TWO LEMMAS ON THE AXIOM (TR4)

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Abstract. We exploit two well-known lemmas on the octahedral axiom (TR4) of triangulated categories. These lemmas are supposed to make the verification of (TR4) somehow easier. We ask a related question at the end.

Let $\mathcal{T}$ be an additive category with an auto-equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$. Denote by $\mathcal{E}$ a class of triangles $X \to Y \to Z \to \Sigma(X)$ in $\mathcal{T}$.

We assume that $(\mathcal{T}, \Sigma, \mathcal{E})$ is a pre-triangulated category, that is, the triple satisfies (TR1), (TR2) and (TR3) in [2]. To prove that $(\mathcal{T}, \Sigma, \mathcal{E})$ is a triangulated category, it suffices to verify the octahedral axiom, that is, the axiom (TR4).

We recall from [2] the axiom (TR4) explicitly.

(TR4) For any given triangles $X \stackrel{u}{\to} Y \stackrel{i}{\to} X' \stackrel{i'}{\to} \Sigma(X)$, $Y \stackrel{v}{\to} Z \stackrel{j}{\to} Z' \stackrel{j'}{\to} \Sigma(Y)$ and $X \stackrel{vu}{\to} Z \stackrel{k}{\to} Y' \stackrel{k'}{\to} \Sigma(X)$ in $\mathcal{E}$, there exist morphisms $u'\colon X' \to Y'$ and $v'\colon Y' \to Z'$ such that the following diagram (*) commutes and that $X' \stackrel{u'}{\to} Y' \stackrel{v'}{\to} Z' \stackrel{\Sigma(i)j'}{\to} \Sigma(X')$ is a triangle in $\mathcal{E}$

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{i} & X' & \xrightarrow{i'} & \Sigma(X) \\
\| & & \downarrow{\scriptstyle v} & & \vdots{\scriptstyle u'} & & \| \\
X & \xrightarrow{vu} & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & \Sigma(X) \\
 & & \downarrow{\scriptstyle j} & & \vdots{\scriptstyle v'} & & \downarrow{\scriptstyle \Sigma(u)} \\
 & & Z' & = & Z' & \xrightarrow{j'} & \Sigma(Y) \\
 & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle \Sigma(i)j'} & & \\
 & & \Sigma(Y) & \xrightarrow{\Sigma(i)} & \Sigma(X'). & &
\end{array}$$

The aim of the note is to provide two lemmas that make the verification of the axiom (TR4) somehow easier in certain situation; compare [3, Chapter 1]. The lemmas are well known to experts.

Date: February 23, 2014.
2010 *Mathematics Subject Classification.* 18E30.
Key words and phrases. the octahedral axiom.
This paper belongs to a series of informal notes, without claim of originality.

We say that a composable pair $X \overset{u}{\to} Y \overset{v}{\to} Z$ of morphisms is *admissible* if it fits into a commutative diagram (**) of the following form

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{i_1} & X_1 & \xrightarrow{i_1'} & \Sigma(X) \\
\| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u_1} & & \| \\
X & \xrightarrow{vu} & Z & \xrightarrow{k_1} & Y_1 & \xrightarrow{k_1'} & \Sigma(X) \\
 & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle \Sigma(u)} \\
 & & Z_1 & = & Z_1 & \xrightarrow{j_1'} & \Sigma(Y) \\
 & & \downarrow{\scriptstyle j_1'} & & \downarrow{\scriptstyle \Sigma(i_1)j_1'} & & \\
 & & \Sigma(Y) & \xrightarrow{\Sigma(i_1)} & \Sigma(X_1) & &
\end{array}$$

such that the first and second rows are triangles in $\mathcal{E}$ and, from the left, the second and third columns are triangles in $\mathcal{E}$.

Lemma 1. ([3, Chapter 1]) *Let $(\mathcal{T}, \Sigma, \mathcal{E})$ be a pre-triangulated category. Then it satisfies* (TR4) *if and only if each composable pair is admissible.*

Proof. Recall that (TR1) implies that each morphism fits into a triangle as the leftmost term. Then the "only if" part follows.

For the "if" part, suppose that we are given any triangles $X \overset{u}{\to} Y \overset{i}{\to} X' \overset{i'}{\to} \Sigma(X)$, $Y \overset{v}{\to} Z \overset{j}{\to} Z' \overset{j'}{\to} \Sigma(Y)$ and $X \overset{vu}{\to} Z \overset{k}{\to} Y' \overset{k'}{\to} \Sigma(X)$ in $\mathcal{E}$. We may assume that the above commutative diagram (**) is given for the composable pair $X \overset{u}{\to} Y \overset{v}{\to} Z$. We apply the first consequence of [2, 1-2 proposition] to the triangles $X \overset{u}{\to} Y \overset{i}{\to} X' \overset{i'}{\to} \Sigma(X)$ and $X \overset{u}{\to} Y \overset{i_1}{\to} X_1 \overset{i_1'}{\to} \Sigma(X)$. Then we obtain an isomorphism $x \colon X' \to X_1$ such that $i_1 = xi$ and $i' = i_1'x$. Similarly, we have isomorphisms $y \colon Y' \to Y_1$ and $z \colon Z' \to Z_1$ such that $k_1 = yk$, $k' = k_1'y$, $j_1 = zj$ and $j' = j_1'z$.

We define $u' = y^{-1}u_1x \colon X' \to Y'$ and $v' = z^{-1}v_1y \colon Y' \to Z'$. The isomorphisms x, y and z yield an isomorphism of triangles from $X' \overset{u'}{\to} Y' \overset{v'}{\to} Z' \overset{\Sigma(i)j'}{\to} \Sigma(X')$ to $X_1 \overset{u_1}{\to} Y_1 \overset{v_1}{\to} Z_1 \overset{\Sigma(i_1)j_1'}{\to} \Sigma(X_1)$. The latter belongs to $\mathcal{E}$ and so does the former, since the class $\mathcal{E}$ is closed under isomorphisms. Then we have the required diagram (*) proving (TR4). □

Two composable pairs $X \overset{u}{\to} Y \overset{v}{\to} Z$ and $A \overset{a}{\to} B \overset{b}{\to} C$ are *isomorphic* if there exist isomorphisms $x \colon X \to A$, $y \colon Y \to B$ and $z \colon Z \to C$ such that $ax = yu$ and $by = zv$.

Lemma 2. *Let the two composable pairs $X \overset{u}{\to} Y \overset{v}{\to} Z$ and $A \overset{a}{\to} B \overset{b}{\to} C$ be isomorphic. Then $X \overset{u}{\to} Y \overset{v}{\to} Z$ is admissible if and only if so is $A \overset{a}{\to} B \overset{b}{\to} C$.*

Proof. We assume that $X \overset{u}{\to} Y \overset{v}{\to} Z$ fits into the above diagram (**). Then we have the following commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{a} & B & \xrightarrow{iy^{-1}} & X' & \xrightarrow{\Sigma(x)i'} & \Sigma(A) \\
\| & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle u'} & & \| \\
A & \xrightarrow{ba} & C & \xrightarrow{kz^{-1}} & Y' & \xrightarrow{\Sigma(x)k'} & \Sigma(A) \\
 & & \downarrow{\scriptstyle jz^{-1}} & & \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle \Sigma(a)} \\
 & & Z' & = & Z' & \xrightarrow{\Sigma(y)j'} & \Sigma(B) \\
 & & \downarrow{\scriptstyle \Sigma(y)j'} & & \downarrow{\scriptstyle \Sigma(i)j'} & & \\
 & & \Sigma(B) & \xrightarrow{\Sigma(iy^{-1})} & \Sigma(X') & &
\end{array}$$

Recall that $\mathcal{E}$ is closed under isomorphisms. Then this diagram contains the required four triangles in $\mathcal{E}$. This proves that $A \overset{a}{\to} B \overset{b}{\to} C$ is admissible. □

Summary To verify the axiom (TR4), it suffices to choose a suitable class $\mathcal{S}$ of composable pairs such that any member of $\mathcal{S}$ fits into a diagram (**) and that any composable pair is isomorphic to some member of $\mathcal{S}$.

We end this note with a question, which might arise in the construction of triangulated categories using separable functors [1].

Let $(\mathcal{T}, \Sigma, \mathcal{E})$ be a pre-triangulated category. Consider the direct sum $X \oplus X' \overset{\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}}{\to} Y \oplus Y' \overset{\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}}{\to} Z \oplus Z'$ of two given composable pairs $X \overset{u}{\to} Y \overset{v}{\to} Z$ and $X' \overset{u'}{\to} Y' \overset{v'}{\to} Z'$. Recall that $\mathcal{E}$ is closed under direct sums. It follows that if the given composable pairs are admissible, so is the direct sum. However, we do not know whether the converse is true or not.

Question Let $(\mathcal{T}, \Sigma, \mathcal{E})$ be a pre-triangulated category. Assume that the direct sum $X \oplus X' \overset{\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}}{\to} Y \oplus Y' \overset{\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}}{\to} Z \oplus Z'$ is admissible. Are the two given composable pairs also admissible?

We mention that if the above question is answered affirmatively, then the construction in [1, Theorem 4.1] might give rise to a triangulated category, not just a pre-triangulated category.

Acknowledgements The author is supported by National Natural Science Foundation of China (No. 11201446) and NCET-12-0507.

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