THE SIGNS IN THE BAR RESOLUTION

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Abstract. We analyze the mysterious signs appearing in the bar resolution of a differential graded algebra.

Let k be a commutative ring. Let A be a dg unital algebra over k. The differential of A is denoted by d_A. In what follows, the unadorned tensors are over k.

For each $n \geq 0$, we always consider $A \otimes A^{\otimes n} \otimes A$ as a dg A-bimodule with the outer action. We identify $A \otimes A^{\otimes 0} \otimes A$ with $A \otimes A$. The following exact sequence of dg A-bimodules is well known:

$$\cdots \to A \otimes A^{\otimes n} \otimes A \xrightarrow{\partial} A \otimes A^{\otimes n-1} \otimes A \to \cdots \to A \otimes A \otimes A \xrightarrow{\partial} A \otimes A \xrightarrow{\mu} A \to 0.$$

Here, μ is the multiplication map of A and ∂ is given by an alternating sum

$$\partial(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}.$$

The above exact sequence implies that A is quasi-isomorphic[1] to the total complex

$$\mathbb{B}' = \bigoplus_{n \geq 0} \Sigma^n (A \otimes A^{\otimes n} \otimes A).$$

The typical element in $\Sigma^n(A \otimes A^{\otimes n} \otimes A)$ will be written as

$$s^n(a_{0,n+1}) = s^n(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}),$$

whose degree is $\sum_{i=0}^{n+1}(|a_i| - 1)$. The differential d' of $\mathbb{B}'$ equals $d'_{\rm in} + d'_{\rm ex}$, where the internal differential $d'_{\rm in}$ is the original differential of the shifted tensor product $\Sigma^n(A \otimes A^{\otimes n} \otimes A)$, and the external one is induced by ∂. More precisely, we have

$$d'_{\rm in} s^n(a_{0,n+1}) = \sum_{i=0}^{n+1} (-1)^{n + \sum_{j=0}^{i-1} |a_j|} \; s^n(a_{0,i-1} \otimes d_A(a_i) \otimes a_{i+1,n}),$$

and

$$d'_{\rm ex} s^n(a_{0,n+1}) = \sum_{i=0}^{n} (-1)^i \; s^{n-1}(a_{0,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,n+1}).$$

In what follows, we write $\Sigma(A)$ as sA; it is a more convenient notation. For each $n \geq 0$, there is an canonical isomorphism of dg A-bimodules.

$$\begin{aligned} \phi_n \colon \Sigma^n(A \otimes A^{\otimes n} \otimes A) &\longrightarrow A \otimes (sA)^{\otimes n} \otimes A \\ s^n(a_{0,n+1}) &\longmapsto (-1)^{\sum_{i=0}^{n}(n-i)|a_i|} a_0 \otimes sa_{1,n} \otimes a_{n+1} \end{aligned}$$

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[1] This reasoning here is not quite right. We observe that the sections in proving the exactness are compatible with internal differentials. Therefore, the induced sequence on cocycles is also exact. Then we apply [USTC Algebra Notes, No. **5**, Proposition 5].

Here, $sa_{1,n} = sa_1 \otimes sa_2 \otimes \cdots \otimes sa_n$ and the sign is due to the Koszul sign rule. We understand ϕ_0 as the identity map.

There is a unique map d_{ex} making the following diagram commute.

$$\begin{array}{ccc} \Sigma^n(A \otimes A^{\otimes n} \otimes A) & \xrightarrow{\phi_n} & A \otimes (sA)^{\otimes n} \otimes A \\ \downarrow{\scriptstyle d'_{\text{ex}}} & & \vdots{\scriptstyle d_{\text{ex}}} \\ \Sigma^{n-1}(A \otimes A^{\otimes n-1} \otimes A) & \xrightarrow{\phi_{n-1}} & A \otimes (sA)^{\otimes n-1} \otimes A \end{array}$$

It is a good exercise to show that d_{ex} is given by the following formula.

$$d_{\text{ex}}(a_0 \otimes sa_{1,n} \otimes a_{n+1}) = (-1)^{|a_0|}\{a_0a_1 \otimes sa_{2,n} \otimes a_{n+1} +$$
$$\sum_{i=1}^{n-1}(-1)^{\epsilon_i} a_0 \otimes sa_{1,i-1} \otimes s(a_ia_{i+1}) \otimes sa_{i+2,n} \otimes a_{n+1} - (-1)^{\epsilon_{n-1}} a_0 \otimes sa_{1,n-1} \otimes a_na_{n+1}\}$$

Here, the rather mysterious signs are determined by $\epsilon_i = \sum_{j=1}^{i}(|a_j| - 1)$ for $1 \leq i \leq n$.

Following [1, Section 1], the *bar resolution* of A is defined as follows:

$$\mathbb{B} = \bigoplus_{n=0}^{\infty} A \otimes (sA)^{\otimes n} \otimes A$$

as a direct sum of graded A-A-bimodules, and its differential $d = d_{\text{in}} + d_{\text{ex}}$. The internal differential d_{in} is given by the ones of the tensor products $A \otimes (sA)^{\otimes n} \otimes A$ and the external differential d_{ex} is described as above. More explicitly, we have

$$d_{\text{in}}(a_0 \otimes sa_{1,n} \otimes a_{n+1}) = d_A(a_0) \otimes sa_{1,n} \otimes a_{n+1} +$$
$$(-1)^{|a_0|}\{\sum_{i=1}^{n}(-1)^{\epsilon_{i-1}+1} a_0 \otimes sa_{1,i-1} \otimes sd_A(a_i) \otimes sa_{i+1,n} \otimes a_{n+1} + (-1)^{\epsilon_n} a_0 \otimes sa_{1,n} \otimes d_A(a_{n+1})\}.$$

We observe that ϕ_n is compatible with the internal differentials. Then we infer an isomorphism of dg A-bimodules

$$\bigoplus_{n \geq 0} \phi_n \colon \mathbb{B}' \longrightarrow \mathbb{B}.$$

In comparison with $\mathbb{B}'$, the bar resolution $\mathbb{B}$ is more convenient.

Denote by δ' the differential of the Hom complex $\text{Hom}_{A^e}(\mathbb{B}, A)$. There is an isomorphism of graded spaces

$$C^*(A, A) = \prod_{n \geq 0} \text{Hom}((sA)^{\otimes n}, A) \longrightarrow \text{Hom}_{A^e}(\mathbb{B}, A), f \longmapsto \tilde{f},$$

which sends $f \in C^{*,n}(A, A) = \text{Hom}((sA)^{\otimes n}, A)$ to $\tilde{f} \colon \mathbb{B} \to A$ given by $1 \otimes sa_{1,m} \otimes 1 \mapsto \delta_{n,m} f(sa_{1,n})$. This isomorphism transfers δ' to a differential on $C^*(A, A)$, denoted by $\delta = \delta_{\text{in}} + \delta_{\text{ex}}$. More precisely, $\delta_{\text{in}}(f)$ lies in $C^{*,n}(A, A)$ and is given by

$$\begin{aligned} \delta_{\text{in}}(f)(sa_{1,n}) &= \delta'_{\text{in}}(\tilde{f})(1 \otimes sa_{1,n} \otimes 1) \\ &= d_A f(sa_{1,n}) - (-1)^{|f|} \tilde{f}(d_{\text{in}}(1 \otimes sa_{1,n} \otimes 1)) \\ &= d_A f(sa_{1,n}) + \sum_{i=1}^{n}(-1)^{|f|+\epsilon_{i-1}} f(sa_{1,i-1} \otimes sd_A(a_i) \otimes sa_{i+1,n}). \end{aligned}$$

We have that $\delta_{\rm ex}(f)$ lies in $C^{*,n+1}(A,A)$ and is given by

$$\begin{aligned}\delta_{\rm ex}(f)(sa_{1,n+1}) &= -(-1)^{|f|}\, \tilde{f}(d_{\rm ex}(1\otimes a_{1,n+1}\otimes 1))\\ &= -(-1)^{\epsilon_1|f|}a_1 f(sa_{2,n+1}) + (-1)^{|f|+\epsilon_n} f(sa_{1,n})a_{n+1}\\ &\quad + \sum_{i=1}^{n}(-1)^{|f|+\epsilon_i+1} f(sa_{1,i-1}\otimes s(a_ia_{i+1})\otimes sa_{i+2,n+1}).\end{aligned}$$

Here, we use the fact that $\tilde{f}(av) = (-1)^{|a|\cdot|f|}af(v)$ for any $a\in A$ and $v \in A\otimes (sA)^{\otimes n}\otimes A$. Then we obtain the *Hochschild cochain complex* $(C^*(A,A),\delta)$ of A.

References

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