^{12}] + \overrightarrow{M^a} \bullet \overrightarrow{\Omega_{2,3}}^{12} + \overrightarrow{M_{1,2}} \bullet \overrightarrow{\Omega_{1,2}}^{12} + \overrightarrow{M_{3,4}} \bullet \overrightarrow{\Omega_{3,4}}^{12}$$

$$= -mg\overrightarrow{e_3^0} \bullet (-\frac{s}{2}\overrightarrow{e_1^4} \times \overrightarrow{e_3^4}) - k\theta_4$$

（4-551）

对于u_{12}的广义惯性力为

$$F_{12}^* = -\sum_{i=1}^{4} m_i \dot{\overrightarrow{v_i}} \bullet \overrightarrow{v_i}^{12} - \sum_{i=1}^{4} \overline{J}_i \bullet \dot{\overrightarrow{\omega_i}} \bullet \overrightarrow{\omega_i}^{12} \qquad (4\text{-}561)$$

式中

$$\dot{\overrightarrow{v_1}} \bullet \overrightarrow{v_1}^{12} = 0 , \dot{\overrightarrow{v_2}} \bullet \overrightarrow{v_2}^{12} = 0 , \dot{\overrightarrow{v_3}} \bullet \overrightarrow{v_3}^{12} = 0$$

$$\dot{\overrightarrow{v_4}} \bullet \overrightarrow{v_4}^{12} = \{\dot{u}_1\overrightarrow{e_1^0} + \dot{u}_2\overrightarrow{e_2^0} + \dot{u}_3\overrightarrow{e_3^0} + E_1\overrightarrow{e_2^2} + E_2\overrightarrow{\omega_2} \times \overrightarrow{e_2^2} + E_3\overrightarrow{e_1^2} + E_4\overrightarrow{\omega_2} \times \overrightarrow{e_1^2} + E_5\overrightarrow{e_1^2} + E_6\overrightarrow{\omega_2} \times \overrightarrow{e_1^2} - \frac{s}{2}[\dot{u}_4\overrightarrow{e_1^2} \times \overrightarrow{e_3^3} + u_4\overrightarrow{\omega_2} \times \overrightarrow{e_1^2} \times \overrightarrow{e_3^3} + u_4\overrightarrow{e_1^2} \times (\overrightarrow{\omega_3} \times \overrightarrow{e_3^3}) + \dot{u}_5\overrightarrow{e_2^2} \times \overrightarrow{e_3^3} + u_5\overrightarrow{\omega_2} \times \overrightarrow{e_2^2} \times \overrightarrow{e_3^3} + u_5\overrightarrow{e_2^2} \times (\overrightarrow{\omega_3} \times \overrightarrow{e_3^3}) + \dot{u}_6\overrightarrow{e_3^2} \times \overrightarrow{e_3^3} + u_6\overrightarrow{\omega_2} \times \overrightarrow{e_3^2} \times \overrightarrow{e_3^3} + u_6\overrightarrow{e_3^2} \times (\overrightarrow{\omega_3} \times \overrightarrow{e_3^3}) + \dot{u}_7\overrightarrow{e_2^2} \times \overrightarrow{e_3^3} + u_7\overrightarrow{\omega_2} \times \overrightarrow{e_2^2} \times \overrightarrow{e_3^3} + u_7\overrightarrow{e_2^2} \times (\overrightarrow{\omega_3} \times \overrightarrow{e_3^3}) - \dot{u}_8\overrightarrow{e_2^3} - u_8\overrightarrow{\omega_3} \times \overrightarrow{e_2^3}] - \frac{s}{2}(\dot{u}_4\overrightarrow{e_1^2} \times \overrightarrow{e_3^4} + u_4\overrightarrow{\omega_2} \times \overrightarrow{e_1^2} \times \overrightarrow{e_3^4} + u_4\overrightarrow{e_1^2} \times (\overrightarrow{\omega_4} \times \overrightarrow{e_3^4}) + \dot{u}_5\overrightarrow{e_2^2} \times \overrightarrow{e_3^4} + u_5\overrightarrow{\omega_2} \times \overrightarrow{e_2^2} \times \overrightarrow{e_3^4} + u_5\overrightarrow{e_2^2} \times (\overrightarrow{\omega_4} \times \overrightarrow{e_3^4}) + \dot{u}_6\overrightarrow{e_3^2} \times \overrightarrow{e_3^4} + u_6\overrightarrow{\omega_2} \times \overrightarrow{e_3^2} \times \overrightarrow{e_3^4} + u_6\overrightarrow{e_3^2} \times (\overrightarrow{\omega_4} \times \overrightarrow{e_3^4}) + \dot{u}_8\overrightarrow{e_1^3} \times \overrightarrow{e_3^4} + u_8\overrightarrow{\omega_3} \times \overrightarrow{e_1^3} \times \overrightarrow{e_3^4} + u_8\overrightarrow{e_1^3} \times (\overrightarrow{\omega_4} \times \overrightarrow{e_3^4}) + \dot{u}_{11}\overrightarrow{e_3^3} \times \overrightarrow{e_3^4} + u_{11}\overrightarrow{\omega_3} \times \overrightarrow{e_3^3} \times \overrightarrow{e_3^4} + u_{11}\overrightarrow{e_3^3} \times (\overrightarrow{\omega_4} \times \overrightarrow{e_3^4}) + \dot{u}_{12}\overrightarrow{e_1^4} \times \overrightarrow{e_3^4} + u_{12}\overrightarrow{\omega_4} \times \overrightarrow{e_1^4} \times \overrightarrow{e_3^4} + u_{12}\overrightarrow{e_1^4} \times (\overrightarrow{\omega_4} \times \overrightarrow{e_3^4})\} \bullet (-\frac{s}{2}\overrightarrow{e_1^4} \times \overrightarrow{e_3^4})$$

$$\overline{J_1}\bullet\dot{\overrightarrow{\omega_1}}\bullet\overrightarrow{\omega_1}^{12}=0$$

$$\overline{J_2}\bullet\dot{\overrightarrow{\omega_2}}\bullet\overrightarrow{\omega_2}^{12}=0$$

$$\overline{J_3}\bullet\dot{\overrightarrow{\omega_3}}\bullet\overrightarrow{\omega_3}^{12}=0$$

$$\begin{aligned}\overline{J_4}\bullet\dot{\overrightarrow{\omega_4}}\bullet\overrightarrow{\omega_4}^{12}=&(J_{41}\overrightarrow{e_1^4}\overrightarrow{e_1^4}+J_{42}\overrightarrow{e_2^4}\overrightarrow{e_2^4}+J_{43}\overrightarrow{e_3^4}\overrightarrow{e_3^4})\bullet(\dot{u}_4\overrightarrow{e_1^2}+\dot{u}_5\overrightarrow{e_2^2}+\dot{u}_6\overrightarrow{e_3^2}+\dot{u}_7\overrightarrow{e_2^2}+\dot{u}_8\overrightarrow{e_1^3}+\\&\dot{u}_{11}\overrightarrow{e_3^3}+\dot{u}_{12}\overrightarrow{e_1^4}u_4\overrightarrow{\omega_2}\times\overrightarrow{e_1^2}+u_5\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+u_6\overrightarrow{\omega_2}\times\overrightarrow{e_3^2}+u_7\overrightarrow{\omega_2}\times\overrightarrow{e_2^2}+u_8\overrightarrow{\omega_3}\times\\&\overrightarrow{e_1^3}+u_{11}\overrightarrow{\omega_3}\times\overrightarrow{e_3^3}+u_{12}\overrightarrow{\omega_4}\times\overrightarrow{e_1^4})\bullet\overrightarrow{e_1^4}\end{aligned}$$

代入凯恩方程

$$F_{12}'+F_{12}^*=0$$

4.2 本章小结

本章对落猫问题进行进一步的研究，考虑了猫体的柔性，把猫体的前后躯又分别地看成两个独立的圆柱体。假设猫体不能扭转，其纵向也不能发生弹性变形，所以前后躯的两个圆柱体之间的接头可以简化为万向铰，再用弯曲弹簧连接在万向铰两侧的柱体上，假设$\theta_1=0$时不产生弹性力矩当万向铰两侧的柱体发生相对运动时就会产生弹性力矩。接下来用凯恩方程对模型进行分析与上一章第三节步骤相同，但由于自由度的增加解题时明显烦琐了。

第 5 章　非线性微分方程的半解析解

5.1 非线性微分方程的半解析解

下面对非线性微分方程式（3-91）进行解析分析。

已知：
$$\frac{d\psi}{d\theta}=\frac{(J/I)S}{(T-1)[1-T+(J/I)(1+T)](1+T)^{\frac{1}{2}}} \tag{5-1}$$

式中，$S=-\sqrt{2}(\cos\alpha\sin\beta+\sin\alpha\cos\beta\cos\theta)\sin\beta$，

$T=\cos\alpha\cos\beta-\sin\alpha\sin\beta\cos\theta$，

α，β 为定值，求解函数 $\psi(\theta)$。

解：式（5-1）的分母中

$$\begin{aligned}T-1&=\cos\alpha\cos\beta-\sin\alpha\sin\beta\cos\theta-1\\&=\cos\alpha\cos\beta-1-\sin\alpha\sin\beta\cos\theta\\&=\frac{1}{\sin\alpha\sin\beta}(\frac{\cos\alpha\cos\beta-1}{\sin\alpha\sin\beta}-\cos\theta)\end{aligned} \tag{5-2}$$

$$\begin{aligned}1-T+(J/I)(1+T)&=(1+J/I)+(J/I-1)T\\&=(1+J/I)+(J/I-1)(\cos\alpha\cos\beta-\sin\alpha\sin\beta\cos\theta)\\&=(1+J/I)+(J/I-1)\cos\alpha\cos\beta-(J/I-1)\sin\alpha\sin\beta\cos\theta\\&=-\frac{1}{(J/I-1)\sin\alpha\sin\beta}[-\frac{(1+J/I)+(J/I-1)\cos\alpha\cos\beta}{(J/I-1)\sin\alpha\sin\beta}+\\&\quad\cos\theta]\end{aligned} \tag{5-3}$$

$$\begin{aligned}1+T&=1+\cos\alpha\cos\beta-\sin\alpha\sin\beta\cos\theta\\&=\frac{1}{\sin\alpha\sin\beta}(\frac{1+\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\cos\theta)\end{aligned} \tag{5-4}$$

式（5-1）分子中

$$S=-\sqrt{2}(\cos\alpha\sin\beta+\sin\alpha\cos\beta\cos\theta)\sin\beta$$

$$=\frac{-\sqrt{2}\sin\beta}{\sin\alpha\cos\beta}(\cot\alpha\tan\beta+\cos\theta) \tag{5-5}$$

把式（5-2）、（5-3）、（5-4）、（5-5）代入式（5-1）得

$$\frac{d\psi}{d\theta}=\frac{\cos\theta+H}{(\cos\theta+P)(\cos\theta+Q)(\cos\theta+R)^{\frac{1}{2}}}K \tag{5-6}$$

式中：

$$H=\cot\alpha\tan\beta$$

$$P=\frac{1-\cos\alpha\cos\beta}{\sin\alpha\sin\beta}$$

$$Q=\frac{-(1+J/I)+(J/I-1)\cos\alpha\cos\beta}{(J/I-1)\sin\alpha\sin\beta}$$

$$R=-\frac{1+\cos\alpha\cos\beta}{\sin\alpha\sin\beta}$$

$$K=\frac{\sqrt{2}\sin^2\alpha\sin^4\beta(J/I-1)}{\cos\beta}$$

（5-7）

把式（5-8）恒等变形为

$$\frac{d\psi}{d\theta}=(\cos\theta+R)^{\frac{1}{2}}[\frac{A}{\cos\theta+P}+\frac{B}{\cos\theta+Q}+\frac{C}{\cos\theta+R}] \tag{5-8}$$

由于（5-6）和（5-8）是恒等变形故得：

$$\begin{aligned}&A(\cos\theta+Q)(\cos\theta+R)+B(\cos\theta+P)(\cos\theta+R)+\\&C(\cos\theta+P)(\cos\theta+Q)=\cos\theta+H\end{aligned}$$

（5-9）

对应系数相等得

$$\begin{cases}A+B+C=0\\A(R+Q)+B(P+R)+C(P+Q)=1\\ARQ+BPR+CPQ=H\end{cases} \tag{5-10}$$

假令：

$$\Delta=\begin{vmatrix}1&1&1\\R+Q&P+R&P+Q\\RQ&PR&PQ\end{vmatrix}\neq0 \tag{5-11}$$

$$\Delta_1=\begin{vmatrix}0 & 1 & 1\\ 1 & P+R & P+Q\\ H & PR & PQ\end{vmatrix} \tag{5-12}$$

$$\Delta_2=\begin{vmatrix}1 & 0 & 1\\ R+Q & 1 & P+Q\\ RQ & H & PQ\end{vmatrix} \tag{5-13}$$

$$\Delta_3=\begin{vmatrix}1 & 1 & 0\\ R+Q & P+R & 1\\ RQ & PR & H\end{vmatrix} \tag{5-14}$$

由 Grame 法则得

$$A=\frac{\Delta_1}{\Delta};B=\frac{\Delta_2}{\Delta};C=\frac{\Delta_3}{\Delta} \tag{5-15}$$

式（5-8）可以写为

$$\frac{d\psi}{d\theta}=A\frac{(\cos\theta+R)^{\frac{1}{2}}}{\cos\theta+P}+B\frac{(\cos\theta+R)^{\frac{1}{2}}}{\cos\theta+Q}+C\frac{1}{(\cos\theta+R)^{\frac{1}{2}}} \tag{5-16}$$

两边积分得

$$\psi=A\int\frac{(\cos\theta+R)^{\frac{1}{2}}}{\cos\theta+P}d\theta+B\int\frac{(\cos\theta+R)^{\frac{1}{2}}}{\cos\theta+Q}d\theta+C\int\frac{1}{(\cos\theta+R)^{\frac{1}{2}}}d\theta \tag{5-17}$$

令

$$\psi_3=C\int\frac{1}{(\cos\theta+R)^{\frac{1}{2}}}d\theta=\frac{C}{\sqrt{R}}\int\frac{1}{(1+\frac{\cos\theta}{R})^{\frac{1}{2}}}d\theta \tag{5-18}$$

假设

$$\left|\frac{\cos\theta}{R}\right|<1$$

对式（5-18）中的积分函数运用二项式定理，有

$$\frac{1}{\sqrt{1+\frac{\cos\theta}{R}}}=1-\frac{1}{2}\left(\frac{\cos\theta}{R}\right)+\frac{1\times3}{2\times4}\frac{\cos^2\theta}{R^2}-\frac{1\times3\times5}{2\times4\times6}\frac{\cos^3\theta}{R^3}+\frac{1\times3\times5\times7}{2\times4\times6\times8}\frac{\cos^4\theta}{R^4}-\cdots\cdots \tag{5-19}$$

对上式积分，其积分分项见下式

$$\begin{aligned}
&\int d\theta=\theta\\
&\int\cos\theta d\theta=\sin\theta\\
&\int\cos^2\theta d\theta=\frac{1}{2}\theta+\frac{1}{4}\sin2\theta\\
&\int\cos^3\theta d\theta=\sin\theta-\frac{\sin^3\theta}{3}\\
&\int\cos^4\theta d\theta=\frac{1}{4}\theta+\frac{1}{4}\sin\theta+\frac{1}{8}\theta+\frac{1}{32}\sin4\theta\\
&\quad\vdots\\
&\quad\vdots
\end{aligned} \tag{5-20}$$

余下不定积分略做。将式（5-20）代入式（5-18）得

$$\psi_3=\frac{C}{\sqrt{R}}\{\theta-\frac{1}{2}\frac{\sin\theta}{R}+\frac{1\times3}{2\times4}\frac{1}{R^2}(\frac{1}{2}\theta+\frac{1}{4}\sin2\theta)-\frac{1\times3\times5}{2\times4\times6}\frac{1}{R^3}(\sin\theta-\frac{1}{3}\sin^3\theta)+\frac{1\times3\times5\times7}{2\times4\times6\times8}\frac{1}{R^4}(\frac{3}{8}\theta+\frac{1}{4}\sin2\theta+\frac{1}{32}\sin4\theta)-\cdots\cdots\} \tag{5-21}$$

令
$$\psi_2=B\int\frac{(\cos\theta+R)^{\frac{1}{2}}}{\cos\theta+Q}d\theta \tag{5-22}$$

有如下式

$$\frac{d\psi_2}{d\theta}=B\frac{(\cos\theta+R)^{\frac{1}{2}}}{\cos\theta+Q} \tag{5-23}$$

变形为
$$\frac{d\theta}{d\psi_2}=B\frac{\cos\theta+R+(Q-R)}{(\cos\theta+R)^{\frac{1}{2}}}$$

$$= B(\sqrt{\cos\theta + R} + \frac{Q-R}{\sqrt{\cos\theta + R}}) \tag{5-24}$$

假设

$$\left|\frac{\cos\theta}{R}\right| < 1, \quad \left|\frac{\cos\theta}{Q}\right| < 1 \tag{5-25}$$

运用二项式定理，有

$$\frac{(\cos\theta + R)^{\frac{1}{2}}}{\cos\theta + Q} = \frac{\sqrt{R}(1+\frac{\cos\theta}{R})^{\frac{1}{2}}}{Q(1+\frac{\cos\theta}{Q})}$$

$$= \frac{\sqrt{R}}{Q}\{1 + \frac{1}{2}\frac{\cos\theta}{R} - \frac{1}{2\cdot4}\frac{\cos^2\theta}{R^2} + \frac{1\cdot3}{2\cdot4\cdot6}\frac{\cos^3\theta}{R^3} - \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\frac{\cos^4\theta}{R^4} + \cdots\}\times$$

$$\{1 - \frac{\cos\theta}{Q} + \frac{\cos^2\theta}{Q^2} - \frac{\cos^3\theta}{Q^3} + \cdots\}$$

$$= \frac{\sqrt{R}}{Q}\{1 + (\frac{1}{2R} - \frac{1}{Q})\cos\theta + (\frac{1}{2\cdot4}\frac{1}{R^2} + \frac{1}{Q^2} - \frac{1}{2RQ})\cos^2\theta + \cdots\} \tag{5-26}$$

对 $\cos\theta$、$\cos^2\theta$、……作不定积分如前（5-19）并相加得

$$\psi_2 = \frac{\sqrt{R}}{Q}\{1 + (\frac{1}{2R} - \frac{1}{Q})\cos\theta + (\frac{1}{2\cdot4}\frac{1}{R^2} + \frac{1}{Q^2} - \frac{1}{2RQ})\cos^2\theta + \cdots\} \tag{5-27}$$

令

$$\psi_1 = A\int\frac{(\cos\theta + R)^{\frac{1}{2}}}{\cos\theta + P}d\theta \tag{5-28}$$

假设

$$\left|\frac{\cos\theta}{R}\right| < 1, \quad \left|\frac{\cos\theta}{P}\right| < 1 \tag{5-29}$$

对式（5-28）中的积分函数运用二项式定理展开得

$$\frac{(\cos\theta + R)^{\frac{1}{2}}}{\cos\theta + P} = \frac{\sqrt{R}}{P}\{1 + (\frac{1}{2R} - \frac{1}{P})\cos\theta + (\frac{1}{2\cdot4}\frac{1}{R^2} + \frac{1}{P^2} - \frac{1}{2RP})\cos^2\theta + \cdots\}$$

$$\psi_1 = A\frac{\sqrt{R}}{P}\{1+(\frac{1}{2R}-\frac{1}{P})\sin\theta+(\frac{1}{2\times4}\frac{1}{R^2}+\frac{1}{P^2}-\frac{1}{2RP})(\frac{1}{2}\theta+\frac{1}{4}\sin2\theta)+\cdots\}$$

对$\cos\theta$、$\cos^2\theta$、……积分如前，将其相加得ψ_1。把ψ_1，ψ_2，ψ_3相加可得

积分

$$\begin{aligned}\psi(\theta)&=\psi_1+\psi_2+\psi_3\\&=A\frac{\sqrt{R}}{P}\{1+(\frac{1}{2R}-\frac{1}{P})\sin\theta+(\frac{1}{2\times4}\frac{1}{R^2}+\frac{1}{P^2}-\frac{1}{2RP})(\frac{1}{2}\theta+\frac{1}{4}\sin2\theta)+\cdots\}+\\&\frac{\sqrt{R}}{Q}\{1+(\frac{1}{2R}-\frac{1}{Q})\cos\theta+(\frac{1}{2\cdot4}\frac{1}{R^2}+\frac{1}{Q^2}-\frac{1}{2RQ})\cos^2\theta+\cdots\}+\\&\frac{C}{\sqrt{R}}\{\theta-\frac{1}{2}\frac{\sin\theta}{R}+\frac{1\times3}{2\times4}\frac{1}{R^2}(\frac{1}{2}\theta+\frac{1}{4}\sin2\theta)-\frac{1\times3\times5}{2\times4\times6}\frac{1}{R^3}(\sin\theta-\frac{1}{3}\sin^3\theta)+\\&\frac{1\times3\times5\times7}{2\times4\times6\times8}\frac{1}{R^4}(\frac{3}{8}\theta+\frac{1}{4}\sin2\theta+\frac{1}{32}\sin4\theta)-\cdots\cdots\}\end{aligned}$$

(5-29)

5.2 本章小结

本章对上文中出现的一个非线性微分方程进行解析分析，把式（5-1）简化成式（5-6），在（5-11）成立的情况下把式（5-6）的右侧化为部分分式的形式，因而（5-6）化成（5-10），然后两边取积分，方程化为三个积分相加的形式（5-17），对每一个积分里的积分函数进行适当地变形，使它们能进行二项式展开，当然这也需在一定的前提下进行，然后对每一个二项式里的分式进行积分，最后把所有的积分相加即得微分方程（5-1）的级数解（5-29）。

结 论

本文把落猫问题化为多体系统模型，在第三章中把猫体简化为多刚体模型，运用第二类拉格朗日方程、牛顿—欧拉法、凯恩方程对其进行动力学分析。通过本文更加看出以上三种方法的特点：

（1）拉格朗日方程提供了建立任意完整系统运动微分方程普遍的、规格化的方法，最重要的是以广义坐标表示系统的运动，因而方程的数目最少，与自由度数目相等，且约束反力不出现在微分方程中，这样就给求解多自由度系统的动力学问题带来了方便，但是拉格朗日方法本身的缺点主要是引入了动力学函数并需求其导数，推导过程比较费力，过于繁冗，而这种困难随着系统的复杂程度而增加。

（2）牛顿欧拉法是牛顿力学描述运动的总称，应用这种方法处理质点系统的运动时候，解除约束、取分离体；然后写出它们的运动方程，是规格方法，推导过程也简单。缺点是微分方程数量多，而系统的约束力包含方程中，必须与约束方程联立求解。

（3）凯恩方法将矢量形式的力与惯性力沿某特殊方向投影，有着清晰的几何直观性。这种方法不但有牛顿力学的优点，还有分析力学的优点：约束反力不出现在运动微分方程中而且得到的也是与自由度数目相等的方程数，是最少的方程数。由于采用了广义速率描述运动，则取独立变量上有较好的选择性，以使方程简洁。凯恩方程是一阶微分方程组，容易化成标准形式，便于编制程序，方法的主要缺点是若对具体系统选择恰当的广义速率以使计算过程简单则需要足够的经验与技巧，而且加速度及惯性力的计算工作也未必轻松。

本文较多的采用了矩阵运算，这可以使行文简洁，最主要的是矩阵便于上机计算。在本文的 3.2 节，我们也比较了矩阵方法与矢量的方法，同时也发现凯恩文章中的控制方程（3-91）为约束方程中的一类——角动量守恒（另

一类方程为质心运动方程）。

在第四章考虑了猫体的柔性，运用有限段建模法把它简化为有限个刚体通过铰与弹簧相连的模型，其实质是把柔体模型化为刚体模型，当然自由度必然会增加，其复杂程度会随着增加。

由于大多数文献中给出的是方程的数值解，本文最后给出凯恩文章中的控制方程（3-91）的半解析解。

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