ADJOINT PAIRS AND CANONICAL MORPHISMS FOR BIMODULES

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Abstract. It is well known that bimodules, rather than one-sided modules, play a central role in algebra. The main reason is that bimodules give rise to adjoint pairs between module categories. We collect the adjoint pairs and canonical morphisms associated to a given bimodule. The composition of these adjoint pairs is studied.

For a unital ring R, we denote by R-Mod and Mod-R the category of left R-modules and the category of right R-modules, respectively. We denote by Hom_R the Hom group in R-Mod and by $\mathrm{Hom}_{\text{-}R}$ the one in Mod-R. We identify right R-modules as left R^{op}-modules, where R^{op} is the opposite ring of R. A left R-module X is indicated by ${}_RX$ and a right R-module Y is indicated by Y_R.

Let R and S be two unital rings. We fix an R-S-bimodule ${}_RM_S$. We will also view M as an S^{op}-R^{op}-bimodule. We denote by $\mathrm{Hom}_{R\text{-}S}$ the Hom group in the category of R-S-bimodules.

Lemma 1. *For ${}_SX$ and ${}_RY$, the canonical isomorphism*

$$\mathrm{Hom}_R(M\otimes_S X, Y)\xrightarrow{\sim}\mathrm{Hom}_S(X,\mathrm{Hom}_R(M,Y)) \tag{0.1}$$

sends f to $(x\mapsto (m\mapsto f(m\otimes x)))$. In other words, we have an adjoint pair

$$S\text{-Mod}\ \underset{\mathrm{Hom}_R(M,-)}{\overset{M\otimes_S -}{\rightleftarrows}}\ R\text{-Mod}.$$

The unit of the adjoint pair is given by

$$X\longrightarrow \mathrm{Hom}_R(M, M\otimes_R X),\quad x\mapsto (m\mapsto m\otimes x),$$

while the counit is given by

$$M\otimes_S \mathrm{Hom}_R(M,Y)\longrightarrow Y,\quad m\otimes g\mapsto g(m).$$

Remark 2. For the given ${}_SX$ and ${}_RY$, the group $\mathrm{Hom}_{\mathbb{Z}}(X,Y)$ is naturally an R-S-bimodule. In view of (0.1), the following isomorphism is of interest

$$\mathrm{Hom}_R(M\otimes_S X, Y)\xrightarrow{\sim}\mathrm{Hom}_{R\text{-}S}(M,\mathrm{Hom}_{\mathbb{Z}}(X,Y)),\quad f\mapsto (m\mapsto (x\mapsto f(m\otimes x))).$$

We observe that the next lemma might be obtained by applying Lemma 1 for the bimodule ${}_{S^{\mathrm{op}}}M_{R^{\mathrm{op}}}$.

Lemma 3. *For Z_R and W_S, the canonical isomorphism*

$$\mathrm{Hom}_{\text{-}S}(Z\otimes_R M, W)\xrightarrow{\sim}\mathrm{Hom}_{\text{-}R}(Z,\mathrm{Hom}_{\text{-}S}(M,W))$$

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sends f to $(z \mapsto (m \mapsto f(z \otimes m)))$. In other words, we have an adjoint pair

$$\mathrm{Mod}\text{-}R \underset{\mathrm{Hom}_{\text{-}S}(M,-)}{\overset{-\otimes_R M}{\rightleftarrows}} \mathrm{Mod}\text{-}S.$$

The unit of the adjoint pair is given by

$$Z \longrightarrow \mathrm{Hom}_{\text{-}S}(M, Z \otimes_R M), \quad z \mapsto (m \mapsto z \otimes m),$$

while the counit is given by

$$\mathrm{Hom}_{\text{-}S}(M, W) \otimes_R M \longrightarrow W, \quad g \otimes m \mapsto g(m).$$

Remark 4. Similar to Remark 2, we have the following isomorphism

$$\mathrm{Hom}_{\text{-}S}(Z \otimes_R M, W) \xrightarrow{\sim} \mathrm{Hom}_{R\text{-}S}(M, \mathrm{Hom}_{\mathbb{Z}}(Z, W)), \quad f \mapsto (m \mapsto (z \mapsto f(z \otimes m))).$$

The following contravariant adjoint pair is less well known. The opposite category of a category $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathrm{op}}$.

Lemma 5. *For ${}_RY$ and W_S, the canonical isomorphism*

$$\mathrm{Hom}_{\text{-}S}(W, \mathrm{Hom}_R(Y, M)) \xrightarrow{\sim} \mathrm{Hom}_R(Y, \mathrm{Hom}_{\text{-}S}(W, M)) \tag{0.2}$$

sends g to $(y \mapsto (w \mapsto g(w)(y)))$. In other words, we have an adjoint pair

$$R\text{-}\mathrm{Mod} \underset{\mathrm{Hom}_{\text{-}S}(-,M)}{\overset{\mathrm{Hom}_R(-,M)}{\rightleftarrows}} (\mathrm{Mod}\text{-}S)^{\mathrm{op}}.$$

The unit is given by

$$Y \longrightarrow \mathrm{Hom}_{\text{-}S}(\mathrm{Hom}_R(Y, M), M), \quad y \mapsto (f \mapsto f(y)),$$

while the counit is given by the following homomorphism in Mod-S

$$W \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_{\text{-}S}(W, M), M), \quad w \mapsto (g \mapsto g(w)).$$

Remark 6. For the given ${}_RY$ and W_S, we have a natural R-S-bimodule $Y \otimes_{\mathbb{Z}} W$. In view of (0.2), we have the following natural isomorphism

$$\mathrm{Hom}_{\text{-}S}(W, \mathrm{Hom}_R(Y, M)) \xrightarrow{\sim} \mathrm{Hom}_{R\text{-}S}(Y \otimes_{\mathbb{Z}} W, M), \quad f \mapsto (y \otimes w \mapsto f(w)(y)).$$

In what follows, we study the composition of these functors.

For the bimodule ${}_RM_S$, $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$ is naturally an S-bimodule and $\mathrm{End}_{\text{-}S}(M) = \mathrm{Hom}_{\text{-}S}(M, M)$ is naturally an R-bimodule. Indeed, the S-bimodule structure on $\mathrm{End}_R(M)$ is induced by the ring homomorphism $S^{\mathrm{op}} \to \mathrm{End}_R(M)$. Similarly, the R-bimodule structure on $\mathrm{End}_{\text{-}S}(M)$ is induced by the ring homomorphism $R \to \mathrm{End}_{\text{-}S}(M)$.

The following observation seems to be of independent interest.

Lemma 7. *There is an isomorphism of S-R-bimodules*

$$\mathrm{Hom}_R(M, \mathrm{End}_{\text{-}S}(M)) \xrightarrow{\sim} \mathrm{Hom}_{\text{-}S}(M, \mathrm{End}_R(M)), \quad f \mapsto (m \mapsto (m' \mapsto f(m')(m))).$$

We might call the above common S-R-bimodule *the dual bimodule* of M, which will be denoted by $M^{\vee}$.

Proof. The isomorphism follows from (0.2). We observe that the isomorphism respects the S-R-bimodule structures. $\square$

Lemma 8. *For $_SX$ and W_S, we have canonical isomorphisms of S-modules*

$$\mathrm{Hom}_R(M \otimes_S X, M) \xrightarrow{\sim} \mathrm{Hom}_S(X, \mathrm{End}_R(M)), f \mapsto (x \mapsto (m \mapsto f(m \otimes x))),$$

and

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\text{-}S}(W, M)) \xrightarrow{\sim} \mathrm{Hom}_{\text{-}S}(W, \mathrm{End}_R(M)), g \mapsto (w \mapsto (m \mapsto g(m)(w))).$$

In other words, we have natural isomorphisms between functors

$$\mathrm{Hom}_R(-, M) \circ M \otimes_S - \xrightarrow{\sim} \mathrm{Hom}_S(-, \mathrm{End}_R(M)),$$

and

$$\mathrm{Hom}_R(M, -) \circ \mathrm{Hom}_{\text{-}S}(-, M) \xrightarrow{\sim} \mathrm{Hom}_{\text{-}S}(-, \mathrm{End}_R(M)).$$

Similarly, we have the following result.

Lemma 9. *For $_RY$ and Z_R, we have canonical isomorphisms of R-modules*

$$\mathrm{Hom}_{\text{-}S}(Z \otimes_R M, M) \xrightarrow{\sim} \mathrm{Hom}_{\text{-}R}(Z, \mathrm{End}_{\text{-}S}(M)), \quad f \mapsto (z \mapsto (m \mapsto f(z \otimes m))),$$

and

$$\mathrm{Hom}_{\text{-}S}(M, \mathrm{Hom}_R(Y, M)) \xrightarrow{\sim} \mathrm{Hom}_R(Y, \mathrm{End}_{\text{-}S}(M)), \quad g \mapsto (y \mapsto (m \mapsto g(m)(y))).$$

In other words, we have natural isomorphisms between functors

$$\mathrm{Hom}_{\text{-}S}(-, M) \circ - \otimes_R M \xrightarrow{\sim} \mathrm{Hom}_{\text{-}R}(-, \mathrm{End}_{\text{-}S}(M))$$

and

$$\mathrm{Hom}_{\text{-}S}(M, -) \circ \mathrm{Hom}_R(-, M) \xrightarrow{\sim} \mathrm{Hom}_R(-, \mathrm{End}_{\text{-}S}(M)).$$

The following general fact is needed.

Lemma 10. *Let T be another ring. Assume that $_SA_T$ and $_RB_T$ are an S-T-bimodule and an R-T-bimodule, respectively. For each $_SX$, there is a canonical isomorphism of right R-modules*

$$\mathrm{Hom}_{\text{-}T}(B, \mathrm{Hom}_S(X, A)) \xrightarrow{\sim} \mathrm{Hom}_S(X, \mathrm{Hom}_{\text{-}T}(B, A))$$

sending f to $(x \mapsto (b \mapsto f(b)(x)))$. In other words, we have a natural isomorphism between functors

$$\mathrm{Hom}_{\text{-}T}(B, -) \circ \mathrm{Hom}_S(-, A) \xrightarrow{\sim} \mathrm{Hom}_S(-, \mathrm{Hom}_{\text{-}T}(B, A)).$$

The following result seems to be not well known.

Proposition 11. *Let $_RM_S$ be the R-S-bimodule as above and $_SM^\vee{}_R$ its dual bimodule. Then there are two commutative diagrams*

$$\begin{array}{ccc}
S\text{-Mod} & \xrightarrow{M \otimes_S -} & R\text{-Mod} \\
{\scriptstyle \mathrm{Hom}_S(-, M^\vee)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_R(-, M)} \\
(\text{Mod-}R)^{\mathrm{op}} & \xleftarrow[\mathrm{Hom}_{\text{-}S}(M, -)]{} & (\text{Mod-}S)^{\mathrm{op}}.
\end{array}$$

and

$$\begin{array}{ccc}
(\text{Mod-}R)^{\mathrm{op}} & \xrightarrow{- \otimes_R M} & (\text{Mod-}S)^{\mathrm{op}} \\
{\scriptstyle \mathrm{Hom}_{\text{-}R}(-, M^\vee)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_{\text{-}S}(-, M)} \\
S\text{-Mod} & \xleftarrow[\mathrm{Hom}_R(M, -)]{} & R\text{-Mod}.
\end{array}$$

Proof. For the first commutative diagram, we just combine the isomorphisms

$$\mathrm{Hom}_R(-, M) \circ M \otimes_S - \xrightarrow{\sim} \mathrm{Hom}_S(-, \mathrm{End}_R(M))$$

and

$$\mathrm{Hom}_{\text{-}S}(M, -) \circ \mathrm{Hom}_S(-, \mathrm{End}_R(M)) \xrightarrow{\sim} \mathrm{Hom}_S(-, \mathrm{Hom}_{\text{-}S}(M, \mathrm{End}_R(M))).$$

We refer to Lemmas 8 and 10 for these isomorphisms. The second diagram is proved in a similar way, which might be deduced by the composition of the above three adjoint pairs. □

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