THE RADICAL VECTORS OF CARTAN MATRICES

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Let $C = (a_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{Z})$ be a symmetric Cartan matrix. We identify C with its diagram Γ. We will assume that C is indecomposable, or equivalently, its diagram Γ is connected.

Consider the root lattice $\mathbb{Z}^n = \bigoplus_{1\leq i\leq n} \mathbb{Z}\mathbf{e}_i$. The Tits form

$$(-,-)\colon \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$$

is a symmetric bilinear form given by $(\mathbf{e}_i, \mathbf{e}_j) = a_{ij}$. The *radical* of the Tits form is given by

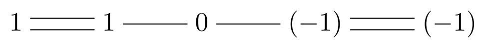
$$\mathrm{rad}(-,-) = \{\alpha \in \mathbb{Z}^n \mid (\alpha, -) = 0\}.$$

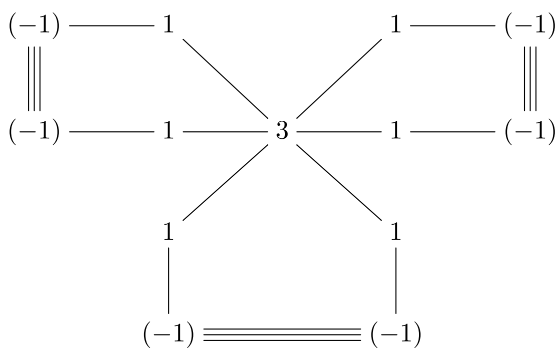
Nonzero vectors in $\mathrm{rad}(-,-)$ are called *radical vectors* of the Cartan matrix C.

Lemma 1. *Let $\alpha = \sum_{i=1}^n a_i\mathbf{e}_i$ be a nonzero vector. Then α is radical if and only if $2a_i = \sum_{j\neq i} |a_{ij}|a_j$ for each $1 \leq i \leq n$.*

Proof. We just observe that $(\alpha, \mathbf{e}_i) = 2a_i + \sum_{j\neq i} a_{ij}a_i$ and that $a_{ij} \leq 0$ for $i \neq j$. □

Example 2. *The following two diagrams correspond to two Cartan matrices. We identify any vector $\sum_{i=1}^n a_i\mathbf{e}_i$ as a map, whose value on the i-th vertex is a_i. We write the value a_i on the i-th vertex. Then the following vectors are both radical.*





In the above examples, the radical vectors have alternating signs. In contrast, the following fact is remarkable; see [1, Section 2].

Proposition 3. *Let C be an indecomposable Cartan matrix with a radical vector $\sum_i a_i\mathbf{e}_i$ such that $a_i \geq 0$ for each i. Then $a_i > 0$ for each i, and C is of affine type. In other words, the diagram Γ is of type $\tilde{A}_n, \tilde{D}_n$ and $\tilde{E}_{6,7,8}$.* □

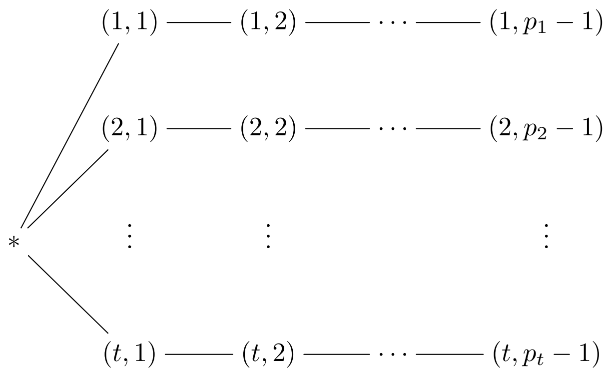
Date: November 2, 2019.

2010 *Mathematics Subject Classification.* 17B67, 15A63.

Key words and phrases. Cartan matrix, radical, stat-shaped graph.

This paper belongs to a series of informal notes, without claim of originality.

Let $t \geq 2$ and $\mathbf{p} = (\mathbf{p_1}, \mathbf{p_2}, \cdots, \mathbf{p_t})$ be a weight sequence such that $p_1 \geq p_2 \geq \cdots \geq p_t \geq 2$. The star-shaped graph $T_{\mathbf{p}}$ is of the following form



Lemma 4. *Consider the star-shaped graph $T_{\mathbf{p}}$. Then the corresponding Cartan matrix has a radical vector if and only if* $\mathbf{p} = (6,3,2), (4,4,2), (3,3,3)$ *or* $(2,2,2,2)$.

Proof. The "if" part is well known, once we observe that $T_{(6,3,2)} = \tilde{E}_8$, $T_{(4,4,2)} = \tilde{E}_7$, $T_{(3,3,3)} = \tilde{E}_6$ and $T_{(2,2,2,2)} = \tilde{D}_4$.

For the "only if" part, we take a radical vector $a_*\mathbf{e}_* + \sum_{i=1}^{t}\sum_{j=1}^{p_i-1} a_{(i,j)}\mathbf{e}_{(i,j)}$. By Lemma 1, we infer that $a_{(i,j)} = ja_{(i,1)}$ and $a_* = p_i a_{(i,1)}$ for any i, j. Therefore, we infer that each $a_{(i,j)}$ is nonzero and of the same sign. The identity $2a_* = \sum_{i=1}^{t} a_{(i,1)}$ yields

$$2 = \sum_{i=1}^{t} \frac{1}{p_i}.$$

The solutions of the above equation are well known. □

Acknowledgements. We thank Xueqing Chen for helpful comments.

References

[1] S. Berman, R. Moody, and M. Wonenburger, *Cartan matrices with null roots and finite Cartan matrices*, Indiana Univ. Math. J. **21** (12) (1972), 1091–1099.

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