THE RELATIVE AUSLANDER FORMULA

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ABSTRACT. We describe the main ingredients in the proof of the relative Auslander formula.

Let $\mathcal{A}$ be an abelian category and $\mathcal{C} \subseteq \mathcal{A}$ a full additive subcategory. Denote by mod-$\mathcal{C}$ the category of finitely presented contravariant additive functors from $\mathcal{C}$ to the category Ab of abelian groups. Here, we recall that a contravariant additive functor $F\colon \mathcal{C} \to \mathrm{Ab}$ is *finitely presented*, if there is an exact sequence of functors

$$\mathcal{C}(-, X_1) \stackrel{\mathcal{C}(-,f)}{\longrightarrow} \mathcal{C}(-, X_0) \longrightarrow F \longrightarrow 0$$

with $X_i \in \mathcal{C}$. The morphism $f\colon X_1 \to X_0$ is called a *representing morphism* of F. Such a functor F is called *effaceable* if its representing morphism can be chosen to be epic in $\mathcal{A}$. Denote by eff-$\mathcal{C}$ the full subcategory formed by effaceable functors.

Recall that mod-$\mathcal{C}$ is an abelian category if and only if $\mathcal{C}$ has weak kernels. We mention that the definition of effaceable functors depends on the embedding $\mathcal{C} \hookrightarrow \mathcal{A}$.

Consider the following additive functor

$$\Theta\colon \text{mod-}\mathcal{C} \longrightarrow \mathcal{A},$$

which sends F to $\mathrm{Cok}(f)$, the cokernel of the representing morphism f of F in $\mathcal{A}$. The functor Θ is right exact, whose essential kernel equals eff-$\mathcal{C}$.

Remark 1. We use the existence of the functor Θ above to infer the following fact: a functor F in mod-$\mathcal{C}$ is effaceable if and only if all its representing morphisms are epic in $\mathcal{A}$.

The subcategory $\mathcal{C}$ is said to *generate* $\mathcal{A}$ if each object A in $\mathcal{A}$ admits an epimorphism $X \to A$ with some $X \in \mathcal{C}$.

The following result, known as the relative Auslander formula, is due to [3, Proposition 3.4]. The original Auslander formula, dealing with the case $\mathcal{C} = \mathcal{A}$, is found in [1, p.205]; see also [2, Proposition 2.3.3].

Theorem 2. *Assume that the subcategory $\mathcal{C}$ has weak kernels and generates $\mathcal{A}$. Then the functor Θ above is exact and induces an equivalence*

$$\overline{\Theta}\colon \frac{\text{mod-}\mathcal{C}}{\text{eff-}\mathcal{C}} \stackrel{\sim}{\longrightarrow} \mathcal{A}.$$

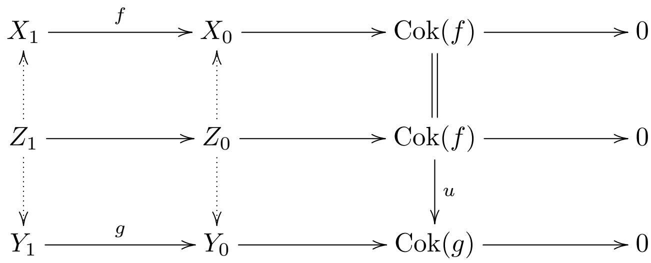
Proof. The assumption implies that the inclusion $\mathcal{C} \hookrightarrow \mathcal{A}$ is weakly left exact. It follows from [2, Lemma 2.1.8] that Θ is exact. Consequently, eff-$\mathcal{C}$ is a Serre subcategory of mod-$\mathcal{C}$.

Since $\mathcal{C}$ generates $\mathcal{A}$, the functor Θ and thus the induced one $\overline{\Theta}$ are dense. Recall a standard fact: any exact functor between two abelian categories, which is full, dense and faithful on objects, is necessarily an equivalence. Since $\overline{\Theta}$ is faithful on

Date: April 29, 2026.
2010 *Mathematics Subject Classification.* 18E35.
Key words and phrases. finitely presented functor, Auslander formula, Serre subcategory.
This paper belongs to a series of informal notes, without claim of originality.

objects, it suffices to show that it is full. This follows immediately from the lemma below, since morphisms in the quotient category are represented by fractions. □

Lemma 3. *Let $\mathcal{C} \subseteq \mathcal{A}$ be a full additive subcategory which generates $\mathcal{A}$. Suppose that $f\colon X_1 \to X_0$ and $g\colon Y_1 \to Y_0$ are two morphisms in $\mathcal{C}$, and that $u\colon \mathrm{Cok}(f) \to \mathrm{Cok}(g)$ is any morphism between their cokernels in $\mathcal{A}$. Then there exists a morphism $Z_1 \to Z_0$ in $\mathcal{C}$ fitting into the following commutative diagram with exact rows in $\mathcal{A}$.*



Proof. This is an elementary exercise in homologial algebra, using pullbacks in $\mathcal{A}$ and the generating assumption on $\mathcal{C}$. □

References

[1] M. Auslander, *Coherent functors*, in: Proc. Conf. Categorical Algebra, 189–231, Springer, New York, 1966.

[2] H. Krause, Homological Theory of Representations, Cambridge Stud. Adv. Math. **195**, Cambridge Univ. Press, Cambridge, 2022.

[3] H. Krause, *Serre's theorem for coherent sheaves via Auslander's techniques*, J. Noncommut. Geom. **20** (2026), 209–220.

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