THE EXTENSION-LIFTING LEMMA VIA TWO-TERM COMPLEXES

XIAO-WU CHEN

Abstract. We give a new proof to the well-known Extension-Lifting Lemma using an observation on two-term complexes.

Let $\mathcal{A}$ be an abelian category. The following well-known lemma is proved in [2, the proof of Proposition 4.2] and [1, Chapter VIII, Lemma 3.1], which plays a central role in establishing model structures on abelian categories. We emphasize that although abelian model categories are of interest, the most important model categories are not abelian, usually not even additive.

Lemma 1. (The Extension-Lifting Lemma) *Suppose that there is a commutative square in* $\mathcal{A}$

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{b} & Y \end{array}$$

with i a monomorphism and p an epimorphism. Assume that $\mathrm{Ext}^1_{\mathcal{A}}(\mathrm{Cok}\, i, \mathrm{Ker}\, p) = 0$*. Then there is a lifting $h\colon B \to X$ for the square, that is, $h \circ i = a$ and $p \circ h = b$.*

The new proof we are giving relies on an observation about two-term complexes. By a *two-term complex* $X^\bullet$ in $\mathcal{A}$ we mean a complex supported on degrees -1 and 0, that is, $X^n = 0$ for $n \neq -1, 0$. We denote by $\mathbf{K}^b(\mathcal{A})$ and $\mathbf{D}^b(\mathcal{A})$ the bounded homotopy category and bounded derived category of $\mathcal{A}$, respectively.

Lemma 2. *Let $X^\bullet$ and $Y^\bullet$ be two-term complexes in $\mathcal{A}$. Then the canonical map*

$$\mathrm{Hom}_{\mathbf{K}^b(\mathcal{A})}(X^\bullet, Y^\bullet) \longrightarrow \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(X^\bullet, Y^\bullet)$$

is injective.

Proof. Take a chain morphism $f^\bullet\colon X^\bullet \to Y^\bullet$. Recall that $f^\bullet$ is zero in $\mathbf{D}^b(\mathcal{A})$ if and only if it factors through an acyclic complex $Z^\bullet$ in $\mathbf{K}^b(\mathcal{A})$. The factorization $X^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{k^\bullet} Y^\bullet$ is visualized as follows.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & {\scriptstyle g^{-1}}\downarrow & & \downarrow{\scriptstyle g^0} & & \downarrow & & \\ \cdots & \longrightarrow & Z^{-2} & \xrightarrow{d_Z^{-2}} & Z^{-1} & \xrightarrow{d_Z^{-1}} & Z^0 & \xrightarrow{d_Z^0} & Z^1 & \longrightarrow & \cdots \\ & & \downarrow & & {\scriptstyle k^{-1}}\downarrow & & \downarrow{\scriptstyle k^0} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & Y^{-1} & \longrightarrow & Y^0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

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Consider the standard factorization $Z^{-1} \xrightarrow{x} I \xrightarrow{y} Z^0$ of the morphism d_Z^{-1}. Then x is the cokernel of d_Z^{-2} and y is the kernel of d_Z^0. Then k^{-1} factors through x and g^0 factors though y uniquely. Set $k^{-1} = h_2 \circ x$ and $g^0 = y \circ h_1$. Then $h = h_2 \circ h_1 \colon X^0 \to Y^{-1}$ yields the required homotopy. $\square$

Proof of Lemma 1. We view the morphisms i and p as two-term complexes in $\mathcal{A}$. Then the commutative square gives rise to a chain morphism $i \to p$. We observe that i is quasi-isomorphic to Cok i and p is quasi-isomorphic to $\Sigma(\text{Ker } p)$, the translation of p. Therefore, we have

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(i, p) \simeq \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\text{Cok } i, \Sigma(\text{Ker } p)) = 0,$$

where the rightmost equality follows from the Ext-vanishing assumption. Now, Lemma 2 implies that the given chain morphism is homotopic to zero, where the homotopy is just the lifting. $\square$

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Xiao-Wu Chen

Key Laboratory of Wu Wen-Tsun Mathematics, Chinese Academy of Sciences,

School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, P.R. China.

URL: http://home.ustc.edu.cn/~xwchen

E-mail: xwchen@mail.ustc.edu.cn.