THE GENERAL FOXBY EQUIVALENCE

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We first recall the general Foxby equivalence from [2, Section 1]. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Assume that (F, G) is an adjoint pair between them, with $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$. Denote the unit by $\eta\colon \mathrm{Id}_{\mathcal{C}} \to GF$ and the counit by $\varepsilon\colon FG \to \mathrm{Id}_{\mathcal{D}}$.

The corresponding *Auslander category* $\mathcal{A} = \mathcal{A}(F, G)$ is defined to the full subcategory of $\mathcal{C}$ formed by those objects C with η_C an isomorphism. Analogously, the *Bass category* $\mathcal{B} = \mathcal{B}(F, G)$ is the full subcategory of $\mathcal{D}$ formed by those objects D with ε_D an isomorphism.

We have the following general fact.

Lemma 1. *The adjoint pair (F, G) induces an equivalence $F|_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{B}$, whose quasi-inverse is given by $G|_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{A}$.*

The obtained equivalence is known as the *general Foxby equivalence*. We observe that (F, G) is an adjoint equivalence if and only if $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \mathcal{D}$.

In practice, it might be nontrivial to describe the subcategories $\mathcal{A}$ and $\mathcal{B}$. In what follows, we describe these subcategories for the Hom-tensor adjoint pair between module categories over artin algebras.

Let A be an artin algebra over a commutative artinian ring. Denote by A-mod the abelian category of finitely generated left A-modules. Let T be a finitely generated left A-module. Set $B = \mathrm{End}_A(T)^{\mathrm{op}}$ to the opposite algebra of its endomorphism algebra. Then T becomes an A-B-bimodule.

We are interested in the following Hom-tensor adjoint pair.

$$A\text{-mod} \underset{\mathrm{Hom}_A(T,-)}{\overset{T\otimes_B -}{\leftrightarrows}} B\text{-mod}$$

It is well known that such an adjoint pair induces an equivalence

$$\mathrm{add}(T) \xrightarrow{\sim} B\text{-proj}.$$

Here, $\mathrm{add}(T)$ denotes the full subcategory of A-mod consisting of direct summands of finite direct sums of T, and B-proj denotes the category of finitely generated projective B-modules. This restricted equivalence is known as the *projectivization*; see [1, II.2].

The corresponding Auslander category is given by

$$\mathcal{A}(T) = \{Y \in B\text{-mod} \mid \eta_Y\colon Y \to \mathrm{Hom}_A(T, T \otimes_B Y) \text{ is an isomorphism}\}.$$

Here, $\eta_Y(y)\colon T \to T \otimes_B Y$ sends a to $a \otimes_B y$. The Bass category is given by

$$\mathcal{B}(T) = \{X \in A\text{-mod} \mid \varepsilon_X\colon T \otimes_B \mathrm{Hom}_A(T, X) \to X \text{ is an isomorphism}\}.$$

Here, $\varepsilon_X(b \otimes_B f) = f(b)$. We observe that B-proj $\subseteq \mathcal{A}(T)$ and $\mathrm{add}(T) \subseteq \mathcal{B}(T)$.

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Recall that fac(T) denotes the full subcategory formed by factor modules of finite direct sums of T. For an A-module M, a *T-presentation* means an exact sequence of A-modules

$$\xi\colon\ T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

with $T_i \in \mathrm{add}(T)$ such that $\mathrm{Hom}_A(T,\xi)$ is still exact. Denote by $\mathrm{fac}_1(T)$ the full subcategory formed by those modules admitting a T-presentation.

Example 2. *Let A be the path algebra of the linear quiver*

$$1 \longrightarrow 2$$

over a field. The unique indecomposable projective-injective module is $P_1 \simeq I_2$. Then we have $\mathrm{fac}(P_1) = \mathrm{add}(P_1 \oplus S_1)$ *and* $\mathrm{fac}_1(P_1) = \mathrm{add}(P_1)$.

Denote by D the Matlis duality. We observe that DT is naturally a B-A-bimodule. Recall that $\mathrm{sub}(DT)$ denotes the full subcategory of B-mod formed by submodules of finite direct sums of DT. For a B-module N, a *DT-copresentation* means an exact sequence of B-modules

$$\kappa\colon\ 0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1$$

with each $E^i \in \mathrm{add}(DT)$ such that $\mathrm{Hom}_B(\kappa, DT)$ is still exact, or equivalently, $T \otimes_B \kappa$ is exact. Denote by $\mathrm{sub}^1(DT)$ the full subcategory of $\mathrm{sub}(DT)$ consisting of those modules admitting a DT-copresentation.

Proposition 3. *Keep the notation as above. Then we have*

$$\mathcal{A}(T) = \mathrm{sub}^1(DT) \text{ and } \mathcal{B}(T) = \mathrm{fac}_1(T).$$

Consequently, we have an ajoint equivalence

$$\mathrm{fac}_1(T) \underset{\mathrm{Hom}_A(T,-)}{\overset{T\otimes_B -}{\rightleftarrows}} \mathrm{sub}^1(DT).$$

Proof. The main idea is to use the projectivization to send projective presentations in B-mod to T-presentations, and injetive copresentations in A-mod to DT-copresentations. We omit the details. $\square$

We apply the consideration above to the A-module $T = DA$. Then we may take $B = A$. The *Nakayama functors* are $\nu = DA \otimes_A -$ and $\nu^- = \mathrm{Hom}_A(DA, -)$, which are endofunctors on A-mod. We identify $D(DA)$ with A. The subcategory $\mathrm{sub}^1(A)$ coincides with A-refl, the category of *reflexive modules*. By duality, the subcategory $\mathrm{fac}_1(DA)$ coincides with A-corefl, the category of *coreflexive modules*. Here, we recall that an A-module X is coreflexive if and only if its dual DX is reflexive.

Corollary 4. *We have an adjoint equivalence*

$$A\text{-corefl} \underset{\nu^-}{\overset{\nu}{\rightleftarrows}} A\text{-refl}.$$

Remark 5. (1) The equivalence above restricts to an adjoint equivalence between A-inj and A-proj.

(2) By duality, the equivalence above follows from the following more well-known duality between reflexive modules.

$$A\text{-refl} \underset{\mathrm{Hom}_A(-,A)}{\overset{\mathrm{Hom}_{A^{\mathrm{op}}}(-,A)}{\rightleftarrows}} A^{\mathrm{op}}\text{-refl}$$

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References

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