PROJECTIVE COVERS AND MINIMAL APPROXIMATIONS

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Let $\mathcal{A}$ be an abelian category. A morphism $f\colon A \to B$ is said to be *right minimal* if any endomorphism $a \in \mathrm{End}_{\mathcal{A}}(A)$ of A satisfying $f = f\circ a$ is necessarily invertible. An epimorphism $f\colon A \to B$ is said to be *essential* if any morphism $t\colon T \to A$ is epic whenever $f \circ t$ is epic.

Let X be an object. Recall that a *projective cover* of X is an epimorphism $\pi\colon P \to X$ such that P is projective and that π is right minimal.

The following characterization of a projective cover is well known; for example, see [3, Lemma 3.4].

Lemma 1. *Let $\pi\colon P \to X$ be a morphism with P projective. Then π is a projective cover of X if and only if it is an essential epimorphism.*

Proof. For the "only if" part, we assume that π is a projective cover. Take any morphism $t\colon T \to P$ with $\pi \circ t$ epic. The projectivity of P yields a morphism $s\colon P \to T$ satisfying $\pi = (\pi \circ t) \circ s$. By the right minimality of π, we infer that $t \circ s$ is invertible. It follows that t is a split epimorphism.

Conversely, we assume that π is an essential epimorphism. Take any endomorphism $a\colon P \to P$ satisfying $\pi = \pi\circ a$. It follows that a is necessarily epic. Since P is projective, a is a split epimorphism. There exists $b\colon P \to P$ satisfying $a \circ b = \mathrm{Id}_P$. We have

$$\pi \circ b = (\pi \circ a) \circ b = \pi \circ (a \circ b) = \pi.$$

Since the epimorphism π is essential, we infer that b is epic. It follows that b is invertible, and so is a. □

Let A be an object in $\mathcal{A}$. We denote by add A the full subcategory formed by direct summands of finite direct sums of A. For any ring R, we denote by proj-R the category of finitely generated projective right R-modules. The *projectivization* at A means the following equivalence

$$\mathrm{Hom}_{\mathcal{A}}(A, -)\colon \mathrm{add}\, A \longrightarrow \text{proj-}\mathrm{End}_{\mathcal{A}}(A).$$

Moreover, we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(A_0, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{End}_{\mathcal{A}}(A)}(\mathrm{Hom}_{\mathcal{A}}(A, A_0), \mathrm{Hom}_{\mathcal{A}}(A, X)) \tag{0.1}$$

sending f to $\mathrm{Hom}_{\mathcal{A}}(A, f)$ for any $A_0 \in \mathrm{add}\, A$ and $X \in \mathcal{A}$.

The projectivization plays a central role in the following proof.

Proposition 2. *Let $\pi\colon P \to X$ be an epimorphism with P projective. Then X has a projective cover if and only if the right $\mathrm{End}_{\mathcal{A}}(P)$-module $\mathrm{Hom}_{\mathcal{A}}(P, X)$ has a projective cover.*

Proof. Write $\Gamma = \mathrm{End}_{\mathcal{A}}(P)$ and $F = \mathrm{Hom}_{\mathcal{A}}(P, -)$. Observe that $F(P) = \Gamma$.

For "only if" part, we assume that $\pi_0\colon P_0 \to X$ is a projective cover. We observe that there is a split epimorphism $t\colon P \to P_0$ such $\pi = \pi_0 \circ t$. In particular,

Date: September 5, 2024.
2020 *Mathematics Subject Classification.* 16D40, 18A20.
Key words and phrases. projective cover, minimal approximation.
This paper belongs to a series of informal notes, without claim of originality.

P_0 is isomorphic to a direct summand of P. Consequently, $F(P_0)$ is a projective Γ-module.

We claim that $F(\pi_0)\colon F(P_0) \to F(X)$ is a projective cover of $F(X)$. It suffices to show that it is right minimal. Take any endomorphism a on $F(P_0)$ with $F(\pi_0) \circ a = F(\pi_0)$. By the projectivization, we have a unique endomorphism s on P_0 with $a = F(s)$. Applying the equation $F(\pi_0 \circ s) = F(\pi_0)$ to the element $t \in F(P_0)$, we obtain

$$\pi_0 \circ s \circ t = \pi_0 \circ t.$$

Since t is epic, we have $\pi_0 \circ s = \pi_0$. Since π_0 is right minimal, we infer that s is invertible, and then so is a.

For the "if" part, we assume that $f\colon Q \to F(X)$ is a projective cover of $F(X)$. Consider the surjective map $F(\pi)\colon F(P) \to F(X)$ of right Γ-modules. There is a split epimorphism $g\colon F(P) \to Q$ satisfying $F(\pi) = f \circ g$. By the projectivization at P, we may assume that $Q = F(P_0)$ with $P_0 \in \operatorname{add} P$; moreover, by (0.1) there are morphisms $a\colon P_0 \to X$ and $b\colon P \to P_0$ satisfying $F(a) = f$ and $F(b) = g$. We have

$$F(\pi) = f \circ g = F(a) \circ F(b) = F(a \circ b).$$

By (0.1) again, we have $\pi = a \circ b$. It follows that $a\colon P_0 \to X$ is epic.

It remains to show that a is right minimal. For this end, we take an endomorphism s on P_0 satisfying $a = a \circ s$. Then we have

$$f = F(a) = F(a \circ s) = F(a) \circ F(s) = f \circ F(s).$$

Since f is right minimal, we infer that $F(s)$ is invertible. By the projectivization at P, the endomorphism s is invertible. This completes the proof. □

Recall that a ring R is *semiperfect* if each finitely generated right R-module has a projective cover. This property is left-right symmetric.

Corollary 3. *Let $\pi\colon P \to X$ be an epimorphism with P projective. Assume further that the endomorphism ring* $\operatorname{End}_{\mathcal{A}}(P)$ *is semiperfect. Then X has a projective cover in $\mathcal{A}$.*

Proof. We observe that the right $\operatorname{End}_{\mathcal{A}}(P)$-module $\operatorname{Hom}_{\mathcal{A}}(P, X)$ is cyclic. Since $\operatorname{End}_{\mathcal{A}}(P)$ is semiperfect, $\operatorname{Hom}_{\mathcal{A}}(P, X)$ has a projective cover. Then the statement follows from Proposition 2. □

Recall that an additive category $\mathcal{C}$ is *Krull-Schmidt* if each object is a finite direct sum of indecomposable objects with local endomorphism rings. By [1, Theorem A.1], $\mathcal{C}$ is Krull-Schmidt if and only if it is idempotent-split and the endomorphism ring of each object is semiperfect; see also [3, Corollary 4.4].

We have the following immediate consequence of Corollary 3.

Corollary 4. *Let $\mathcal{A}$ be a Krull-Schmidt abelian category with enough projective objects. Then each object has a projective cover.* □

Let $\mathcal{C}$ be a skeletally small additive category. By a right $\mathcal{C}$-module we mean a contravariant additive functor from $\mathcal{C}$ to the category of abelian groups. Denote by Mod-$\mathcal{C}$ the abelian category of right $\mathcal{C}$-modules. A right $\mathcal{C}$-module M is *finitely generated* if there is an epimorphism $\mathcal{C}(-, A) \to M$ for some object A in $\mathcal{C}$.

The following result is implicitly used in [2, the proof of 1.3 Lemma].

Corollary 5. *Let $\mathcal{C}$ be a Krull-Schmidt category, which is skeletally small. Then each finitely generated $\mathcal{C}$-module has a projective cover.*

Proof. Let M be a finitely generated $\mathcal{C}$-module with an epimorphism $\mathcal{C}(-,A) \to M$. The representable module $\mathcal{C}(-,A)$ is projective in Mod-$\mathcal{C}$. By Yoneda Lemma the endomorphism ring of $\mathcal{C}(-,A)$ is isomorphic to $\mathrm{End}_{\mathcal{C}}(A)$, which is semiperfect. Then we are done by Corollary 3. □

Remark 6. The smallness condition above is superfluous. Indeed, Proposition 2 holds true for idempotent-split exact categories. For the proof of Corollary 5 in the general case, we just work with the exact category formed by finitely generated $\mathcal{C}$-modules.

Let $\mathcal{C}$ be an additive category. Consider a full additive subcategory $\mathcal{M}$. Let X be an object in $\mathcal{C}$. By a *right $\mathcal{M}$-approximation* of X, we mean a morphism $f\colon M \to X$ with $M \in \mathcal{M}$ and every morphism from an object in $\mathcal{M}$ to X factors through f. Such a right $\mathcal{M}$-approximation is *minimal*, provided that it is right minimal.

It is well known that an object X has a right $\mathcal{M}$-approximation if and only if the $\mathcal{M}$-module $\mathcal{C}(-,X)|_{\mathcal{M}}$ is finitely generated.

The following lemma is proved using Yoneda Lemma.

Lemma 7. *Let $f\colon M \to X$ be a morphism with $M \in \mathcal{M}$. Then f is right minimal if and only if the induced morpshim $(-,f)|_{\mathcal{M}}\colon \mathcal{M}(-,M) \to \mathcal{C}(-,X)|_{\mathcal{M}}$ between $\mathcal{M}$-modules is right minimal.* □

In view of Lemma 7, we have the following fact.

Lemma 8. *An object X has a right minimal $\mathcal{M}$-approximation if and only if the $\mathcal{M}$-module $\mathcal{C}(-,X)|_{\mathcal{M}}$ is finitely generated and has a projective cover.* □

The following result is a variant of Proposition 2.

Proposition 9. *Let $f\colon M \to X$ be a right $\mathcal{M}$-approximation of X. Then X has a right minimal $\mathcal{M}$-approximation if and only if the right $\mathrm{End}_{\mathcal{C}}(M)$-module $\mathrm{Hom}_{\mathcal{C}}(M,X)$ has a projective cover.*

Proof. The approximation f yields an epimorphism $\mathcal{M}(-,M) \to \mathcal{C}(-,X)|_{\mathcal{M}}$ between $\mathcal{M}$-modules. By Lemma 8 X has a right minimal $\mathcal{M}$-approximation if and only if $\mathcal{C}(-,X)|_{\mathcal{M}}$ has a projective cover.

We will apply Proposition 2 to the category of $\mathcal{M}$-modules; compare Remark 6. We observe that the endomorphism ring of $\mathcal{M}(-,M)$ is isomorphic to $\mathrm{End}_{\mathcal{C}}(M)$, and that the Hom group from $\mathcal{M}(-,M)$ to $\mathcal{C}(-,X)|_{\mathcal{M}}$ is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(M,X)$. Based on these observations and Proposition 2, we infer that $\mathcal{C}(-,X)|_{\mathcal{M}}$ has a projective cove if and only if the right $\mathrm{End}_{\mathcal{C}}(M)$-module $\mathrm{Hom}_{\mathcal{C}}(M,X)$ has a projective cover. □

We mention that one easily deduces Proposition 2 from the proposition above.

References

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