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Charging a supercapacitor through a lamp: A power-law *RC* decay

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A circuit involving a charging supercapacitor in series with a non-Ohmic tungsten lamp displays a wealth of interesting behavior. Most notably, the current through the lamp decreases in time according to a power-law function as opposed to the exponential time dependence observed in *RC* circuits with Ohmic resistors. We use a combination of computational and analytical techniques to model this power-law behavior as well as the behavior of the filament's temperature and resistance as the supercapacitor charges. Our results agree well with experiment, and the experiment described here can be modified to be appropriate for physics courses at a wide range of levels.
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I. INTRODUCTION

The availability of low-cost, high-capacitance supercapacitors has made it possible to construct *RC* circuits with time constants on the order of tens of seconds, even with relatively modest resistance values. In one popular lecture demonstration, the resistor is replaced by a small incandescent lamp, allowing students to visualize how the current changes as the capacitor charges and discharges. This circuit differs from an idealized *RC* circuit in that the supercapacitor has a non-negligible internal resistance, and the lamp filament's resistance varies significantly with temperature.

Even a qualitative description of this circuit should address the effect of the supercapacitor's internal resistance, which, being on the order of a few ohms, is comparable to the resistance of the lamp.[1] In addition, the effective capacitance of a supercapacitor can depend on whether the capacitor is being charged or discharged and can vary significantly from its nominal capacitance.[2] For a given supercapacitor, the internal resistance R_C and the capacitance C can be determined experimentally by measuring the time constants for a series of *RC* circuits with various Ohmic external resistors (see Appendix A).

The non-Ohmic nature of the lamp poses an additional challenge for quantitative descriptions of this circuit. When the capacitor is being charged, for example, the charge buildup on the plates of the capacitor combined with decreasing resistance of the lamp as it cools results in the current decaying more slowly than the familiar exponential relationship. Previous research has produced either numerical results for the current-time relationship[3] or a transcendental equation that must be solved numerically for the current at a given time.[4] In this study, we assume that both the lamp resistance and the power emitted by the filament can be modeled as having a polynomial dependence on the filament temperature. This allows us to obtain a simple, closed-form analytic expression for the current's dependence on time.

In Sec. II, we discuss our experimental results that show the lamp resistance is a linearly decreasing function of time for a significant portion of the capacitor's charging period. This is a new result, to the best of our knowledge. We also show that, as a consequence, the measured current through the lamp is well described by a power-law relationship with time of the form $I(t) = I_0(1-\beta t)^{\gamma}$, for constants I_0, β, and γ. In Sec. III, we demonstrate that the linear time dependence of the resistance is a robust phenomenon, occurring for a wide range of circuit parameters. The linear behavior emerges from a numerical solution of coupled differential equations for the circuit, obtained by applying Kirchhoff's loop rule and conservation of energy for the lamp filament. In Sec. IV, we explore analytically how the parameters β and γ in the power law depend on circuit parameters. Using an analytic result for the lamp's resistance as a function of time, we discuss how the duration of the linear regime depends on the supercapacitor's capacitance, its internal resistance, and any external resistance in the circuit. We conclude with a discussion of pedagogical implications and potential areas of further research.

II. EXPERIMENT

A. Procedure

Our circuit, shown in Fig. 1, included a supercapacitor (NEC/TOKIN FYH0H105ZF) with a measured capacitance of $C = 1.158 \pm 0.016$ F and an internal resistance of $R_C = 2.54 \pm 0.14$ Ω (see Appendix A for details). The capacitor was connected in series with a No. 47 lamp of variable resistance R, a decade resistor (General Radio 1433-B) $R_{\rm ext}$, and a DC power supply (Agilent E3633A) set to an emf of $\varepsilon = 5.0 \pm 0.1$ V.

Two multimeters (HP E2373A) were used as voltmeters. One was connected across the lamp, and the other was connected across the decade resistor to determine the voltage drop across it, which was used to measure the current in the loop. This method was chosen because we found that these multimeters included some significant but unknown resistance when used as milliammeters. The resistance $R_{\rm ext}$ of the decade resistor was found to be 12.01 ± 0.01 Ω using an *LCR* meter (BK Precision 878).

We recorded a video of the displays of both multimeters, a stopwatch, and the power supply. We found it easiest to begin the stopwatch and video recording first and then toggle the output of the power supply on. This allows for a more accurate determination of $t = 0$ for the charging process, which we also define to be the time when the lamp is at its maximum temperature and, thus, has a maximum resistance R_0. In reality, the lamp must first warm from room temperature after the power supply is toggled on, but we find this process to be nearly instantaneous. The more significant source of systematic error is the determination of $t = 0$ from

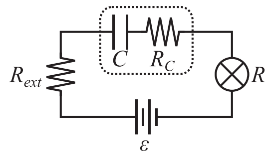

Fig. 1. A diagram of the circuit under study.

the video. We also note that these multimeters have a relatively low refresh rate of about 2 Hz, which limits our precision.

The video was used to obtain the time t since charging began, the voltage drops V_{lamp} across the lamp, and the voltage drops $V_{R_{\text{ext}}}$ across the external resistor. The current I through the loop was determined from the values of $V_{R_{\text{ext}}}$ and the measurement of R_{ext}, and then the values of I and V_{lamp} were used to determine the resistance R of the lamp at each time.

B. Results

Figure 2 demonstrates that the lamp's resistance decays linearly with time for nearly a full minute while the capacitor charges, obeying the equation

$$R(t) = R_0 - at. \tag{1}$$

A linear fit, also shown in Fig. 2, produces values of $a = 0.296 \pm 0.003\ \Omega/\text{s}$ and $R_0 = 31.34 \pm 0.10\ \Omega$. In Fig. 2, as well as all figures containing data, error bars were estimated from the uncertainty in each constituent measurement. Because the error bars were estimated to be slightly smaller than the symbol size used, they are not included.

For convenience, the values of measured constants used throughout this paper are displayed in Table I. These values allow us to determine the appropriate functional form with which to fit the current vs time data. We begin by rewriting Kirchhoff's loop rule for the circuit in Fig. 1 in terms of previously defined quantities as well as the charge Q on the capacitor and the current I through the loop

$$\begin{aligned} \varepsilon &= V_C + V_{R_C} + V_{R_{\text{ext}}} + V_{\text{lamp}} \\ &= \frac{Q}{C} + I(R_C + R_{\text{ext}}) + IR. \end{aligned} \tag{2}$$

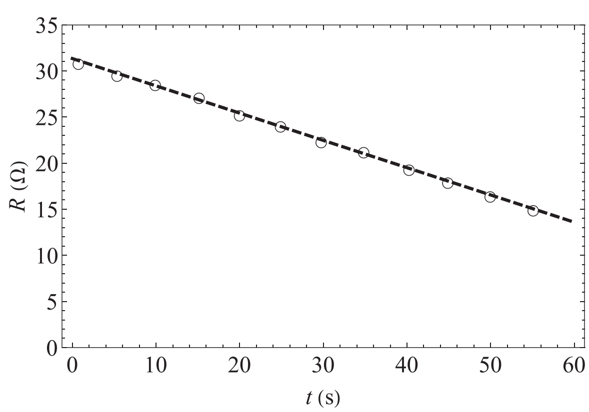

Fig. 2. Lamp resistance as a function of time for $C = 1.158$ F, $R_C = 2.54\ \Omega$, and $R_{\text{ext}} = 12.01\ \Omega$. The dashed line shows a linear fit to the data, $R = R_0 - at$, with $a = 0.296 \pm 0.003\ \Omega/\text{s}$ and $R_0 = 31.34 \pm 0.10\ \Omega$.

Table I. Values of measured constants used throughout this paper.

Symbol	Definition	Value
C	Capacitance	1.158 ± 0.016 F
ε	Power supply setting	5.0 ± 0.1 V
a	Slope of R vs t	$0.296 \pm 0.003\ \Omega/\text{s}$
R_C	Internal resistance of supercapacitor	$2.54 \pm 0.14\ \Omega$
R_{ext}	External resistance	$12.01 \pm 0.01\ \Omega$
R_0	Maximum filament resistance	$31.4 \pm 0.10\ \Omega$

Taking the time derivative of Eq. (2) and assuming that ε, C, R_C, and R_{ext} are constant produces a differential equation involving $I(t)$ and $R(t)$,

$$0 = \frac{I}{C} + \frac{dI}{dt}(R_C + R_{\text{ext}}) + \frac{dI}{dt}R + I\frac{dR}{dt}. \tag{3}$$

If the lamp resistance is modeled according to Eq. (1), then Eq. (3) becomes

$$0 = \left(\frac{1}{C} - a\right)I + (R_C + R_{\text{ext}} + R_0 - at)\frac{dI}{dt}. \tag{4}$$

This differential equation can be solved by separation of variables, which leads to

$$I(t) = I_0(1 - \beta t)^{\gamma}, \tag{5}$$

where $\beta = a/R_{\text{net}}$, with $R_{\text{net}} = R_C + R_{\text{ext}} + R_0$, and the exponent is given by $\gamma = (aC)^{-1} - 1$. The initial current is $I_0 = \varepsilon/R_{\text{net}}$. It is interesting to note that taking the limit $a \to 0$ in Eq. (5) results in constant filament resistance and, thus, produces the usual exponentially decaying current $I(t) = I_0 e^{-t/\tau}$, where the time constant $\tau = R_{\text{net}}C$.

Manipulating Eq. (5) and using the values of a and R_0 from Table I to determine values of I_0 and β allows us to produce a linear plot with the current and time data, shown in Fig. 3. A linear fit produces a slope of $\gamma = 1.944 \pm 0.010$, which is in agreement with the value of $\gamma = 1.92 \pm 0.03$ calculated using the relationship $\gamma = (aC)^{-1} - 1$.

To emphasize visually the non-exponential behavior of the current, Fig. 4 shows a semi-log plot of the current vs time as well as the familiar exponential decay and the more appropriate power-law expression given in Eq. (5). Constants from Table I were used to produce both the

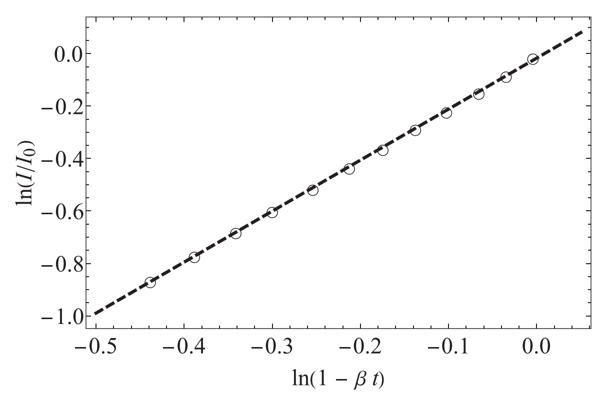

Fig. 3. The dashed line shows a linear fit to the manipulated current and time data with slope $\gamma = 1.944 \pm 0.010$ and intercept -0.017 ± 0.002.

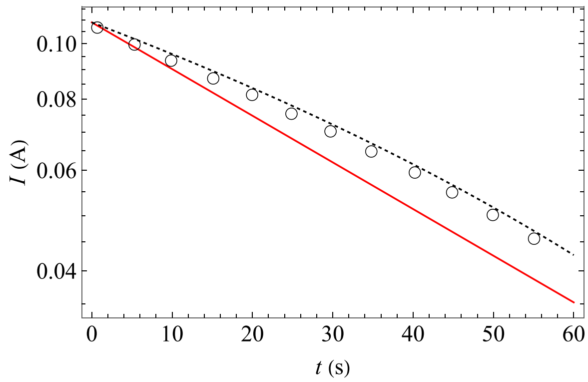

Fig. 4. A semi-log plot of the current through a non-Ohmic lamp in series with a resistor and a charging supercapacitor reveals that the data are not well described by the familiar exponential decay equation $I(t) = I_0 e^{-t/\tau}$ (solid red line) but are well described by a power-law relationship $I(t) = I_0(1-\beta t)^{\gamma}$ (dashed black line).

exponential and power-law graphs. It is clear that the power-law relationship better describes the data using the same circuit parameters. While the exponential relationship could be forced to better agree with the data using a different value of the time constant τ, we believe the most appropriate choice is $\tau = R_{\rm net}C$. In an RC circuit involving a lamp and a typical capacitor, the lamp would not have time to cool significantly and, therefore, would exhibit its original resistance R_0 throughout the decay. The significantly longer discharge time of a supercapacitor causes the power-law behavior to be apparent.

The small systematic shift between the power-law relationship and the experimental data evident in Fig. 4 is likely a result of inaccurately determining $t = 0$ for the charging process. We took $t = 0$ to be the time at which the power supply signaled that the output had been turned on, but this may occur *after* current is supplied to the circuit. This would cause our time measurements to be systematically small, accounting for some of the disagreement in Fig. 4. In addition, systematically small time measurements would cause the value of R_0 produced in Fig. 2 to be systematically small. This would, in turn, result in a slightly high value of I_0 for the models.

III. EXPLORING THE LINEAR LAMP RESISTANCE

A natural question to ask is whether the linear time dependence of the lamp resistance (which is essential to the power-law decay of the current) is particular to certain circuit parameters, or whether this behavior occurs more generally. By numerically solving a system of coupled differential equations, we show that this behavior is robust for a range of C and $R_{\rm ext}$ values. We also gain intuition about the duration of the linear regime for $R(t)$.

We return to Eq. (3) with the goal of solving for R without the assumption that it is linear in time. As in previous studies,[3,4] we assume that the thermal relaxation time of the filament is insignificant compared to the time over which the current (and therefore filament temperature T) is changing significantly, allowing us to replace dR/dt with $(dR/dT)(dT/dt)$. This assumption is reasonable because of the small mass of the filament and slow decay of current through the lamp, which is due in part to the large capacitance in the circuit. We also assume that the resistance is at most cubic in temperature,[5] $R = \sum_{n=0}^{3} k_n T^n$ and produce the coefficients k_n in the supplementary material.[10] With these substitutions, Eq. (3) becomes a differential equation involving $I(t)$ and $T(t)$,

$$0 = \frac{I}{C} + \frac{dI}{dt}(R_C + R_{\rm ext}) + \frac{dI}{dt}\left(\sum_{n=0}^{3} k_n T^n\right) + I\left(\sum_{n=1}^{3} n k_n T^{n-1}\right)\frac{dT}{dt}. \tag{6}$$

A full solution requires a second equation involving $I(t)$ and $T(t)$, which we find from conservation of energy for the system of the tungsten filament. The electrical power entering the filament, I^2R, serves to increase the temperature, while any power leaving the filament $P_{\rm out}$ serves to decrease the temperature, such that the rate of temperature change dT/dt obeys

$$mc\frac{dT}{dt} = I^2R - P_{\rm out}, \tag{7}$$

where m is the mass of the filament and c is the specific heat capacity of tungsten. The mass of the filament is most precisely measured by measuring the dimensions of the filament with a microscope. In our case, a scanning electron microscope image of a No. 47 lamp filament revealed a total volume of 1.43×10^{-11} m^3, for a mass of 2.76×10^{-7} kg, assuming a mass density of 19.3 g/cm^3 for tungsten.[6] The specific heat capacity of tungsten was taken to be 132 J/kg K.[6]

As discussed in the supplementary material,[10] we may treat the relationship between $P_{\rm out}$ and T as quintic, $P_{\rm out} = \sum_{n=0}^{5} j_n T^n$, and the determination of the coefficients j_n involves repeating the familiar laboratory exercise verifying the Stefan– Boltzmann law.[7–9] While our data are sufficiently well fit by the simpler Stefan–Boltzmann law, in which $P_{\rm out} \propto T^4$, other terms are included to account for the effects of conduction, convection, and filament oxidation that may be significant for other types of bulbs. Inserting the polynomial relationships for both R and $P_{\rm out}$ into Eq. (7) gives

$$mc\frac{dT}{dt} = I^2\left(\sum_{n=0}^{3} k_n T^n\right) - \sum_{n=0}^{5} j_n T^n. \tag{8}$$

Equations (6) and (8) constitute a system of coupled, first-order, non-linear differential equations which can be solved numerically for $I(t)$ and $T(t)$. The initial temperature is taken to be room temperature, 293 K, and designated as T_{293}, and the initial current $I_{293} = \varepsilon/(R_C + R_{\rm ext} + R_{293})$ for room temperature resistance R_{293}.

Once the solution for $T(t)$ is known, it can be substituted into the expression for $R(T)$ to find the resistance of the lamp as a function of time. As seen in Fig. 5, the combined effects of the capacitor charging and the resistance of the lamp decreasing as it cools leads to a linear $R(t)$ curve (after the initial rapid warming from room temperature) for a significant portion of the capacitor's charging period. Larger capacitance values allow the decay to occur more slowly and, therefore, produce a longer linear regime for $R(t)$. Even a 0.5 F supercapacitor produces a linear regime of roughly 30 s, which can be extended by increasing $R_{\rm ext}$ if necessary. The slope and duration of this linear regime will be further explored in Sec. IV.

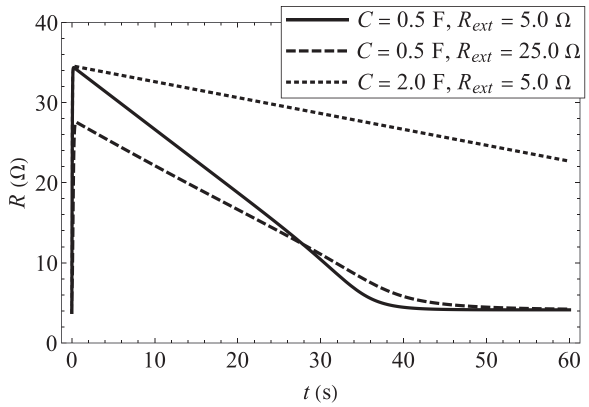

Fig. 5. Numerical solutions indicate that the lamp resistance varies linearly with time for a range of C and R_{ext} values. Here, $R_C = 2.0\,\Omega$, $\varepsilon = 5.0\,\text{V}$, and $R_{293} = 4.0\,\Omega$.

For the circuit parameters used in this experiment (see Table I), the numerical solution for $R(t)$ is in excellent agreement with the experimental data, as shown in Fig. 6. Because the numerical solution begins with the lamp at room temperature, we also measured $R_{293} = 3.971 \pm 0.001\,\Omega$ using an *LCR* meter. The rapid warming to reach the maximum resistance of R_0 is demonstrated by the numerical solution but is not feasible to observe in our data because of limited time sampling.

IV. DEPENDENCE OF γ AND β ON CIRCUIT PARAMETERS

In this section, we present analytic expressions for both the exponent γ and the coefficient β appearing in Eq. (5). For the reader who is interested in the details of the calculation, please refer to Appendix B. We find it possible to express γ in terms of R_C, R_{ext}, R_0, and p_0. The quantity p_0 appears in a linear relationship between filament resistance R and temperature T, $R = p_0 + p_1 T$. This relationship is discussed in Appendix B, and in the supplementary material,[10] it is found that $p_0 = -4.12 \pm 0.12\,\Omega$ and $p_1 = (2.215 \pm 0.013) \times 10^{-2}\,\Omega/\text{K}$. Our result for γ is

$$\gamma = \frac{1}{2}\left(1 + \frac{R_C + R_{\text{ext}}}{R_0}\right)\left(\frac{3R_0 + p_0}{R_0 - p_0}\right). \tag{9}$$

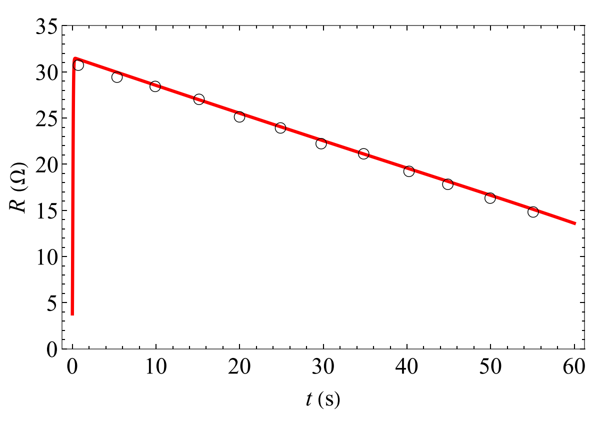

Fig. 6. Numerical solutions (solid red line) agree well with experimental data during the linear R vs t regime.

The above equation is independent of C, which tells us that the exponent in the power-law decay of the current is, in fact, universal with respect to capacitance, which is perhaps surprising given the expression $\gamma = (aC)^{-1} - 1$ discussed in Sec. II. As described in Appendix B, the maximum filament resistance, R_0, depends on ε, R_C, and R_{ext}, and is found by solving the following equation implicitly:

$$\frac{\varepsilon^2 R_0}{(R_C + R_{\text{ext}} + R_0)^2} = g\left(\frac{R_0 - p_0}{p_1}\right)^4, \tag{10}$$

where $g = 5.55 \times 10^{-14}\,\text{W/K}^4$ is the coefficient of the Stefan–Boltzmann Law for the filament power output, $P_{\text{out}} = gT^4$ (see the supplementary material[10] for a discussion of P_{out}).

The analytic result of Eq. (9) is plotted in Fig. 7 along with a numerical solution for γ beginning from Eqs. (6) and (8), but with the simplification that $R = p_0 + p_1 T$ and $P_{\text{out}} = gT^4$ instead of the full polynomial forms. Once the solution for $R(t)$ is known, a is determined from its time derivative, and γ is found via $\gamma = (aC)^{-1} - 1$. The results are plotted in Fig. 7 for $C = 5.0\,\text{F}$. The agreement between the numerical and analytical results is respectable. The difference in the signs of the (slight) concavities in the two plots is presumably a result of the approximations made to obtain Eq. (9) and discussed in Appendix B.

Figure 7 shows γ is an increasing function of $R_C + R_{\text{ext}}$, which is not surprising given Eq. (9) and the term $(R_C + R_{\text{ext}})/R_0$. Note, additionally, that the maximum filament resistance R_0 is found from Eq. (10) to be a *decreasing* function of $R_C + R_{\text{ext}}$ and independent of C, consistent with Fig. 5. Perhaps more unexpected is the dependence of γ on the ratio $(3R_0 + p_0)/(R_0 - p_0)$. This implies that, for sufficiently large $R_C + R_{\text{ext}}$, and thus small value of R_0, γ should peak and then approach zero (for $R_0 = -p_0/3$, where $p_0 = -4.12\,\Omega$ for our tungsten filament). This behavior of γ is *not* observed in the numerical solution to Eqs. (6) and (8). However, the difference between the behaviors of Eq. (9) and the numerical solution is not surprising in this situation, because Eq. (9) requires that $at/R_0 \ll 1$. As R_0 becomes sufficiently small and/or t become sufficiently large, this condition is no longer satisfied, and Eq. (9) is no longer valid.

Using the relationship from Sec. II B, $\gamma = (aC)^{-1} - 1$, we can obtain an analytic expression for the time rate of decrease a in the resistance. Substituting Eq. (9) for γ gives

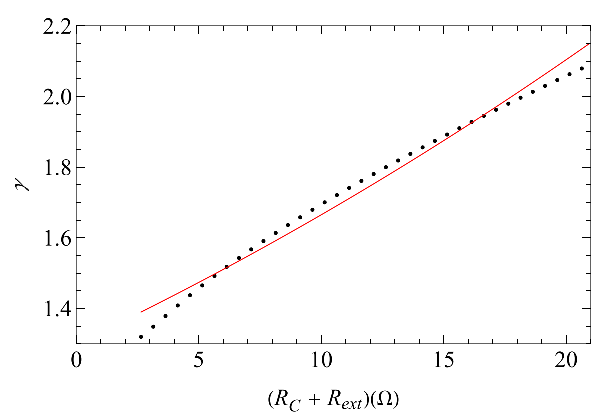

Fig. 7. The dependence of γ on the combination $R_C + R_{\text{ext}}$, showing both the numerical solution for γ using the simplified Eqs. (6) and (8) (black dots) as well as the analytical expression given in Eq. 9 (solid red line). Both plots were made with $C = 5.0\,\text{F}$.

$$a = \frac{1}{C(\gamma+1)} = \frac{1}{C\left[1+\frac{1}{2}\left(1+\frac{R_C+R_{\rm ext}}{R_0}\right)\left(\frac{3R_0+p_0}{R_0-p_0}\right)\right]}, \tag{11}$$

which shows a simple inverse relationship between a and C. This result, in turn, provides insight into the duration of the filament resistance's linear (in time) regime. As demonstrated in Fig. 5, the numerical solution for the lamp resistance is, to a good approximation, linear in time until decreasing back to its room-temperature value, R_{293}. Since a is the slope of the resistance vs time plot, the smaller the value of a, the longer the duration of the linear regime until the value of R_{293} is reached for a given R_0. We can see from the analytic result that the slope is inversely proportional to C, so a larger capacitance results in a longer linear regime, which is in agreement with Fig. 5. Equation (11) is also consistent with the observation in Fig. 5 that increasing $R_{\rm ext}$ for a fixed C results in a longer linear regime.

For pedagogical reasons, it is interesting to contrast the slower power-law current decay with the more common exponential decay. To make such a comparison, consider some characteristic decay time, such as the half-life. In the power-law current decay expression, $I(t) = I_0(1-\beta t)^\gamma$, the coefficient β has dimensions of inverse time, where $\beta = a/R_{\rm net}$. Using Eq. (11), we obtain

$$\beta = \frac{1}{R_{\rm net}C\left[1+\frac{1}{2}\left(1+\frac{R_C+R_{\rm ext}}{R_0}\right)\left(\frac{3R_0+p_0}{R_0-p_0}\right)\right]}. \tag{12}$$

Let us denote β^{-1} as a characteristic time $\tau_{\rm eff}$ for the current decay,

$$\tau_{\rm eff} = \beta^{-1} = R_{\rm net}C\left[1+\frac{1}{2}\left(1+\frac{R_C+R_{\rm ext}}{R_0}\right)\left(\frac{3R_0+p_0}{R_0-p_0}\right)\right]. \tag{13}$$

Using this expression along with Eq. (5), the time for the current to decay to one half of I_0 is

$$t_{1/2} = \left[1-\left(\frac{1}{2}\right)^{1/\gamma}\right]\tau_{\rm eff}, \tag{14}$$

which is linear in C through $\tau_{\rm eff}$ while being a complicated function of R_C and $R_{\rm ext}$. For comparison, the half-life for *exponentially* decaying current in an RC circuit with only Ohmic resistors is

$$t_{1/2,\,\exp} = \ln(2)\tau, \tag{15}$$

where $\tau = R_{\rm net}C$. For $R_{\rm net} = 45.89\,\Omega$ and $C = 1.158$ F, Eq. (15) gives $t_{1/2,\,\exp} = 36.8$ s, while Eq. (14) gives $t_{1/2} = 45.9$ s. Both values can be checked visually by referring to the plots in Fig. 4. An instructor can, in turn, use this to reinforce the idea that the charge buildup on the capacitor and the decreasing resistance of the lamp combine to produce truly non-exponential behavior.

V. CONCLUSION

In this paper, we show with a combination of experiment, computation, and analytical work that the current in an RC circuit consisting of a charging supercapacitor, and an incandescent lamp is described by a power-law function of the form $I(t) = I_0(1-\beta t)^\gamma$. This is a direct result of the lamp resistance decreasing linearly with time, a phenomenon we observe for a range of circuit parameters. We are able to both measure and compute analytically the exponent γ, and we find that γ depends in a complicated manner on the supercapacitor's internal resistance R_C as well as filament parameters and any external Ohmic resistance $R_{\rm ext}$ in the circuit, but that γ is independent of the capacitance C. We are also able to determine how the prefactor β depends on R_C, $R_{\rm ext}$, R_0 (the maximum resistance of lamp filament), and C.

The experiment described in Sec. II (and the supporting experiments described in Appendix A and the supplementary material[10]) is suitable for inclusion in an advanced lab course. The experiment offers a new take on a more traditional RC circuit experiment, in which exponential time dependence is observed. For additional study, students could test Eq. (11) by studying the duration of the linear-in-time regime of the lamp resistance as a function of the supercapacitor properties C and R_C. They could also extend the experiment described in Appendix A to explore in more detail how the measured values of C and R_C depend on whether the supercapacitor is being charged or discharged and whether these values are constant during the entirety of the charging/discharging process. While the experiment was designed to be low-cost and accessible, it can be improved by developing more accurate or automated measurement techniques. By providing students more scaffolding (and the results from Appendix A and the supplementary material[10]), the experiment could also be used as an introductory or intermediate lab exercise, especially if the instructor wishes to provide an example of power-law behavior in contrast to exponential behavior. Although the analysis depends on the product mc (mass of lamp filament multiplied by its specific heat), our numerical solution to the coupled system (Eqs. (6) and (8)) shows that the resulting current behavior is not a sensitive function of mc. Thus, measuring m, for example, need not be a concern. If necessary, using our provided value for the mass of the filament of a No. 47 lamp should suffice for an introductory lab. This variation of a standard experiment provides a wealth of interesting physics for students to pursue. We encourage instructors to take advantage of it.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

APPENDIX A: DETERMINING SUPERCAPACITOR QUANTITIES

There is a good deal of subtlety in defining and measuring the properties of supercapacitors due to the way in which they store charge.[2] For our purposes, it is sufficient to follow

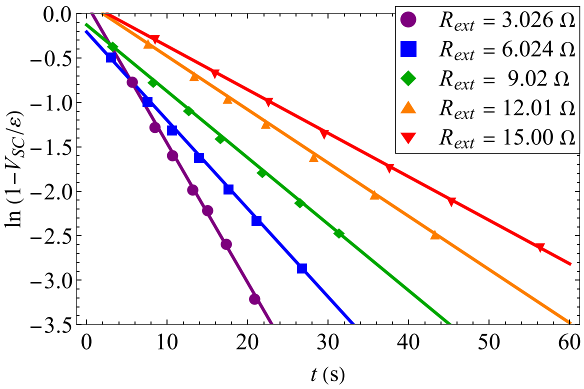

Fig. 8. Measuring the voltage drop across the supercapacitor V_{SC} as it charges in series with a known external resistance R_{ext} allows us to determine the time constant for each circuit.

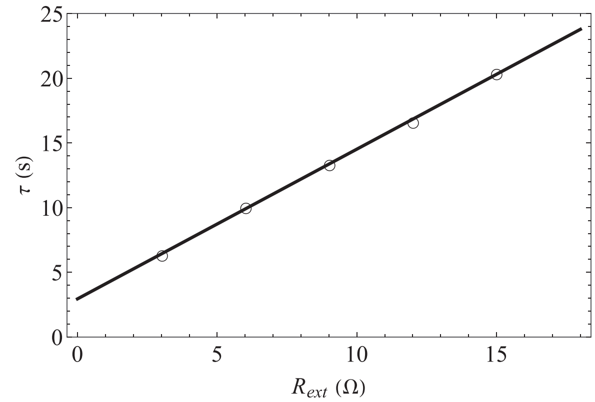

Fig. 9. The solid black line shows a linear fit to the data with a slope of $C = 1.158 \pm 0.016$ F and intercept $CR_C = 2.94 \pm 0.16$ s.

a previous publication[1] in modeling the supercapacitor as an ideal capacitor of capacitance C in series with an Ohmic resistor of resistance R_C. By charging the supercapacitor in series with a power supply of emf ε and external resistor of resistance R_{ext}, the voltage drop across the supercapacitor V_{SC} can be analyzed to determine the time constant τ for the process. Repeating this experiment with several values of R_{ext} allows for the determination of both C and R_C.

The total voltage drop across the supercapacitor is the sum of the capacitive and resistive terms and so becomes

$$V_{SC} = V_C + V_{R_C} = \varepsilon(1 - e^{-t/\tau}) + \frac{R_C}{R_C + R_{\text{ext}}}\varepsilon e^{-t/\tau} \quad \text{(A1)}$$

when the supercapacitor is charging. Note that if the resistors are Ohmic, exponential behavior is observed since the circuit is effectively an ideal RC circuit with total resistance $R_C + R_{\text{ext}}$. Rearranging Eq. (A1) and taking the natural log of both sides produces

$$\ln\left(1 - \frac{V_{SC}}{\varepsilon}\right) = -\left(\frac{1}{\tau}\right)t + \ln\left(\frac{R_{\text{ext}}}{R_C + R_{\text{ext}}}\right), \quad \text{(A2)}$$

such that the time constant can be determined from the slope of a plot of $\ln(1 - V_{SC}/\varepsilon)$ vs time, as shown in Fig. 8. Note that it is unnecessary to determine the exact time at which the charging process begins, because the physical quantity of interest is determined only from the slope.

The calculated time constants for each of the R_{ext} values shown Fig. 8 are summarized in Table II. The time constant depends on R_{ext} according to

$$\tau = (R_{\text{ext}} + R_C)C, \quad \text{(A3)}$$

such that C can be determined from the slope of a plot of τ vs R_{ext} and R_C can be determined from its intercept, as shown in Fig. 9. For the nominal 1.0 F supercapacitor used in our experiment (NEC/TOKIN FYH0H105ZF), we found a capacitance $C = 1.158 \pm 0.016$ F and an effective resistance $R_C = 2.54 \pm 0.14$ Ω.

Table II. Summary of results from Fig. 8.

R_{ext} (Ω)	Slope (s^{-1})	τ (s)
3.026	−0.157 ± 0.003	6.37 ± 0.12
6.024	−0.0994 ± 0.0012	10.06 ± 0.12
9.02	−0.0747 ± 0.0007	13.39 ± 0.13
12.01	−0.0600 ± 0.0005	16.67 ± 0.14
15.00	−0.0490 ± 0.0004	20.41 ± 0.17

APPENDIX B: DETERMINING AN EXPRESSION FOR γ

In this appendix, we derive the analytic expression for the power-law exponent γ given in Eq. (9). We begin with the established linear relationship between filament resistance and time

$$R = R_0 - at, \quad \text{(B1)}$$

as discussed in Sec. II B. We combine this with a linear relation between resistance and *temperature* for metals

$$R = p_0 + p_1 T, \quad \text{(B2)}$$

where p_0 and p_1 are known constants for our tungsten filament. We note Eq. (B2) is a simplification from an expression that is cubic in temperature used in Sec. III (see the supplementary material[10]). In the temperature regime of interest to us, however, a linear temperature relationship fits the data well. Combining Eqs. (B1) and (B2) shows that the lamp temperature must also be linear in time

$$T(t) = T_0 - \eta t, \quad \text{(B3)}$$

where $\eta = a/p_1$, and the initial (maximum) temperature is $T_0 = (R_0 - p_0)/p_1$. The accuracy of Eq. (B3) is demonstrated by the plot in Fig. 10. This graph was obtained from the numerical solution of Eqs. (6) and (8).

To proceed, we make use of Eq. (B3) in Eq. (7) by substituting $dT/dt = -\eta$ on the left side

$$-mc\eta = I^2 R - P_{\text{out}}. \quad \text{(B4)}$$

We assume that radiative power loss dominates the output power, and so we write $P_{\text{out}} = gT^4$. (This quartic expression for output power is a simplification that we justify in the supplementary material.) Next, we solve Eq. (B4) for the square of the current as a function of time

$$I^2(t) = \frac{g(T_0 - \eta t)^4 - mc\eta}{R_0 - at}, \quad \text{(B5)}$$

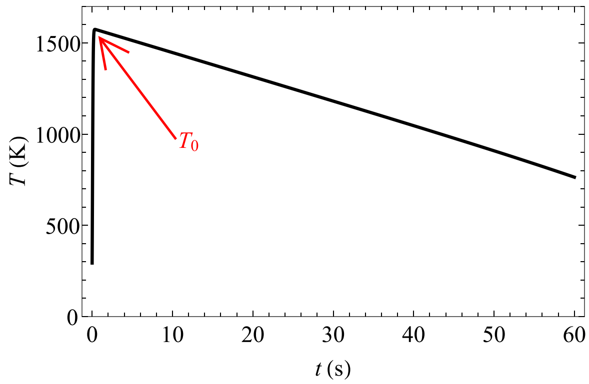

Fig. 10. Filament temperature vs time. Numerical solution of Eqs. (6) and (8) demonstrates a sizable time interval over which the filament's temperature is a linearly decreasing function of time. We emphasize that this linear time dependence is obtained with no simplifications to the polynomials appearing in both equations.

where we used $R = R_0 - at$ and $T = T_0 - \eta t$. Squaring both sides of Eq. (5) gives us a second expression for $I^2(t)$,

$$I^2(t) = I_0^2(1-\beta t)^{2\gamma}. \tag{B6}$$

Equating the right sides of Eqs. (B5) and (B6) gives, after some factoring, the following:

$$I_0^2(1-\beta t)^{2\gamma} = \frac{gT_0^4}{R_0}\left[\left(1-\frac{\eta t}{T_0}\right)^4 - \frac{mc\eta}{gT_0^4}\right]\left(1-\frac{at}{R_0}\right)^{-1}. \tag{B7}$$

Knowledge of the values of the constants in Eq. (B7) allows us to employ useful approximations. The interested reader can refer to the values provided in Table III, which are obtained from a combination of measurements and numerical computations described throughout this paper. The conclusions we draw from the values in Table III are twofold: that we may drop the dimensionless term $mc\eta/gT_0^4$ because it is small compared to one; that all three binomial expressions in Eq. (B7) are of the form $(1+\epsilon)^n$, where over the 30–60 s time interval in which Eq. (B1) holds it is true that ϵ is also small compared to one. Thus, dropping the term $mc\eta/gT_0^4$ and performing binomial expansions for all binomials in Eq. (B7) gives

$$I_0^2\left(1-2\beta\gamma t+\mathcal{O}(\epsilon^2)\right) = \frac{gT_0^4}{R_0}\left(1-\frac{4\eta t}{T_0}+\mathcal{O}(\epsilon^2)\right) \times\left(1+\frac{at}{R_0}+\mathcal{O}(\epsilon^2)\right). \tag{B8}$$

As time t or circuit parameter values cause the ratios $2\beta\gamma t$, $4\eta t/T_0$, or at/R_0 to become comparable to unity in size, the expressions derived from Eq. (B8) by ignoring all terms of order $\mathcal{O}(\epsilon^2)$, specifically Eqs. (B10) and (B14), become less accurate.

When we equate terms of order t^0 on the two sides of Eq. (B8), we obtain the equality $I_0^2 = gT_0^4/R_0$. Using $T_0 = (R_0 - p_0)/p_1$, we can express this as

$$I_0^2 R_0 = g\left(\frac{R_0 - p_0}{p_1}\right)^4. \tag{B9}$$

Next, because $I_0 = \varepsilon/(R_C + R_{\text{ext}} + R_0)$, this becomes

$$\frac{\varepsilon^2 R_0}{(R_C + R_{\text{ext}} + R_0)^2} = g\left(\frac{R_0 - p_0}{p_1}\right)^4, \tag{B10}$$

which implicitly determines the value of R_0 once the values of ε and $R_C + R_{\text{ext}}$ are specified. We solve Eq. (B10) for R_0 numerically. For example, if $\varepsilon = 5.0$ V, $R_C = 2.54\,\Omega$, and $R_{\text{ext}} = 12.01\,\Omega$, then we find that $R_0 = 31.4\,\Omega$, as quoted in Table I, which agrees with the value obtained experimentally as shown in Fig. 2.

Next, we equate terms of order t^1 on both sides of Eq. (B8) to obtain

$$2\beta\gamma I_0^2 = \frac{gT_0^4}{R_0}\left(\frac{4\eta}{T_0} - \frac{a}{R_0}\right). \tag{B11}$$

Make the substitutions $\beta = a/R_{\text{net}}$ and $\eta = a/p_1$ and cancel factors of a from both sides

$$\frac{2\gamma I_0^2}{R_{\text{net}}} = \frac{gT_0^4}{R_0}\left(\frac{4}{p_1 T_0} - \frac{1}{R_0}\right). \tag{B12}$$

Table III. Values of constants used throughout this paper.

Symbol	Definition	How obtained	Value
a	Slope of R vs t	Fit to data (Fig. 2)	$0.296\,\Omega$/s
R_C	Internal resistance of supercapacitor	Fit to data (Fig. 9)	$2.94\,\Omega$
R_{ext}	External resistance	Measured with *LCR* meter	$12.01\,\Omega$
p_0	Coefficient in Eq. (B2)	Fit to data (Fig. S1)	$-4.12\,\Omega$
p_1	Coefficient in Eq. (B2)	Fit to data (Fig. S1)	$0.0221\,\Omega$/K
R_{293}	Room-temperature filament resistance	Measured with *LCR* meter	$3.971\,\Omega$
R_0	Maximum filament resistance	Numerical solution of Eq. (B10)	$31.4\,\Omega$
T_0	Maximum filament temperature	From R_0, p_0, p_1 and Eq. (B2)	1607 K
β	Coefficient of t in Eq. (5)	$\beta = a/(R_C + R_{\text{ext}} + R_0)$	$6.44 \times 10^{-3}\,\text{s}^{-1}$
η	Coefficient of t in Eq. (B3)	$\eta = a/p_1$	13.4 K/s
m	Mass of tungsten filament	Measured with SEM (Sec. II)	2.76×10^{-7} kg
c	Specific heat of tungsten	Known value (Ref. 6)	132 J/kg K
g	Coefficient of filament power in $P_{\text{out}} = gT^4$	Averaging data (Fig. S3)	$5.55 \times 10^{-14}\,\text{W/K}^4$

Next, substitute $I_0^2 = gT_0^4/R_0$ on the left side of Eq. (B12) and solve for γ,

$$\gamma = \frac{R_{\rm net}}{2}\left(\frac{4}{p_1 T_0} - \frac{1}{R_0}\right). \tag{B13}$$

Recalling that $R_{\rm net} = R_C + R_{\rm ext} + R_0$ along with $R_0 = p_0 + p_1 T_0$ allows this to be expressed as

$$\gamma = \frac{1}{2}\left(1 + \frac{R_C + R_{\rm ext}}{R_0}\right)\left(\frac{3R_0 + p_0}{R_0 - p_0}\right), \tag{B14}$$

which is provided as Eq. (9) in Sec. IV.

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[10]See the supplementary material at https://www.scitation.org/doi/suppl/10.1119/5.0065500 for a determination of the coefficients in the resistance-temperature and power-temperature relationships.

Culpepper Microscope

The Culpepper design dates from 1725-30, and is a relatively simple and inexpensive type of microscope. The viewing stage has been pushed up; it belongs between the objective lens at the bottom of the tube and the diagonal mirror that is used to direct light through the specimens. It was made by the London optical firm of W.S. Jones, in business from 1776 to 1859. This microscope is one of the few items in the Amherst College collection that dates from before the 1882 fire that destroyed the collection put together by Prof. E.S. Snell. This was originally owned by Snell, and is the Amherst College Collection. (Amherst College picture and text by Thomas B. Greenslade, Jr., Kenyon College)