ON WEAK IDEMPOTENT COMPLETIONS

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Let $\mathcal{A}$ be a category. An endomorphism $e\colon X \to X$ is called an *idempotent* if $e^2 = e$. The idempotent e *splits* if there are morphisms $r\colon X \to Y$ and $s\colon Y \to X$ satisfying $e = s \circ r$ and $\mathrm{Id}_Y = r \circ s$. The category $\mathcal{A}$ is said to be *idempotent-split* if each idempotent splits.

It is well known that each category $\mathcal{A}$ has a canonical idempotent completion $\mathcal{A}^\natural$; see [2, Chapter I, Theorem 6.10]. The objects of $\mathcal{A}^\natural$ are pairs (X, e) such that X is an object in $\mathcal{A}$ and that $e\colon X \to X$ is an idempotent. The morphisms $f\colon (x, e) \to (X', e')$ are given by morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(X, X')$ satisfying $f = e' \circ f \circ e$. We have a canonical embedding $\mathcal{A} \hookrightarrow \mathcal{A}^\natural$, which sends $f\colon X \to X'$ to $f\colon (X, \mathrm{Id}_X) \to (X', \mathrm{Id}_{X'})$.

Lemma 1. *Let $e\colon X \to X$ be an idempotent in $\mathcal{A}$. Then e splits in $\mathcal{A}$ if and only if (X, e) is isomorphic to some object (Y, Id_Y) in $\mathcal{A}^\natural$.* □

In what follows, we assume that $\mathcal{A}$ is pre-additive. The following two lemmas are well known.

Lemma 2. *Let $e\colon X \to X$ be an idempotent. Then e splits if and only if $\mathrm{Id}_X - e$ admits a kernel, if and only if $\mathrm{Id}_X - e$ admits a cokernel.* □

Lemma 3. *Assume that $e\colon X \to X$ is an idempotent such that e splits as $X \xrightarrow{r} Y \xrightarrow{s} X$ and that $\mathrm{Id}_X - e$ splits as $X \xrightarrow{r'} Y' \xrightarrow{s'} X$. Then $(s, s')\colon Y \oplus Y' \to X$ is an isomorphism, whose inverse is $\binom{r}{r'}$.* □

The following example shows that e and $\mathrm{Id}_X - e$ do not split simultaneously in general.

Example 4. *Let k be a field. Denote by $\mathcal{C}$ the category formed by finite dimensional k-spaces with dimension at least two. Consider the idempotent $e = \mathrm{diag}(1, 1, 0)\colon k^3 \to k^3$. Then e splits in $\mathcal{C}$, but $\mathrm{Id}_{k^3} - e$ does not split in $\mathcal{C}$.*

A pre-additive category $\mathcal{A}$ is said to be *weakly idempotent-split*, if any idempotent $e\colon X \to X$ splits whenever $\mathrm{Id}_X - e$ splits. We mention that a weakly idempotent-split category necessarily has a zero object.

Example 5. *(1) The category $\mathcal{C}$ in Example 4 is not weakly idempotent-split.*

(2) Let k be a field. Denote by $\mathcal{C}'$ the category of even dimensional k-spaces. Then $\mathcal{C}'$ is weakly idempotent-split, but not idempotent-split.

(3) Any triangulated category is weakly idempotent-split; see [3, Lemma 2.2].

Lemma 6. *Let $\mathcal{A}$ be a pre-additive category. Then $\mathcal{A}$ is weakly idempotent-split if and only if any split idempotent admits a kernel, if and only if any split idempotent admits a cokernel.*

Proof. Apply Lemmas 2. □

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We define the *weak idempotent completion* $\mathcal{A}^\flat$ of $\mathcal{A}$ to be the full subcategory of $\mathcal{A}^\natural$ formed by the pairs (X,e) such that $\mathrm{Id}_X - e = 0$ or $\mathrm{Id}_X - e$ is a split idempotent in $\mathcal{A}$.

The following fact is somehow non-trivial.

Proposition 7. *The category $\mathcal{A}^\flat$ is indeed weakly idempotent-split.*

Proof. Let (X,e) be an object in $\mathcal{A}^\flat$ such that $\mathrm{Id}_X - e$ splits as $X \xrightarrow{r} Y \xrightarrow{s} X$. Let (X',e') be another object such that $\mathrm{Id}_{X'} - e'$ splits as $X' \xrightarrow{r'} Y' \xrightarrow{s'} X$. We observe the following identities

$$r \circ e = 0 = e \circ s \text{ and } r' \circ e' = 0 = e' \circ s'. \tag{0.1}$$

Let $f\colon (X,e) \to (X',e')$ and $g\colon (X',e') \to (X,e)$ be two morphisms such that $f \circ g = e' = \mathrm{Id}_{(X',e')}$. Then we have

$$f = e' \circ f \circ e \text{ and } g = e \circ g \circ e'. \tag{0.2}$$

The idempotent $g \circ f\colon (X,e) \to (X,e)$ splits in $\mathcal{A}^\flat$.

We have to show that $\mathrm{Id}_{(X,e)} - g \circ f = e - g \circ f\colon (X,e) \to (X,e)$ also splits. For this end, we first observe that

$$(X \oplus Y', \begin{pmatrix} e - g \circ f & 0 \\ 0 & 0 \end{pmatrix})$$

is an object in $\mathcal{A}^\flat$. Indeed, using (0.1) and (0.2), we infer that $\begin{pmatrix} \mathrm{Id}_X - e + g \circ f & 0 \\ 0 & \mathrm{Id}_{Y'} \end{pmatrix}$ splits as

$$X \oplus Y' \xrightarrow{\begin{pmatrix} r & r \circ g \circ s' \\ f & s' \end{pmatrix}} Y \oplus X' \xrightarrow{\begin{pmatrix} s & g \\ r' \circ f \circ s & r' \end{pmatrix}} X \oplus Y'.$$

We further observe that $e - g \circ f$ splits in $\mathcal{A}^\flat$ as follows

$$(X,e) \xrightarrow{\begin{pmatrix} e - g \circ f \\ 0 \end{pmatrix}} (X \oplus Y', \begin{pmatrix} e - g \circ f & 0 \\ 0 & 0 \end{pmatrix}) \xrightarrow{\begin{pmatrix} e - g \circ f & 0 \end{pmatrix}} (X,e).$$

This completes the proof. $\square$

Example 8. *Let R be any unital ring. Denote by $\mathcal{A}$ the category of finitely generated free R-modules. Then $\mathcal{A}^\natural$ is equivalent to the category of finitely generated projective R-modules, and $\mathcal{A}^\flat$ is equivalent to the category of stably free R-modules.*

The following result might be of interest. For any additive category $\mathcal{A}$, we denote by $\mathbf{K}^b(\mathcal{A})$ the homotopy category of bounded cochain complexes in $\mathcal{A}$.

Proposition 9. *Let $F\colon \mathcal{A} \to \mathcal{A}'$ be an additive functor between two additive categories. Then the induced functor $\mathbf{K}^b(F)\colon \mathbf{K}^b(\mathcal{A}) \to \mathbf{K}^b(\mathcal{A}')$ is a triangle equivalence if and only if the induced functor $F^\flat\colon \mathcal{A}^\flat \to \mathcal{A}'^\flat$ is an equivalence.*

Proof. Apply [1, Theorem 4.1 and Corollary 4.3(1)]. $\square$

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References

[1] M.V. Bondarko, and S.V. Vostokov, *The hearts of weight structures are the weakly idempotent complete categories*, arXiv:2005.11518, 2020.
[2] M. Karoubi, K-Theory–An introduction, Grund. Math. Wiss. **226**, Springer, 1978.
[3] J. Le, and X.W. Chen, *Karoubianness of a triangulated category*, J. Algebra **310** (2007), 452–457.

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