THE TORSION SUBGROUP OF AN ABELIAN GROUP

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Abstract. We give a detailed proof on an explicit presentation by generators and relations for the torsion subgroup of an abelian group, which includes the Picard group of a weighted projective line.

Let $\mathbf{p} = (p_1, p_2, \cdots, p_t)$ and $\mathbf{d} = (d_1, d_2, \cdots, d_t)$ be two sequences of positive integers with $t \geq 1$. Let $L = L(\mathbf{p}, \mathbf{d})$ be an abelian group defined by generators $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_t, \vec{c}$ subject to the relations $p_i\vec{x}_i = d_i\vec{c}$ for $1 \leq i \leq t$. We denote by $t(L)$ its torsion subgroup.

Set $p = \mathrm{lcm}(p_1, p_2, \cdots, p_t)$ and $d = \gcd(\frac{pd_1}{p_1}, \frac{pd_2}{p_2}, \cdots, \frac{pd_t}{p_t})$, where lcm and gcd denote the least common multiple and the greatest common divisor, respectively.

The main goal is to provide a detailed and elementary proof for the following result.

Proposition 1. *Let $L = L(\mathbf{p}, \mathbf{d})$ be as above. Then the following statements hold.*

(1) *The cardinality of $t(L)$ equals $\frac{p_1p_2\cdots p_t}{p}\gcd(p, d)$.*
(2) $\gcd(p, d) = 1$ *if and only if $t(L)$ is isomorphic to the abelian group with generators $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_t$ and relations $p_1\vec{x}_1 = p_2\vec{x}_2 = \cdots = p_t\vec{x}_t = \vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_t = 0$.*

We comment on the special case with each $d_i = 1$, in which case the group L is isomorphic to the Picard group of a weighted projective line; see [2, Proposition 2.1]. Since $d = 1$, (2) yields an explicit presentation for the torsion subgroup $t(L)$ by generators and relations. Indeed, this presentation is implicitly stated in [4, p.1693]. For a general sequence $\mathbf{d}$, the group $L = L(\mathbf{p}, \mathbf{d})$ appears in the study of a canonical algebra over an arbitrary field; see [3, Section 12].

We denote by $\mathbb{Z}_n$ the additive group of integers modulo n. Consider the cyclic subgroup $\mathbb{Z}\vec{c}$ of L generated by $\vec{c}$. We observe an isomorphism of groups

$$\pi\colon L/\mathbb{Z}\vec{c} \stackrel{\sim}{\longrightarrow} \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \mathbb{Z}_{p_t},$$

sending $\vec{x}_i$ to the unit row vector ε_i with $\bar{1}$ at the i-th coordinate. Here, we use the fact that the quotient group $L/\mathbb{Z}\vec{c}$ is generated by the same generators of L but with the relations $p_1\vec{x}_1 = p_2\vec{x}_2 = \cdots = p_t\vec{x}_t = 0 = \vec{c}$.

The following homomorphism of groups

$$\delta\colon L \longrightarrow \mathbb{Z}, \quad \vec{x}_i \mapsto \frac{pd_i}{p_i}, \vec{c} \mapsto p$$

is well defined. Set $a = \gcd(p, d)$. Then the image of δ is $\mathbb{Z}a$.

Lemma 2. *We have $p\vec{x} = \delta(\vec{x})\vec{c}$ for each $\vec{x} \in L$. Consequently, we have $t(L) = \mathrm{Ker}\,\delta$.*

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Proof. For the first identity, using the additivity it suffices to prove it for each generator $\vec{x}_i$ and $\vec{c}$, which is trivial.

We observe that δ sends torsion elements to zero, thus $t(L) \subseteq \mathrm{Ker}\ \delta$. On the other hand, each element $\vec{x}$ in Ker δ satisfies $p\vec{x} = 0$, thus is torsion. □

We denote by H the abelian group with generators $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_t$ and relations $p_1\vec{x}_1 = p_2\vec{x}_2 = \cdots = p_t\vec{x}_t = \vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_t = 0$. Then H is isomorphic to the quotient group of $L/\mathbb{Z}\vec{c}$ by the cyclic subgroup generated by $\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_t$.

Using the isomorphism π, we infer that H is isomorphic to the quotient group $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_t})/\mathbb{Z}(\bar{1}, \bar{1}, \cdots, \bar{1})$. Here, we use the fact that $\pi(\vec{x}_1+\vec{x}_2+\cdots+\vec{x}_t) = (\bar{1}, \bar{1}, \cdots, \bar{1})$. We observe that the orders of these two elements equal p. We infer by Lagrange's Theorem that the cardinality of H equals $\frac{p_1p_2\cdots p_t}{p}$.

Proof of Proposition 1(1). By Lemma 2, the homomorphism δ induces an isomorphism $\bar{\delta}\colon L/t(L) \xrightarrow{\sim} \mathbb{Z}a$, which restricts to an isomorphism $(\mathbb{Z}\vec{c} + t(L))/t(L) \simeq \mathbb{Z}p$. Therefore, we have an induced isomorphism $L/(\mathbb{Z}\vec{c} + t(L)) \simeq \mathbb{Z}a/\mathbb{Z}p$. In particular, the cardinality of $L/(\mathbb{Z}\vec{c} + t(L))$ is $\frac{p}{a}$.

By $\delta(\vec{c}) = p$, we infer that the element $\vec{c}$ is torsionless. Then we have $t(L) \cap \mathbb{Z}\vec{c} = 0$. Therefore, $t(L)$ is naturally viewed as a subgroup of $L/\mathbb{Z}\vec{c}$, and the corresponding quotient group is identified with $L/(\mathbb{Z}\vec{c} + t(L))$. Recall by the isomorphism π that the cardinality of $L/\mathbb{Z}\vec{c}$ equals $p_1p_2\cdots p_t$. It follows from Lagrange's Theorem that the cardinality of $t(L)$ equals $p_1p_2\cdots p_t\frac{a}{p}$. □

Recall that the image of δ equals $\mathbb{Z}a$. Take $\vec{y} \in L$ with $\delta(\vec{y}) = a$.

Lemma 3. *There is an isomorphism* Ker $\delta \simeq L/\mathbb{Z}\vec{y}$ *of groups.*

Proof. We observe $L = \mathrm{Ker}\ \delta \oplus \mathbb{Z}\vec{y}$. Then the isomorphism follows immediately. □

Lemma 4. *Assume that $a = 1$. Then we have $\mathbb{Z}\vec{c} \subseteq \mathbb{Z}\vec{y}$, and that the order of $\vec{y}$ in $L/\mathbb{Z}\vec{c}$ equals p.*

This observation allows us to identify $L/\mathbb{Z}\vec{y}$ with the quotient group of $L/\mathbb{Z}\vec{c}$ by the cyclic subgroup generated by $\vec{y}$ in $L/\mathbb{Z}\vec{c}$.

Proof. By Lemma 2, we have $p\vec{y} = \vec{c}$. Moreover, if $m\vec{y}$ lies in $\mathbb{Z}\vec{c}$, by applying δ we infer that p divides m. □

Proof of Proposition 1(2). We have already computed the cardinalities of H and $t(L)$. Then the "if" part follows.

For the "only if " part, we assume that $a = 1$. Combing the three lemmas above, we infer that $t(L)$ is isomorphic to the quotient group of $L/\mathbb{Z}\vec{c}$ by the cyclic subgroup generated by $\vec{y}$. Recall that H is isomorphic to the quotient group of $L/\mathbb{Z}\vec{c}$ by the cyclic subgroup generated by $\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_t$. The orders of both the elements $\vec{y}$ and $\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_t$ in $L/\mathbb{Z}\vec{c}$ are p. The following well-known lemma implies that $t(L)$ is isomorphic to H. Then we are done. □

Let p be a positive integer. We say that a finite abelian group A is p-torsion if the order of each element divides p.

Lemma 5. *Let A be a finite abelian group which is p-torsion. Assume that x and y are two elements in A with order p. Then there is an isomorphism $A/\mathbb{Z}x \simeq A/\mathbb{Z}y$ of groups.*

Proof. We identity p-torsion groups with finite $\mathbb{Z}_p$-modules. The ring $\mathbb{Z}_p$ is quasi-Frobenius, in particular, the regular module $\mathbb{Z}_p$ is injective; see [1, §30]. Both $\mathbb{Z}x$ and $\mathbb{Z}y$ are isomorphic to $\mathbb{Z}_p$, thus are injective $\mathbb{Z}_p$-modules. Therefore, both submodules of A are direct summands. Then we have isomorphisms $A \simeq \mathbb{Z}x \oplus A/\mathbb{Z}x \simeq$

$\mathbb{Z}y \oplus A/\mathbb{Z}y$. The required isomorphism follows from Krull-Schmidt Theorem; see [1, Theorem 12.9]. □

Remark 6. We assume that $\gcd(p, d) = 1$.

(1) We consider the following embedding of $t(L)$ in $L/\mathbb{Z}\vec{c}$

$$t(L) \xrightarrow{\text{inc}} L \xrightarrow{\text{can}} L/\mathbb{Z}\vec{c},$$

where "inc" denotes the inclusion and "can" denotes the canonical projection. Indeed, this identifies $t(L)$ as a direct summand of $L/\mathbb{Z}\vec{c}$. This statement follows from the decomposition $L = t(L) \oplus \mathbb{Z}\vec{y}$ and the inclusion $\mathbb{Z}\vec{c} \subseteq \mathbb{Z}\vec{y}$.

We mention that if $\gcd(p, d) \neq 1$, $t(L)$ might not be a direct summand of $L/\mathbb{Z}\vec{c}$. Take $\mathbf{p} = (4, 4)$ and $\mathbf{d} = (2, 2)$ for example.

(2) Proposition 1(2) implies that the group $t(L)$ is isomorphic to H. However, the following natural homomorphism

$$t(L) \xrightarrow{\text{inc}} L \xrightarrow{\text{can}} H$$

is in general not an isomorphism. Here, we identify H with the quotient group $L/(\mathbb{Z}\vec{c} + \mathbb{Z}(\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_t))$. For the non-isomorphism example, we may take $\mathbf{p} = (2, 2)$ and $\mathbf{d} = (1, 1)$.

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