THE AUSLANDER ALGEBRA OF THE TRUNCATED POLYNOMIAL ALGEBRA

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Let k be field and $p \geq 1$. Denote by $A = k[t]/(t^{p+1})$ the truncated polynomial algebra. Denote by A-mod the category of finite dimensional left A-modules. Then $E = \bigoplus_{i=1}^{p+1} k[t]/(t^i)$ is an additive generator for A-mod.

Consider the Auslander algebra $\Lambda = \mathrm{End}_A(E)$. It is well known that Λ is given by the following quiver

$$1 \underset{b_1}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_2}{\overset{a_2}{\rightleftarrows}} 3 \rightleftarrows \cdots \rightleftarrows p \underset{b_p}{\overset{a_p}{\rightleftarrows}} p+1$$

subject to the relations $b_1 a_1 = 0$ and $a_i b_i = b_{i+1} a_{i+1}$ for $1 \leq i \leq p-1$. Here, each vertex i corresponds to the A-module $k[t]/(t^i)$. Therefore, the arrow a_i corresponds to the embedding $k[t]/(t^i) \hookrightarrow k[t]/(t^{i+1})$, sending 1 to t. The arrow b_i corresponds to the canonical projection $k[t]/(t^{i+1}) \twoheadrightarrow k[t]/(t^i)$.

For each vertex i, denote by e_i the corresponding primitive idempotent of Λ. We have a natural identification

$$e_j \Lambda e_i \simeq \mathrm{Hom}_A(k[t]/(t^i), k[t]/(t^j)).$$

Recall that each homomorphism of A-modules has a unique factorization as the composition of an epimorphism followed by a monomorphism. Therefore, the following terminology is natural: a path from i to j is *normal*, if it is of the form

$$a_{j-1} \cdots a_{l+1} a_l b_l \cdots b_{i-2} b_{i-1}$$

for some $l \leq \min(i, j)$. By convention, if $l = i$, the path is $a_{j-1} \cdots a_i$; if $l = j$, the path is $b_j \cdots b_{i-1}$; if $l = i = j$, it is the trivial path e_i.

The above identification yields the following observation. In this way, we obtain a basis of Λ consisting of normal paths.

Lemma 1. *The set of normal paths from i to j forms a basis for $e_j \Lambda e_i$.* □

The following fact is standard. For an algebra B, denote by $Z(B)$ its center.

Lemma 2. *There is an isomorphism*

$$A \xrightarrow{\sim} Z(\Lambda), \quad t \mapsto \sum_{i=1}^{p} a_i b_i.$$

The central element $\sum_{i=1}^{p} a_i b_i$ is denoted by ϕ.

Proof. The isomorphism $Z(A) \simeq Z(\Lambda)$ holds for any algebra A of finite representation type and its Auslander algebra Λ.

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Since $Z(\Lambda)$ is isomorphic to $Z(\text{proj-}\Lambda)$, the center of the category of finite dimensional projective right Λ-modules. By the equivalence

$$A\text{-mod} \xrightarrow{\sim} \text{proj-}\Lambda, \quad X \mapsto \mathrm{Hom}_A(E, X),$$

we identify $Z(\text{proj-}\Lambda)$ with $Z(A\text{-mod})$. The latter is well known to be isomorphic to $Z(A)$. □

Recall that an *involution* σ on an algebra B means an isomorphism $\sigma\colon B \to B^{\mathrm{op}}$ of algebras satisfying $\sigma^2 = \mathrm{Id}_B$.

Lemma 3. *There is an involution σ on Λ satisfying $\sigma(e_i) = e_i$, $\sigma(b_i) = a_i$ and $\sigma(a_i) = b_i$.* □

We are interested in the indecomposable projective Λ-modules Λe_{p+1} and $e_{p+1}\Lambda$, both of which are also injective. The linear map

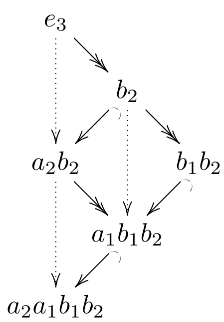
$$\delta\colon e_{p+1}\Lambda e_{p+1} \longrightarrow k$$

is defined such that its value on the longest normal path is one, and its values on other normal paths are zero.

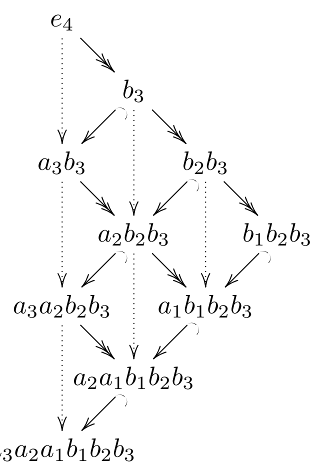
Lemma 4. *There is an isomorphism of left Λ-modules*

$$\Lambda e_{p+1} \xrightarrow{\sim} \mathrm{Hom}_k(e_{p+1}\Lambda, k), \quad x \mapsto \delta(-x).$$

The left Λ-module Λe_{p+1} takes the following shape, as illustrated for the cases $p = 2$ and $p = 3$. Here, we denote the left action of a_i's by $\hookrightarrow$, and the left action of b_i's by $\twoheadrightarrow$.



the case $p = 2$



the case $p = 3$

Here, the left action by the central element ϕ is given by the dotted arrows.

The following filtration of Λe_{p+1}:

$$\Lambda b_1 b_2 \cdots b_p \subseteq \Lambda b_2 \cdots b_p \subseteq \cdots \subseteq \Lambda b_p \subseteq \Lambda e_{p+1}$$

consists of projective Λ-modules, whose factors are the Δ-modules in the quasi-hereditary structure of Λ. There is another filtration

$$\Lambda a_p \cdots a_2 a_1 b_1 b_2 \cdots b_p \subseteq \Lambda a_p \cdots a_2 b_2 \cdots b_p \subseteq \cdots \subseteq \Lambda a_p b_p \subseteq \Lambda e_{p+1},$$

whose corresponding quotient modules are all injective and whose factors are the ∇-modules.

We have an exact sequence

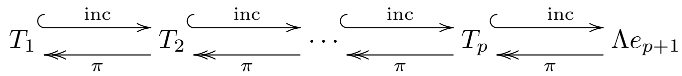
$$0 \longrightarrow \Lambda b_{p-i+1} \cdots b_{p-1} b_p \xrightarrow{\text{inc}} \Lambda e_{p+1} \xrightarrow{\pi_i} \Lambda a_p \cdots a_i b_i \cdots b_p \longrightarrow 0$$

for each $1 \leq i \leq p$, where $\pi_i(e_{p+1}) = a_p \cdots a_i b_i \cdots b_p$; it is indeed a projective resolution of $\Lambda a_p \cdots a_i b_i \cdots b_p$. Therefore, we have an epimorphism

$$\pi \colon \Lambda a_p \cdots a_i b_i \cdots b_p \longrightarrow \Lambda a_p \cdots a_i a_{i-1} b_{i-1} b_i \cdots b_p$$

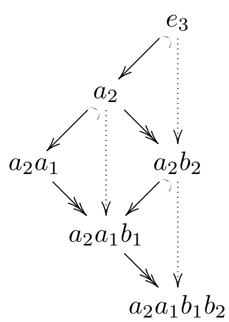
given by $\pi(a_p \cdots a_i b_i \cdots b_p) = a_p \cdots a_i a_{i-1} b_{i-1} b_i \cdots b_p$ for $2 \leq i \leq p$; if $i = p+1$, we mean the projection cover $\pi \colon \Lambda e_{p+1} \to \Lambda a_p b_p$ given by $\pi(e_{p+1}) = a_p b_p$. Here, we abuse the notation for π.

We set $T_i = \Lambda a_p \cdots a_i b_i \cdots b_p$ and $T_{p+1} = \Lambda e_{p+1}$. Consequently, we have the following diagram of morphisms.

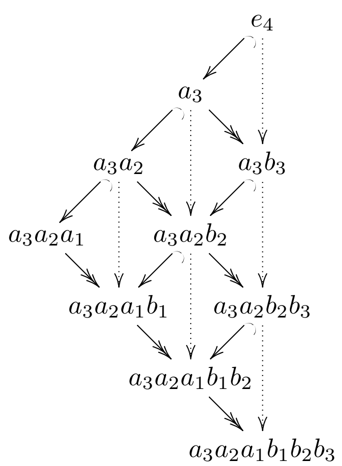


We observe that $T = \bigoplus_{i=1}^{p+1} T_i$ is the characteristic tilting module over Λ. Hence, by the above diagram, it is not hard to see that $\mathrm{End}_\Lambda(T)$ is isomorphic to Λ, that is, the algebra Λ is Ringel self-dual. We refer to [1, Section 7] for more details.

Dually, the right Λ-module $e_{p+1}\Lambda$ has the following shape, where the right action by ϕ is denoted by the dotted arrows.



the case $p = 2$



the case $p = 3$

Remark 5. The shapes of the Λ-module Λe_{p+1} and $e_{p+1}\Lambda$ coincide with the Auslander-Reiten quiver for the path algebra of a linear quiver. This seems to be mysterious.

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References

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