PREPROJECTIVE ALGEBRAS

XIAO-WU CHEN

ABSTRACT. We give a detailed proof on the 2-Calabi-Yau property of a preprojective algebra.

Let $\mathbb{K}$ be a field and $Q = (Q_0, Q_1; s, t)$ be a finite quiver. Denote by $\mathbb{K}Q$ the path algebra. For each vertex i, we denote by e_i the corresponding idempotent. In what follows, the unadorned tensor $\otimes$ means the tensor over $\mathbb{K}$.

Lemma 1. *We have a short exact sequence of* $\mathbb{K}Q$-$\mathbb{K}Q$*-bimodules.*

$$0 \longrightarrow \bigoplus_{\alpha\in Q_1} \mathbb{K}Qe_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}\mathbb{K}Q \xrightarrow{d} \bigoplus_{i\in Q_0} \mathbb{K}Qe_i \otimes e_i\mathbb{K}Q \xrightarrow{\mu} \mathbb{K}Q \longrightarrow 0 \tag{0.1}$$

Here, μ is given by the multiplication of $\mathbb{K}Q$, and d is uniquely determined by

$$d(e_{t(\alpha)} \otimes \alpha \otimes e_{s(\alpha)}) = \alpha \otimes e_{s(\alpha)} - e_{t(\alpha)} \otimes \alpha.$$

Proof. We only point out that d admits the following retract.

$$\begin{aligned} r\colon \bigoplus_{i\in Q_0} \mathbb{K}Qe_i \otimes e_i\mathbb{K}Q &\longrightarrow \bigoplus_{\alpha\in Q_1} \mathbb{K}Qe_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}\mathbb{K}Q \\ p\otimes q &\longmapsto \sum_{p=p_1\alpha p_2} p_1 \otimes \alpha \otimes p_2 q \end{aligned} \tag{0.2}$$

Here, p and q are paths satisfying $s(p) = t(q)$. If p is trivial, $r(p\otimes q)$ is set to be zero. We mention that r is a homomorphism of right $\mathbb{K}Q$-modules. $\square$

We mention the following *universal derivation* of the extension $\mathbb{K}Q_0 \to \mathbb{K}Q$.

$$\begin{aligned} \Delta\colon \mathbb{K}Q &\longrightarrow \bigoplus_{\alpha\in Q_1} \mathbb{K}Qe_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}\mathbb{K}Q \\ p &\longmapsto \sum_{p=p_1\alpha p_2} p_1 \otimes \alpha \otimes p_2 \end{aligned}$$

Here, p is a path of positive length. We set $\Delta(e_i) = 0$ for each vertex i.

Let $A = \mathbb{K}Q/I$ be a quotient algebra of $\mathbb{K}Q$ with respect to a two-sided ideal I. Denote by $\pi\colon \mathbb{K}Q \to A$ the canonical projection. By definition, we have a short exact sequence of $\mathbb{K}Q$-$\mathbb{K}Q$-bimodules.

$$\xi\colon 0 \longrightarrow I \xrightarrow{\text{inc}} \mathbb{K}Q \xrightarrow{\pi} A \longrightarrow 0. \tag{0.3}$$

Consider the following map

$$\Big(\bigoplus_{\alpha\in Q_1} \pi\otimes \mathbb{K}\alpha \otimes \pi\Big) \circ \Delta\colon I \longrightarrow \bigoplus_{\alpha\in Q_1} Ae_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A.$$

Date: September 23, 2025.

2010 *Mathematics Subject Classification.* 16E30, 16E05.

Key words and phrases. resolution, preprojective algebra, Calabi-Yau property.

This paper belongs to a series of informal notes, without claim of originality.

It is a pleasant exercise to prove that it is a homomorphism of $\mathbb{K}Q$-$\mathbb{K}Q$-bimodules. Since it vanishes on I^2, we have the following induced homomorphism of A-A-bimodules.

$$c\colon I/I^2 \longrightarrow \bigoplus_{\alpha\in Q_1} Ae_{t(\alpha)}\otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A$$

The following result is standard; see [2, (1.2)].

Proposition 2. *We have the following exact sequence of A-A-bimodules.*

$$(0.4)\quad 0\longrightarrow I/I^2 \stackrel{c}{\longrightarrow} \bigoplus_{\alpha\in Q_1} Ae_{t(\alpha)}\otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A \stackrel{d'}{\longrightarrow} \bigoplus_{i\in Q_0} Ae_i\otimes e_iA \stackrel{\mu'}{\longrightarrow} A\longrightarrow 0$$

Here, μ' is given by the multiplication of A and d' is induced from d.

Proof. Applying $-\otimes_{\mathbb{K}Q} A$ to (0.1), we obtain the following exact sequence of $\mathbb{K}Q$-A-bimodules.

$$\eta\colon 0\longrightarrow \bigoplus_{\alpha\in Q_1} \mathbb{K}Qe_{t(\alpha)}\otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A \stackrel{d\otimes_{\mathbb{K}Q}A}{\longrightarrow} \bigoplus_{i\in Q_0} \mathbb{K}Qe_i\otimes e_iA \stackrel{\mu\otimes_{\mathbb{K}Q}A}{\longrightarrow} A\longrightarrow 0$$

Consider the tensor bicomplex $\xi\otimes_{\mathbb{K}Q}\eta$. We obtain the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \bigoplus_{\alpha\in Q_1} Ie_{t(\alpha)}\otimes \mathbb{K}\alpha\otimes e_{s(\alpha)}A & \longrightarrow & \bigoplus_{i\in Q_0} Ie_i\otimes e_iA & \longrightarrow & I/I^2 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \bigoplus_{\alpha\in Q_1} \mathbb{K}Qe_{t(\alpha)}\otimes \mathbb{K}\alpha\otimes e_{s(\alpha)}A & \stackrel{d\otimes_{\mathbb{K}Q}A}{\longrightarrow} & \bigoplus_{i\in Q_0} \mathbb{K}Qe_i\otimes e_iA & \longrightarrow & A & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
 & & \bigoplus_{\alpha\in Q_1} Ae_{t(\alpha)}\otimes \mathbb{K}\alpha\otimes e_{s(\alpha)}A & \longrightarrow & \bigoplus_{i\in Q_0} Ae_i\otimes e_iA & \longrightarrow & A & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here, we identify $I\otimes_{\mathbb{K}Q} A$ with I/I^2. The desired sequence follows by applying the Snake Lemma to the upper part. To obtain the map c, we recall (0.2) and implicitly use the fact that $r\otimes_{\mathbb{K}Q} A$ is a retract of $d\otimes_{\mathbb{K}Q} A$. $\square$

In what follows, we apply the general consideration above to preprojective algebras.

Denote by $\overline{Q}$ the *double quiver* of Q, which is obtained from Q by adding an arrow α^* in the converse direction for each $\alpha\in Q_1$; we will later identify α with $(\alpha^*)^*$. For any arrow $\beta\in\overline{Q}_1$, we set $\epsilon(\beta)=1$ if $\beta\in Q_1$; otherwise, we set $\epsilon(\beta)=-1$.

The *preprojective algebra* of Q is defined as follows.

$$\Pi(Q)=\mathbb{K}\overline{Q}/(\Sigma_{\alpha\in Q_1}[\alpha,\alpha^*])=\mathbb{K}\overline{Q}/(\Sigma_{\beta\in\overline{Q}_1}\epsilon(\beta)\beta\beta^*)$$

In what follows, we write Π for $\Pi(Q)$.

Denote by I the defining ideal above. For each $i\in Q_0$, we set

$$\rho_i=\sum_{\{\beta\in\overline{Q}_1\ \mid\ t(\beta)=i\}}\epsilon(\beta)\beta\beta^*.$$

Since the set $\{\rho_i \mid i \in Q_0\}$ generates I, we have a surjective homomorphism of bimodules.

$$\bigoplus_{i\in Q_0} \Pi e_i \otimes e_i\Pi \longrightarrow I/I^2,\ e_i\otimes e_i \longmapsto \rho_i + I^2$$

By connecting this homomorphism with the sequence (0.4), we obtain the following exact sequence of Π-Π-bimodules.

(0.5)

$$\bigoplus_{i\in Q_0} \Pi e_i \otimes e_i\Pi \xrightarrow{c'} \bigoplus_{\beta\in\overline{Q}_1} \Pi e_{t(\beta)} \otimes \mathbb{K}\beta \otimes e_{s(\beta)}\Pi \xrightarrow{d'} \bigoplus_{i\in Q_0} \Pi e_i \otimes e_i\Pi \xrightarrow{\mu} \Pi \longrightarrow 0$$

Here, μ is given by the multiplication of Π. The homomorphism c' is uniquely determined by the following identity.

$$c'(e_i\otimes e_i) = \sum_{\{\beta\in\overline{Q}_1 \mid s(\beta)=i\}} \epsilon(\beta^*)\beta^*\otimes\beta\otimes e_i + \sum_{\{\beta\in\overline{Q}_1 \mid t(\beta)=i\}} \epsilon(\beta)e_i\otimes\beta\otimes\beta^*$$

The following result is fundamental; compare [1, Propositions 3.3 and 3.1], [3, Lemma 1] and [4, Subsection 8.5]. Denote by D the usual $\mathbb{K}$-duality.

Proposition 3. *Let M and N be finite dimensional Π-modules. Then we have a functorial isomorphism*

$$\mathrm{Ext}^1_\Pi(M,N) \simeq D\mathrm{Ext}^1_\Pi(M,N).$$

Proof. By (0.5), we have the following exact sequence of left Π-modules.

$$\bigoplus_{i\in Q_0} \Pi e_i \otimes e_iM \xrightarrow{f} \bigoplus_{\beta\in\overline{Q}_1} \Pi e_{t(\beta)} \otimes \mathbb{K}\beta \otimes e_{s(\beta)}M \xrightarrow{g} \bigoplus_{i\in Q_0} \Pi e_i \otimes e_iM \longrightarrow M \longrightarrow 0$$

Here, f is induced by c' and g is induced by d'. Consequently, $\mathrm{Ext}^1_\Pi(M,N)$ is isomorphic to the middle cohomolgy of the following complex.

(0.6)
$$\bigoplus_{i\in Q_0}(e_iM,e_iN) \xrightarrow{g^*} \bigoplus_{\beta\in\overline{Q}_1}(\mathbb{K}\beta\otimes e_{s(\beta)}M, e_{t(\beta)}N) \xrightarrow{f^*} \bigoplus_{i\in Q_0}(e_iM,e_iN)$$

Here, we write $(-,-)$ for $\mathrm{Hom}_\mathbb{K}(-,-)$. In particular, $\mathrm{Ext}^1_\Pi(M,N)$ is finite dimensional. By interchanging M and N and taking duality, we infer that $D\mathrm{Ext}^1_\Pi(N,M)$ is isomorphic to the middle cohomology of the following complex.

(0.7)

$$\bigoplus_{i\in Q_0} D(e_iN,e_iM) \xrightarrow{D(f^*)} \bigoplus_{\beta\in\overline{Q}_1} D(\mathbb{K}\beta\otimes e_{s(\beta)}N, e_{t(\beta)}M) \xrightarrow{D(g^*)} \bigoplus_{i\in Q_0} D(e_iN,e_iM)$$

For the desired result, it suffices to show that the two complexes (0.6) and (0.7) are isomorphic. To this end, we recall that for each vertex i, we have the following canonical isomorphism

$$\phi_i\colon (e_iM,e_iN) \xrightarrow{\sim} D(e_iN,e_iM), \theta \longmapsto (\theta' \mapsto \mathrm{Tr}_{e_iM}(\theta'\circ\theta)).$$

Here, $\theta\colon e_iM\to e_iN$ and $\theta'\colon e_iN\to e_iM$ are linear maps, and Tr denotes the trace. Similarly, for each arrow $\beta\in\overline{Q}_1$, we have an isomorphism

$$\psi_\beta\colon (\mathbb{K}\beta\otimes e_{s(\beta)}M, e_{t(\beta)}N) \xrightarrow{\sim} D(\mathbb{K}\beta^*\otimes e_{s(\beta^*)}N, e_{t(\beta^*)}M),$$

which sends $\theta\colon \mathbb{K}\beta\otimes e_{s(\beta)}M\to e_{t(\beta)}N$ to the following map

$$\theta' \longmapsto \mathrm{Tr}_{e_{s(\beta)}M}(\theta'(\beta^*\otimes\theta(\beta\otimes -))).$$

Here, the blank takes values in $e_{s(\beta)}M$. We use the fact that $t(\beta^*) = s(\beta)$.

It remains to verify that the following two squares

$$\begin{array}{ccc}
\bigoplus_{i\in Q_0}(e_iM, e_iN) & \xrightarrow{g^*} & \bigoplus_{\beta\in \overline{Q}_1}(\mathbb{K}\beta\otimes e_{s(\beta)}M, e_{t(\beta)}N) \\
{\scriptstyle \bigoplus_{i\in Q_0}\phi_i}\downarrow & & \downarrow{\scriptstyle \bigoplus_{\beta\in\overline{Q}_1}\epsilon(\beta)\psi_\beta} \\
\bigoplus_{i\in Q_0} D(e_iN, e_iM) & \xrightarrow{D(f^*)} & \bigoplus_{\beta\in \overline{Q}_1} D(\mathbb{K}\beta\otimes e_{s(\beta)}N, e_{t(\beta)}M)
\end{array}$$

and

$$\begin{array}{ccc}
\bigoplus_{\beta\in \overline{Q}_1}(\mathbb{K}\beta\otimes e_{s(\beta)}M, e_{t(\beta)}N) & \xrightarrow{f^*} & \bigoplus_{i\in Q_0}(e_iM, e_iN) \\
{\scriptstyle \bigoplus_{\beta\in\overline{Q}_1}\epsilon(\beta)\psi_\beta}\downarrow & & \downarrow{\scriptstyle \bigoplus_{i\in Q_0}\phi_i} \\
\bigoplus_{\beta\in \overline{Q}_1} D(\mathbb{K}\beta\otimes e_{s(\beta)}N, e_{t(\beta)}M) & \xrightarrow{D(g^*)} & \bigoplus_{i\in Q_0} D(e_iN, e_iM)
\end{array}$$

commute.

We only verify the commutativity of the second square. For this, we fix $\alpha\in\overline{Q}_1$, $j\in Q_0$, and two linear maps $\theta\colon \mathbb{K}\alpha\otimes e_{s(\alpha)}M\to e_{t(\alpha)}N$ and $h\colon e_jN\to e_jM$. The linear function $(\bigoplus_{i\in Q_0}\phi_i)\circ f^*(\theta)$ sends h to

$$\delta_{s(\alpha),j}\ \epsilon(\alpha^*)\mathrm{Tr}_{e_jM}(h\circ\alpha^*\theta(\alpha\otimes -)) + \delta_{t(\alpha),j}\ \epsilon(\alpha)\mathrm{Tr}_{e_jM}(h\circ\theta(\alpha\otimes\alpha^* -)).$$

Here, the blank takes values in e_jM. On the other hand, the linear function $D(g^*)\circ(\bigoplus_{\beta\in\overline{Q}_1}\epsilon(\beta)\psi_\beta)$ sends h to

$$\delta_{s(\alpha^*),j}\ \epsilon(\alpha)\mathrm{Tr}_{e_{s(\alpha)}M}(\alpha^*h\circ\theta(\alpha\otimes -)) - \delta_{t(\alpha^*),j}\ \epsilon(\alpha)\mathrm{Tr}_{e_jM}(h\circ\alpha^*\theta(\alpha\otimes -)).$$

Using the well-known symmetric property of trace maps, we infer that the two elements above coincide. □

Remark 4. (1) Recall that the symmetric bilinear form

$$(-,-)\colon \mathbb{Z}Q_0\times\mathbb{Z}Q_0\longrightarrow\mathbb{Z}$$

is defined such that $(\mathbf{x},\mathbf{y})=\sum_{i\in Q_0}2x_iy_i-\sum_{\beta\in\overline{Q}_1}x_{s(\beta)}y_{t(\beta)}$. Consider the complex (0.6). The isomorphism between (0.6) and (0.7) above implies that the rightmost cohomology of (0.6) is isomorphic to $D\mathrm{Hom}_\Pi(N,M)$. The leftmost cohomology is clearly isomorphic to $\mathrm{Hom}_\Pi(M,N)$. Combining these facts, we infer the following identity in [3, Lemma 1].

$$(\mathbf{dim}\ M, \mathbf{dim}\ N) = \dim\mathrm{Hom}_\Pi(M,N)+\dim\mathrm{Hom}_\Pi(N,M)-\dim\mathrm{Ext}^1_\Pi(M,N)$$

(2) Assume that Q is connected. It is well known that $\Pi(Q)$ is finite dimensional if and only if Q is Dynkin. In this situation, the isomorphism in Proposition 3 implies that $\Pi(Q)$ is selfinjective and stably 2-Calabi-Yau.

References

[1] M. Auslander, and I. Reiten, *DTr-periodic modules and functors*, in: Representation theory of algebras, CMS Conf. Proc. **18**, 39–50, Amer. Math. Soc., Providence, RI, 1996.

[2] M.C.R. Bulter, and A.D. King, *Minimal resolutions of algebras*, J. Algebra **212** (1) (1999), 323–362.

[3] W. Crawley-Boevey, *On the exceptional fibres of Kleinian singularities*, Amer. J. Math. **122** (2000), 1027–1037.

[4] B. Keller, *On triangulated orbit categories*, Doc. Math. **10** (2005), 551–581.

Xiao-Wu Chen

School of Mathematical Sciences, University of Science and Technology of China

No. 96 Jinzhai Road, Hefei, Anhui Province, 230026, P. R. China.

URL: http://home.ustc.edu.cn/~xwchen, E-mail: xwchen@mail.ustc.edu.cn.