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Graphical analysis of an oscillator with constant magnitude sliding friction

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We treat a horizontal oscillator damped by constant-magnitude sliding friction by extending the analogy between the simple harmonic motion of a mass on a spring and the uniform circular motion of a mass attached to the end of a string. In the presence of sliding friction, the motion of the mass on a spring becomes the horizontal projection of the path of a mass attached to a string winding around two nails separated by a well-defined distance; this path is a spiral consisting of connected semi-circles of diminishing radii. This graphical analysis is very simple and pedagogically useful. It can also be generalized to any oscillation affected by other forces of constant magnitude but not necessarily constant direction.
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I. INTRODUCTION

The standard presentation of harmonic motion in an introductory physics course begins with a mass moving on a horizontal surface whilst attached to a spring. Under ideal no-friction conditions, the mass performs simple harmonic motion (SHM). The next natural step should be relaxing the idealization of the system by adding sliding friction, but this is nearly never done. Instead one usually adds drag, a velocity-dependent force representing the resistance of a fluid (possibly air). The resulting behavior is damped harmonic motion (DHM).

Up to this point, however, drag is routinely ignored in the introductory course, even where it would be natural, as in projectile motion. One common way to motivate its introduction in harmonic motion is to first treat the vertical spring system in which a block is hung from the ceiling by a spring and oscillates vertically. When solid surfaces are absent, sliding friction is irrelevant and drag becomes a natural consideration. However, the vertical spring system requires a bit of care because gravitation shifts the system's equilibrium position.

Some (admittedly a minority) of our more curious students wonder why we do not also treat the apparently natural case of solid-on-solid friction. Furthermore, once exposed to DHM, they sometimes ask whether a horizontal spring with sliding friction also settles eventually at its relaxed length, whether it requires an infinite time to stop, whether its period is affected, and whether it has a constant period at all. This motivated us to look for a simple approach to the horizontal spring with sliding friction.

The oscillator with solid-on-solid friction has been treated several times before.[1–7] Unfortunately, most teachers seem to feel that the problem is either conceptually challenging, mathematically tedious, or both. Indeed, previous treatments use either work-energy considerations, in which every semi-cycle must be treated independently and anew, or differential equations that produce a split-function, and are possibly a little advanced for introductory courses.

In this paper, we offer an elementary treatment of the problem based on an intuitive visualization of the solution and analogies with SHM. Qualitative properties become transparent, and many quantitative results can also be obtained easily. One important pedagogical advantage of this treatment is in teaching the power of analogies. The idea of using simple systems as analogies of more complex ones is an important tool for any student of science. The present approach is a very good example of how to analyze a much more complex problem (oscillations with solid-on-solid friction) by using only the solution of simple harmonic motion, which is far better known and more accessible. We consider this pedagogical aspect to be one of the important advantages of our treatment.

We begin in Sec. II by recalling the fundamental relation between SHM and circular motion. Section III then extends this analogy to the case of an oscillator with solid-on-solid friction and presents the fundamental visualization we apply throughout the paper. Sections IV and V use this visualization to obtain quantitative results, including the number of cycles the system performs before stopping and the time-dependence of the motion. Section VI extends the treatment to arbitrary initial conditions, which is much easier to do in the present approach than in previous ones. Section VII summarizes the main advantages of our treatment.

II. HARMONIC OSCILLATIONS AS PROJECTIONS OF CIRCULAR MOTION

It is well known that SHM can be viewed as the projection of the circular motion of a "virtual" mass m attached to a string of length A, and rotating (counterclockwise, by convention) at a constant angular velocity ω (see Fig. 1), hence the angle of the radius-vector of the virtual mass to the positive x-axis is $\phi(t) = \omega t$.

The horizontal component of the centripetal force is $F_x = m\omega^2 A \cos\phi(t) = m\omega^2 x(t)$, which is identical to the force of a horizontal spring, hence the horizontal projection of the circular motion is a SHM, i.e.,

$$x(t) = A\cos\phi(t) = A\cos(\omega t). \tag{1}$$

We will make an important analogy with the vertical spring, where gravity is an additional constant force. The effect is to shift the equilibrium point from the spring's relaxation point (the point at which the spring's length is at its unstretched value), by an amount

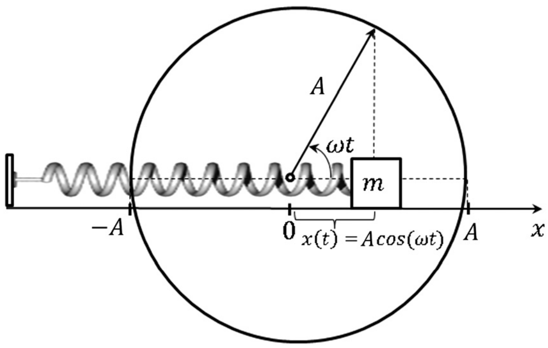

Fig. 1. SHM as a projection on the x-axis of a uniform circular motion in the x-y plane. The angular velocity of the circular motion is the same as the angular frequency of the SHM. The equilibrium position of the SHM coincides with the center of the circular rotation.

$$D = \frac{mg}{k}. \tag{2}$$

The visualization of the system is the same as before, except that the virtual string is now attached to the equilibrium point, a distance D from the spring's relaxation point. Note that the system's period is unchanged

$$T = 2\pi\sqrt{\frac{m}{k}}, \tag{3}$$

and the system's motion is symmetrical with respect to the equilibrium position D.

III. APPLICATION TO A SYSTEM WITH SLIDING FRICTION

We now consider a mass m attached to a horizontal spring of stiffness k. The mass is constrained to move along a straight line on a horizontal surface with static and kinetic coefficients of friction μ_s and μ_k, respectively. We have as usual $\mu_k < \mu_s$. For simplicity, we assume here that the mass is initially at rest and the spring is stretched in the positive direction by some amount x_0 (see Sec. VI for the general case).

We must distinguish two cases, as in all motions containing sliding friction:

Case 1. Static friction overcomes the elastic force.

This happens if

$$k|x_0| < \mu_s mg. \tag{4}$$

Defining

$$D_s = \mu_s \frac{mg}{k}, \tag{5}$$

we see that if $|x_0| < D_s$, the system remains at rest and no oscillations occur.

Case 2. Elastic force overcomes static friction.

If $|x_0| > D_s$, the mass begins to move and friction becomes kinetic. Under our assumption of an initial stretch, the motion begins in the negative direction and the total force is

$$F = -kx + \mu_k mg. \tag{6}$$

This has the form of the force in a vertical spring system. By analogy, the resulting motion is simply harmonic around a new "equilibrium" point

$$x = D_k = \frac{\mu_k mg}{k}. \tag{7}$$

It is non-trivial but clear that, as in the case of the vertical spring, the time between extremal positions (local maxima and minima of the position) remains unchanged from the frictionless case, Eq. (3). This is not the case, for example, in DHM, where that time, although constant, is different from the undamped case.

From the analysis of the vertical spring system, the initial motion of the spring with sliding friction is the x-projection of a uniform circular motion centered on the point $x = D_k$, with radius $A_0 = x_0 - D_k$. To visualize this easily, we imagine a virtual string, stretched along the positive x-axis, and fixed with a nail at the point $x = D_k$. (Alternatively, one can also imagine the string to be fixed at the origin and merely passing under the nail at $x = D_k$.) The string rotates counterclockwise at a constant speed and traces a circular arc until it is horizontal again, and the mass is at the point $x_1 = D_k - A_0$. As long as $|x_1| > D_s$, the block will move again, this time in the positive direction. Friction reverses direction, and the total force is now

$$F = -kx - \mu_k mg. \tag{8}$$

This force describes a SHM, but the "equilibrium" point is shifted to $x = -D_k$, as seen in Fig. 2.

It is at this point that usual treatments become disjointed, treating the next leg of the motion as a separate problem from the first. The graphical analogy of a winding string that we present here offers, by contrast, an appealing alternative in which the motion is seen to be one continuous process. To see how, imagine that we have a second nail driven at the point $x = -D_k$. When the mass reaches point x_1, the string is horizontal and touches that nail. As the string continues to rotate, the second nail now serves as a new pivot around which the string starts winding. The next phase of the motion is, thus, clear: it is another semi-circle, centered on $x = -D_k$. At the same time, the string is shortened, and the new radius of motion is $A_1 = A_0 - 2D_k$.

The entire motion can, thus, be visualized as the projection of a spiral described by a string wrapping itself around two nails hammered at $x = \pm D_k$, as seen in Fig. 3. This is

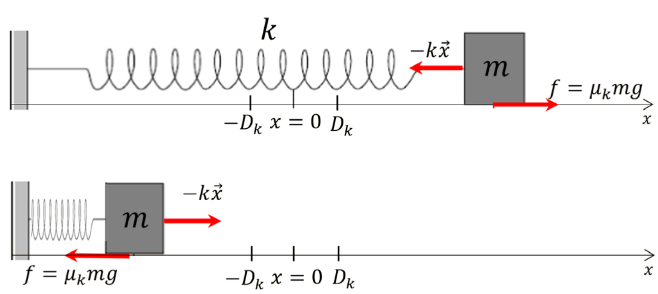

Fig. 2. Motion of a horizontal spring with added solid-on-solid friction. In the top figure, the mass moves to the left, and the corresponding SHM is centered on $x = D_k$. In the bottom figure, the mass moves to the right, and the corresponding SHM is centered on $x = -D_k$. In both cases, $\vec{x}$ is the displacement of the mass, and the magnitude of which is equal to the string's extension or compression relative to its relaxed state.

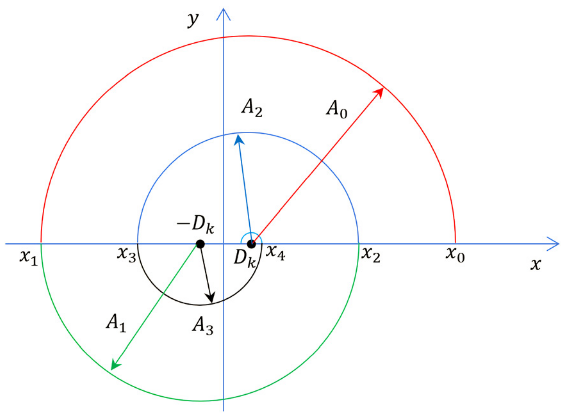

Fig. 3. The spiral-like trajectory of the virtual string is composed of semi-circles with amplitudes A_p, each centered alternately on the "kinetic equilibrium" positions $\pm D_k$.

relatively easy to visualize, and all the motion's properties can be deduced from this picture with minimal calculations.

It may seem surprising that the radius of motion remains constant during each half-cycle despite the presence of friction. In the string visualization, the radius of motion changes abruptly and instantaneously at every turning point when the mass comes to momentary rest. However, this is true only with respect to the current center of motion. With respect to the spring's unstretched state (its relaxation point), the maximum stretch and contraction in each half cycle are not equal. Each turning point is closer to the relaxation point than the last, in accordance with intuition.

IV. QUANTITATIVE ANALYSIS

Let us denote by $\{x_0, x_1, x_2, \ldots\}$ the series of positions where the mass is instantaneously at rest, x_0 being the initial position. All other $\{x_i\}$ for $i \geq 1$ are the turning points at which the mass changes directions and the virtual string touches one of the nails and switches its winding center.

Next, let $\{A_0, A_1, A_2, \ldots\}$ be the series of consecutive radii of the semi-circles described by the virtual string. The first semi-circle extends from the position x_0 to x_1. It is centered at $x = D_k$, and its radius is, thus,

$$A_0 = x_0 - D_k. \tag{9}$$

Thereafter, the string is shortened by $2D_k$, the distance between the two nails, every semi-revolution. Thus,

$$A_{p+1} = A_p - 2D_k, \quad p \geq 0. \tag{10}$$

Since the string's length is finite (in fact equal to A_0), there is some definite p for which A_p becomes negative, which is impossible. Thus, the number of windings must be finite, which means that the block will stop after a finite number of semi-cycles, unlike the case of DHM where the number of cycles is theoretically infinite. Combining Eqs. (9) and (10), we have that

$$A_p = A_0 - 2pD_k = x_0 - (2p+1)D_k. \tag{11}$$

Let $p_{\max}$ be the index of the final turning point, i.e., the index for which $A_{p_{\max}} \leq 0$ and the motion cannot continue. This condition implies that

$$p_{\max} \geq \frac{A_0}{2D_k} = \frac{x_0}{2D_k} - \frac{1}{2} \tag{12}$$

or equivalently,

$$p_{\max} = \text{ceil}\left[\frac{A_0}{2D_k}\right], \tag{13}$$

where ceil[], the ceiling function, denotes the smallest integer equal to or larger than the function's argument.

However, although this sets an upper limit on the number of windings, the motion may stop earlier. The reason is that once $|x_p| \leq D_s$, the elastic force cannot overcome the static friction and the block will not start another revolution, even if there is enough leftover string to apparently allow it. The instantaneous center of rotation shifts between D_k and $-D_k$ every semi-cycle. Since the turning points are measured with respect to the origin, i.e., the string's relaxation point, they verify the relation

$$|x_p| = |x_{p-1}| - 2D_k = x_0 - 2pD_k. \tag{14}$$

If the motion ceases at the N-th turning point, then the stopping criterion $|x_N| \leq D_s$ implies that

$$N \geq \frac{x_0 - D_s}{2D_k}. \tag{15}$$

Since $\mu_s \geq \mu_k$, Eq. (15) is more restrictive than Eq. (12). Based on the definitions of D_s and D_k, from Eqs. (5) and (7), the motion stops at the N-th turning point, i.e., after N half-cycles, where N is

$$N = \text{ceil}\left[\frac{x_0}{2D_k} - \frac{D_s}{2D_k}\right] = \text{ceil}\left[\frac{x_0}{2D_k} - \frac{\mu_s}{2\mu_k}\right] \tag{16a}$$

or

$$N = \text{ceil}\left[\frac{A_0}{2D_k} - \frac{D_s - D_k}{2D_k}\right] = \text{ceil}\left[\frac{A_0}{2D_k} - \frac{\mu_s - \mu_k}{2\mu_k}\right]. \tag{16b}$$

The last form of the relation is obtained from Eq. (9).

We can easily calculate the total distance traveled by the mass attached to the spring, since every half cycle it goes over a distance of two amplitudes, i.e.,

$$L = 2\sum_{p=0}^{N-1} A_p = 2\sum_{p=0}^{N-1} [A_0 - 2pD_k]. \tag{17}$$

This is an arithmetic series, so that

$$L = 2N[A_0 - (N-1)D_k]. \tag{18}$$

In the last two equations, N is given in Eq. (16).

V. DESCRIPTION OF THE MOTION IN TIME

The winding string visualization allows us to write down with relative ease the time evolution of the block's position without the need for differential equations.

During each semi-cycle, the block performs a SHM centered on alternating pivots. The centers and amplitudes change at every turning point. At $t_0 = 0$, the block is at x_0, the first point where the velocity vanishes. Hereafter, it comes to momentary rest every half cycle. Since the period remains constant throughout the motion, the p-th turning point occurs at

$$t_p = pT_{1/2}, \tag{19}$$

where

$$T_{1/2} = \frac{T}{2} = \pi\sqrt{\frac{m}{k}}. \tag{20}$$

We now write the rotating radius-vector of the endpoint of the virtual string, around its instantaneous center

$$\mathbf{r}(t) = (A_p \cos\omega t, A_p \sin\omega t) \quad p = 0, 1, 2, \ldots, \tag{21}$$

where the amplitude from Eq. (11) is

$$A_p = x_0 - (2p+1)D_k \quad \text{for} \quad pT_{1/2} \le t \le (p+1)T_{1/2}. \tag{22}$$

To obtain the actual harmonic motion of the block, we project this vector on the horizontal axis and add the displacement of the center of rotation. Since this center alternates between D_k and $-D_k$ every half-cycle, we have that during the p-th cycle,

$$x_{\text{center, p}} = (-1)^p D_k. \tag{23}$$

Thus, the final result is

$$x(t) = (-1)^p D_k + [x_0 - (2p+1)D_k]\cos(\omega t) \quad \text{for} \quad pT_{1/2} \le t \le (p+1)T_{1/2}. \tag{24}$$

This solution holds for $p \le N$, where N is given in Eq. (16). For larger times, the mass remains stationary at its stopping point. Our solution agrees with that appearing in Refs. 1, 2, and 6 but without the need to solve differential equations.

The function $x(t)$ is drawn in Fig. 4. The two dotted inner lines represent the alternating instantaneous centers of the motion, $x = \pm D_k$. Each extremum of the function (either a crest or a trough) represents a turning point. The two dashed outer lines represent the positions $x = \pm D_s$. When a turning point falls between these lines, the systems stops.

From Eq. (19), the times of the turning points verify the relation $\omega t_p = p\pi$, so that their positions are (from Eq. (24))

$$x_p = (-1)^p \left[x_0 - \frac{2\omega D_k}{\pi} t_p\right]. \tag{25}$$

These are the crests (even p) and troughs (odd p) of the graph. Notice that the envelope following these turning points decays linearly in time, not exponentially as found in damped harmonic motion.

We immediately see that the oscillator stops after a finite number of periods, because when the envelope crosses the horizontal axis, the amplitude of the motion vanishes. In contrast, the envelope of the damped oscillator only tends asymptotically to zero but never actually reaches it, thus accommodating (in theory) an infinite number of oscillations.

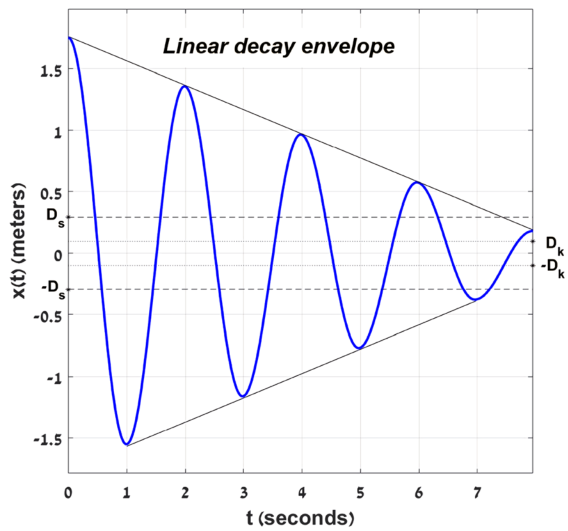

Fig. 4. The block's position as a function of time. The pair of outer dashed lines are the stopping borders at $x = \pm D_s$. When the blocks comes to rest between these two lines, the motion ends. The pair of inner dotted lines are the positions of the alternating instantaneous centers of motions, $x = \pm D_k$. The envelope of the position as a function of time decays linearly and is determined by the extremal points of the graph. In this graph, we took $m = 1\,\text{kg}, k = 10\,\text{N/m}, A_0 = 1.75\,\text{m}, \mu_s = 0.3$, and $\mu_k = 0.1$.

Note that the distance from crest [trough] to the origin differs from the distance from the origin to trough [crest], thus showing that with respect to the spring's relaxed state, the motion of the block is asymmetrical when it is to the right of the origin compared to when it is to its left. When measured with respect to the appropriate red inner dashed line, however, the distances are identical. These are the amplitudes of the SHM with respect to alternating instantaneous centers, and this motion is indeed symmetric with respect to these centers during each half-cycle. The changes in amplitudes with respect to the instantaneous centers of motion occur only from one half-cycle to the next, each time the friction force flips direction.

VI. GENERAL INITIAL CONDITIONS

The winding string visualization allows a simple treatment of general initial conditions. We can represent the initial position x_0 and velocity v_0 by shifting the initial position of the virtual string along its circular path. Instead of imagining that the virtual mass at the end of the string starts on the x-axis, we position it along an initial angle ϕ_0, as seen in Fig. 5.

As before, we assume by convention that the virtual mass rotates counterclockwise along its circular path. Choosing $0 \le \phi_0 < 2\pi$, the relevant ranges depend on the signs of the initial position and velocity

$$\begin{aligned} &\phi_0 = 0 && \text{if } v_0 = 0 \text{ and } x_0 > 0, \\ &0 < \phi_0 < \pi && \text{if } v_0 < 0, \\ &\phi_0 = \pi && \text{if } v_0 = 0 \text{ and } x_0 < 0, \\ &\pi < \phi_0 < 2\pi && \text{if } v_0 > 0. \end{aligned} \tag{26}$$

For convenience, define the parameter

$$\eta = \text{int}\left(\frac{\phi_0}{\pi}\right), \tag{27}$$

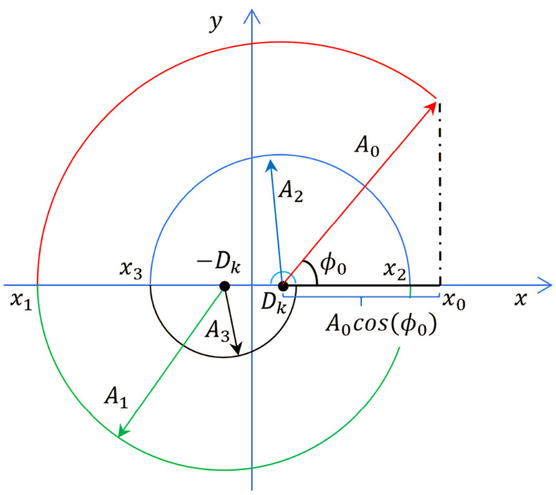

Fig. 5. The path of a virtual string representing the frictional oscillator with general initial conditions. The virtual string starts from an angle ϕ_0, which determines the initial position x_0 and velocity v_0. The first sequence of the spiral is not a complete half circle, but all the following ones are identical to the above considered special case, when $v_0 = 0$.

where int( ) is the floor function, i.e., the largest integer that is smaller than or equal to the function's argument. We now have that

$$x_0 = (-1)^{\eta} D_k + A_0 \cos \phi_0, \quad v_0 = -A_0 \omega \sin \phi_0. \tag{28}$$

From this, we obtain the relations

$$A_0 = \sqrt{\left[x_0 + (-1)^{\eta+1} D_k\right]^2 + \frac{v_0^2}{\omega^2}}, \tag{29a}$$

$$\tan \phi_0 = -\frac{v_0}{\omega\left[x_0 + (-1)^{\eta+1} D_k\right]}. \tag{29b}$$

Figure 5 shows the spiral drawn by the virtual string under general initial conditions. The string does not complete a full half-cycle before reaching the first turning point x_1 (the first point of instantaneous rest), which occurs at the time t_1. Since the string rotates at a constant angular speed ω, however, the time at which it is horizontal again is easily found to be

$$t_1 = \begin{cases} \dfrac{\pi - \phi_0}{\omega} & \text{if} \quad \phi_0 < \pi, \\[2ex] \dfrac{2\pi - \phi_0}{\omega} & \text{if} \quad \phi_0 \geq \pi, \end{cases} \tag{30}$$

or equivalently,

$$t_1 = \left(1 + \eta - \frac{\phi_0}{\pi}\right) T_{1/2}. \tag{31}$$

One of our visualization's advantages is that a look at Fig. 5 suffices to see that from x_1 onwards, the spiral behaves exactly as in the case treated previously. This means that the next turning points $\{t_2, t_3, \ldots\}$, all occur $T_{1/2}$ after one another, so that

$$t_p = t_1 + (p-1) T_{1/2} \quad (p \geq 2), \tag{32}$$

and the amplitude of each semi-cycle is

$$A_p = A_0 - 2pD_k \quad \text{for} \quad t_{p-1} \leq t \leq t_p. \tag{33}$$

In the $p - th$ semi-cycle, the radius-vector of the endpoint of the virtual string is centered on $x_{\text{center, p}} = (-1)^{\eta+p} D_k$, as seen in Eq. (28), and its value is $\mathbf{r}(t) = (A_p \cos(\omega t + \phi_0), A_p \sin(\omega t + \phi_0))$. The motion of the mass on the spring is the projection of this vector on the horizontal axis with the displacement of the center of rotation added. The final result is

$$x(t) = (-1)^{\eta+p} D_k + [A_0 - 2pD_k] \cos(\omega t + \phi_0) \quad \text{for} \quad t_{p-1} \leq t \leq t_p. \tag{34}$$

This solution reduces to Eq. (24) when $\phi_0 = 0$, which implies that $\eta = 0$, $v_0 = 0$, and $A_0 = x_0 - D_k$.

As before, this solution holds for $p \leq N$, and the maximal turning point index N is still given in Eq. (16b), i.e.,

$$N = \text{ceil}\left[\frac{A_0}{2D_k} - \frac{\mu_s - \mu_k}{2\mu_k}\right]. \tag{35}$$

The value of A_0 from Eq. (29a) differs from the above treated case, which is why Eq. (16a) no longer holds. For times larger than t_N, the mass remains stationary at its stopping point.

Although the solution looks fairly elaborate, the virtual string visualization makes its derivation quite simple, and the meaning of each term is geometrically clear. This is a great advantage over alternative methods of solution. Indeed the general case has not been treated previously and to the best of our knowledge, its solution is obtained here for the first time.

One can also reduce the general case to the above treated special one by introducing negative times. Some students find this approach helpful (others do not, so it is a matter of preference). Instead of the actual start of the motion, $t = 0$ is now considered the start of observation, which occurs whilst the block is already moving. The "true" initial condition is again taken to be $v_0 = 0$ and $x_0 = A_0 + D_k$, as before. However, this occurs at the time

$$t_0 = -\frac{\phi_0}{\omega} \tag{36}$$

in accordance with the visualization of the "virtual block" moving at a constant angular speed ω. The virtual block's radius-vector becomes a generalization of Eq. (21)

$$\mathbf{r}(t) = (A_p \cos \omega(t - t_0), A_p \sin \omega(t - t_0)) \quad p = 0, 1, 2, \ldots, \tag{37}$$

which is of course identical to the above solution with a different notation.

VII. CONCLUSIONS

In this work, the motion of a harmonic oscillator with sliding friction is seen as the projection of a two dimensional spiral motion created by a string winding itself around two nails. This visualization permits a simple geometric analysis of the oscillator's motion, using only elementary algebra and basic trigonometry, with no reliance on differential equations. Furthermore, the treatment of general initial conditions is incorporated with minimal changes, and a very minor increase in mathematical complexity, which is a significant advantage over previous approaches. All simulations of the motion presented here were made using the MATLAB software, which is an added pedagogical advantage, as it can be given as a project for students to perform themselves.[8] Furthermore, the whole visualization is extremely concrete. One of us (V.R.) built a physical model using only a board, two nails and an string, to show the behavior of the system. Students who actually hold the "visualization" tool in their hands are inclined to play with it and try to extract more information from it.

The visualization presented here allows the treatment of any oscillator with added constant-magnitude force. For example, one can treat easily the case of an oscillator on an incline plane. The added component of the gravitational force represents a single shift of the center of the winding string to a "gravitational equilibrium" position. The addition of friction to this case can now be easily accommodated by having two winding pivots placed symmetrically with respect to the gravitational equilibrium position. Other generalizations are possible, all using basically the same visualization. This represents an important pedagogical message on the power of analogies and their usefulness, which transcends the specific system analyzed here to exemplify it.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

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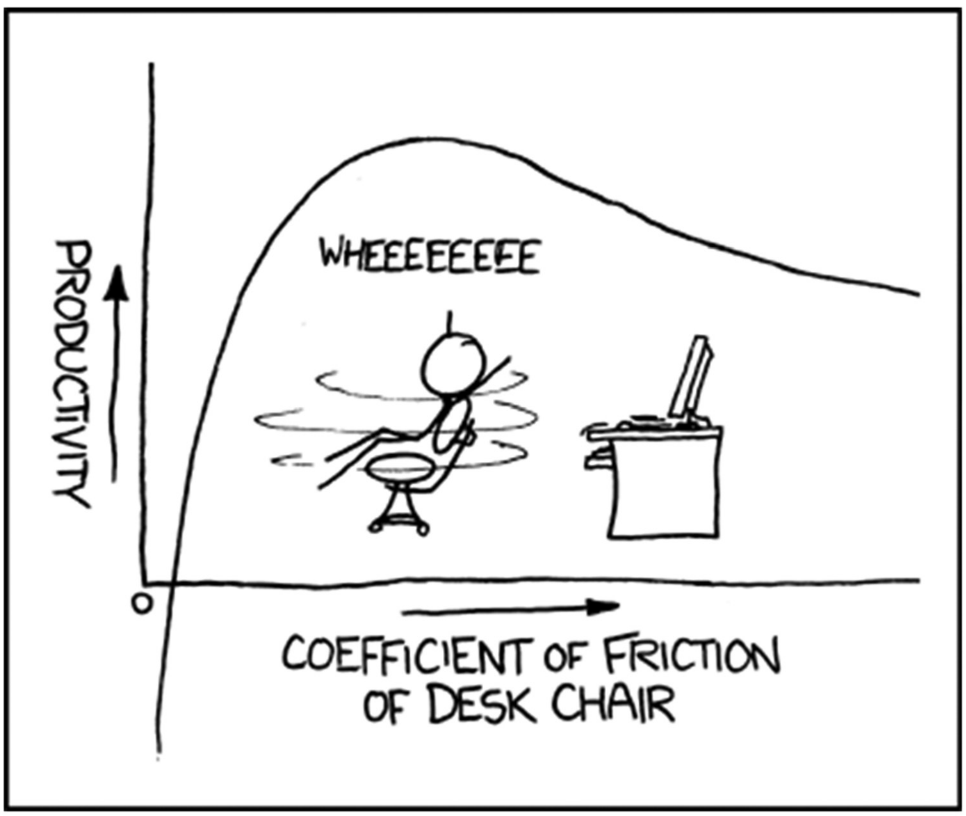
[8]See supplementary material at https://www.scitation.org/doi/suppl/10.1119/5.0073812 some explanations about the MATLAB code we used to plot the block's motion as appears in Fig. 4 as well as the code itself.



As the CoKF approaches 0, productivity goes negative as you pull OTHER people into chair-spinning contests.

(Source: https://xkcd.com/815/)