DOUBLE COMPLEXES AND TOTAL COMPLEXES

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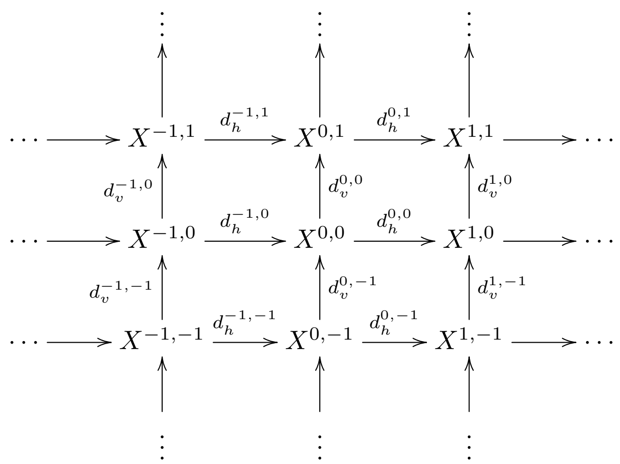
Abstract. The total complex of a double complex with acyclic columns is not necessarily acyclic. We discuss various situations where the total complex is indeed acyclic.

Let $\mathcal{A}$ be an abelian category. We denote by $C(\mathcal{A})$ the abelian category of cochain complexes in $\mathcal{A}$.

Recall that a *double complex* $X = (X^{p,q}, d_h^{p,q}, d_v^{p,q})_{p,q\in\mathbb{Z}}$ consists of bigraded objects $X^{p,q}$ and two kinds of differentials: the horizontal differentials $d_h^{p,q}\colon X^{p,q} \to X^{p+1,q}$ and the vertical differentials $d_v^{p,q}\colon X^{p,q} \to X^{p,q+1}$, which are subject to the relations

$$d_h^{p+1,q} \circ d_h^{p,q} = 0 = d_v^{p,q+1} \circ d_v^{p,q} \text{ and } d_v^{p+1,q} \circ d_h^{p,q} = d_h^{p,q+1} \circ d_v^{p,q}.$$

We visualize a double complex as follows



We denote by $DC(\mathcal{A})$ the abelian category of double complexes in $\mathcal{A}$.

We observe that each column $X^{p,\bullet}$ of X is a cochain complex and that $d_h^{p,\bullet}\colon X^{p,\bullet} \to X^{p+1,\bullet}$ is a cochain morphism. Then we have the following observation.

Lemma 1. *There is an isomorphism of categories*

$$DC(\mathcal{A}) \xrightarrow{\sim} C(C(\mathcal{A})),$$

sending a double complex X to the cochain complex $\cdots \to X^{i,\bullet} \xrightarrow{d_h^{i,\bullet}} X^{i+1,\bullet} \xrightarrow{d_h^{i+1,\bullet}} X^{i+2,\bullet} \to \cdots$ in $C(\mathcal{A})$.

Date: June 6, 2018.

2010 *Mathematics Subject Classification.* 18G35, 18E10.

Key words and phrases. double complex, total complex, acyclic complex.

This paper belongs to a series of informal notes, without claim of originality.

We say that a double complex X is *locally finite*, provided that for each $n \in \mathbb{Z}$, there are only finitely many nonzero objects $X^{i,n-i}$. In other words, there are only finitely many nonzero objects on each diagonal.

For example, if X has only finite many nonzero columns, then X is locally finite. In this case, we define the *width* $w(X) = n - m + 1$, where n is the largest number with $X^{n,\bullet} \neq 0$ and m is the smallest number with $X^{m,\bullet} \neq 0$.

For a locally finite double complex X, its *total complex* $\mathrm{tot}(X)$ is a cochain complex in $\mathcal{A}$ such that $\mathrm{tot}(X)^n = \bigoplus_{n=p+q} X^{p,q}$ and the restriction of the differential $d^n \colon \mathrm{tot}(X)^n \to \mathrm{tot}(X)^{n+1}$ to $X^{p,q}$ is given by $d_h^{p,q} + (-1)^p d_v^{p,q}$. We observe that $d^n(X^{p,q})$ is contained in $X^{p+1,q} \oplus X^{p,q+1}$.

The *transpose* X^t of a double complex X is defined such that $(X^t)^{p,q} = X^{q,p}$ and its horizontal (*resp.* vertical) differentials are the vertical (*resp.* horizontal) differentials of X. Then X is locally finite if and only if so is X^t.

The following observation allows us to switch rows and columns in the consideration of total complexes.

Lemma 2. *There is an isomorphism of cochain complexes*

$$\mathrm{tot}(X) \xrightarrow{\sim} \mathrm{tot}(X^t),$$

which acts on $X^{p,q}$ by $(-1)^{pq}$.

The main concern is when the total complex is acyclic.

Proposition 3. *Let X be a locally finite double complex such that each column $X^{p,\bullet}$ is acyclic. Then the total complex $\mathrm{tot}(X)$ is acyclic.*

Proof. For each $n \in \mathbb{Z}$, we will show the exactness of the sequence

$$\mathrm{tot}(X)^{n-1} \xrightarrow{d^{n-1}} \mathrm{tot}(X)^n \xrightarrow{d^n} \mathrm{tot}(X)^{n+1}.$$

There are only finitely many nonzero $X^{p,q}$ involved in this sequence. So, we may assume that X has only finitely many nonzero columns. We use induction on the width $w(X)$.

If $w(X) = 1$, then $\mathrm{tot}(X)$ is isomorphic to the translation of the nonzero column, and hence is acyclic. In general, we observe that the rightmost nonzero column of X yields a sub-double complex X_1 of X. Then $w(X_1) = 1$ and $w(X/X_1) < w(X)$; moreover, both X_1 and X/X_1 have acyclic columns. There is an exact sequence of complexes

$$0 \to \mathrm{tot}(X_1) \to \mathrm{tot}(X) \to \mathrm{tot}(X/X_1) \to 0.$$

By induction, we infer the acyclicity of $\mathrm{tot}(X)$. □

For a double complex X, we denote by $Z_h^{p,q} = \mathrm{Ker} d_h^{p,q}$, $B_h^{p,q} = \mathrm{Im} d_h^{p-1,q}$ and $H_h^{p,q} = Z_h^{p,q}/B_h^{p,q}$. Set $C_h^{p,q} = X^{p,q}/B_h^{p,q}$. We observe that $Z_h^{p,\bullet}$ and $B_h^{p,\bullet}$ are subcomplexes of the p-th column $X^{p,\bullet}$. Hence, we have the subquotient complex $H_h^{p,\bullet}$ and the quotient complex $C_h^{p,\bullet}$.

Lemma 4. *Assume that the double complex X has acyclic columns. Then all $Z_h^{p,\bullet}$ are acyclic if and only if all $B_h^{p,\bullet}$ are acyclic, if and only if all $C_h^{p,\bullet}$ are acyclic. In this situation, each $H_h^{p,\bullet}$ is acyclic.*

Proof. For the proof, we just apply the following three short exact sequences of complexes

$$0 \longrightarrow Z_h^{p,\bullet} \xrightarrow{\mathrm{inc}} X^{p,\bullet} \xrightarrow{\partial^{p,\bullet}} B_h^{p+1,\bullet} \longrightarrow 0,$$

and

$$0 \longrightarrow B_h^{p,\bullet} \xrightarrow{\mathrm{inc}} X^{p,\bullet} \longrightarrow C_h^{p,\bullet} \longrightarrow 0,$$

and
$$0 \longrightarrow B_h^{p,\bullet} \xrightarrow{\text{inc}} Z_h^{p,\bullet} \longrightarrow H_h^{p,\bullet} \longrightarrow 0.$$
Here, the morphisms $\partial^{p,q}$ are induced by $d_h^{p,q}$. □

We will assume that the abelian category $\mathcal{A}$ has countable coproducts. Then for any double complex X, its total complex $\mathrm{tot}^{\amalg}(X)$ is defined such that $\mathrm{tot}^{\amalg}(X)^n = \coprod_{p+q=n} X^{p,q}$.

Recall that an infinite sequence of morphisms
$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} A_4 \to \cdots$$
yields a direct system induced by $\mathbb{N}$. The morphisms ϕ_i are called the structure morphisms.

Assumption (†): Assume that $\mathcal{A}$ has countable coproducts. For any exact sequence
$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$$
of direct systems indexed by $\mathbb{N}$ with injective structures morphisms, the induced sequence
$$0 \longrightarrow \mathrm{colim} A_i \longrightarrow \mathrm{colim} B_i \longrightarrow \mathrm{colim} C_i \longrightarrow 0$$
is required to be always exact.

We observe that the sequence of colimits is always right exact. Hence, the assumption really requires the injectivity of the morphism $\mathrm{colim} A_i \to \mathrm{colim} B_i$.

We mention that any Grothendieck category and the opposite category of a module category satisfy assumption (†), where in the latter example, the assumption follows from the exactness of inverse limits under the Mittag-Leffler condition.

Proposition 5. *We assume that $\mathcal{A}$ satisfies assumption (†). Assume that X is a double complex with acyclic columns such that each $Z_h^{p,\bullet}$ is also acyclic. Then the total complex $\mathrm{tot}^{\amalg}(X)$ is acyclic.*

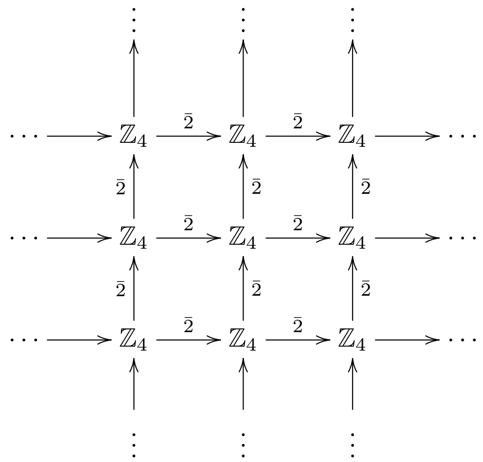
Proof. For each $k \geq 1$, we denote by X_k the following sub-double complex of X
$$0 \to X^{-k,\bullet} \to X^{1-k,\bullet} \to \cdots \to X^{k-1,\bullet} \to Z_h^{k,\bullet} \to 0.$$
Here, we identify double complexes with complexes in $C(\mathcal{A})$; see Lemma 1. Since each columns of X_k is acyclic, by Proposition 3 $\mathrm{tot}(X_k)$ is acyclic. We observe that $\{\mathrm{tot}(X_k)\}_{k\geq 1}$ is an ascending chain of sub-complexes of $\mathrm{tot}^{\amalg}(X)$. Moreover, we have $\mathrm{colim}_{k\geq 1}\, \mathrm{tot}(X_k) \simeq \mathrm{tot}^{\amalg}(X)$. By assumption (†), the sequence $\mathrm{tot}^{\amalg}(X)$, as the colimit of acyclic complexes with injective structure morphisms, is acyclic. □

We assume now that $\mathcal{A}$ has both countable coproducts and products. For a double complex X, we have the total complexes $\mathrm{tot}^{\amalg}(X)$ and $\mathrm{tot}^{\Pi}(X)$, where $\mathrm{tot}^{\Pi}(X)$ is given by $\mathrm{tot}^{\Pi}(X)^n = \prod_{p+q=n} X^{p,q}$. Moreover, there is a canonical cochain morphism
$$\mathrm{can}\colon \mathrm{tot}^{\amalg}(X) \longrightarrow \mathrm{tot}^{\Pi}(X),$$
which is in general not a quasi-isomorphism.

We mention that the total complex $\mathrm{tot}^{\Pi}(X)$ might be viewed as the total complex of the double complex X in the opposite category $\mathcal{A}^{\mathrm{op}}$. More precisely, the double complex X induces a double complex X^{op} in $\mathcal{A}^{\mathrm{op}}$, where $(X^{\mathrm{op}})^{p,q} = X^{-p,-q}$, the horizontal differential $(X^{\mathrm{op}})^{p,q} \to (X^{\mathrm{op}})^{p+1,q}$ is given by $d_h^{-p-1,-q}$ and the vertical differential $(X^{\mathrm{op}})^{p,q} \to (X^{\mathrm{op}})^{p,q+1}$ is given by $d_v^{-p,-q-1}$. Then we have
$$\mathrm{tot}^{\amalg}(X^{\mathrm{op}}) = \mathrm{tot}^{\Pi}(X)^{\mathrm{op}}.$$

Here, we identify $C(\mathcal{A})^{\rm op}$ with $C(\mathcal{A}^{\rm op})$. Indeed, a complex Y in $\mathcal{A}$ gives rise to a complex $Y^{\rm op}$ in $\mathcal{A}^{\rm op}$ such that $(Y^{\rm op})^n = Y^{-n}$ and the differential $(Y^{\rm op})^n \to (Y^{\rm op})^{n+1}$ is given by the differential d_Y^{-n-1} of Y.

Example 6. We assume that $\mathcal{A} = \mathbb{Z}_4$-Mod is the abelian category of left $\mathbb{Z}_4$-modules. We consider the following double complex X, whose differentials are given by the multiplication of $\bar{2}$.



The columns and rows are acyclic. However, none of $\mathrm{tot}^{\amalg}(X)$ and $\mathrm{tot}^{\Pi}(X)$ is acyclic, and the canonical morphism can: $\mathrm{tot}^{\amalg}(X) \to \mathrm{tot}^{\Pi}(X)$, which is an injective cochain morphism, is not a quasi-isomorphism.

We denote by Y the sub-double complex X obtained by all the columns of non-negative degrees, and by Z the corresponding quotient double complex. We observe that $\mathrm{tot}^{\amalg}(Y)$ is not acyclic but $\mathrm{tot}^{\Pi}(Y)$ is acyclic. On the other hand, $\mathrm{tot}^{\amalg}(Z)$ is acyclic but $\mathrm{tot}^{\Pi}(Z)$ is not acyclic.

Acknowledgements. We thank Jian Liu for Example 6. Proposition 5 is used to construct the dg-projective and dg-injective resolutions for dg modules. The treatment here avoids spectral sequences. However, we point out that spectral sequences provide more useful information for the total complexes of a general double complex.

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