

## PREPROJECTIVE ALGEBRAS

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ABSTRACT. We give a detailed proof on the 2-Calabi-Yau property of a preprojective algebra.

Let  $\mathbb{K}$  be a field and  $Q = (Q_0, Q_1; s, t)$  be a finite quiver. Denote by  $\mathbb{K}Q$  the path algebra. For each vertex  $i$ , we denote by  $e_i$  the corresponding idempotent. In what follows, the unadorned tensor  $\otimes$  means the tensor over  $\mathbb{K}$ .

**Lemma 1.** *We have a short exact sequence of  $\mathbb{K}Q$ - $\mathbb{K}Q$ -bimodules.*

$$(0.1) \quad 0 \longrightarrow \bigoplus_{\alpha \in Q_1} \mathbb{K}Q e_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)} \mathbb{K}Q \xrightarrow{d} \bigoplus_{i \in Q_0} \mathbb{K}Q e_i \otimes e_i \mathbb{K}Q \xrightarrow{\mu} \mathbb{K}Q \longrightarrow 0$$

Here,  $\mu$  is given by the multiplication of  $\mathbb{K}Q$ , and  $d$  is uniquely determined by

$$d(e_{t(\alpha)} \otimes \alpha \otimes e_{s(\alpha)}) = \alpha \otimes e_{s(\alpha)} - e_{t(\alpha)} \otimes \alpha.$$

*Proof.* We only point out that  $d$  admits the following retract.

$$(0.2) \quad r: \bigoplus_{i \in Q_0} \mathbb{K}Q e_i \otimes e_i \mathbb{K}Q \longrightarrow \bigoplus_{\alpha \in Q_1} \mathbb{K}Q e_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)} \mathbb{K}Q$$

$$p \otimes q \longmapsto \sum_{p=p_1 \alpha p_2} p_1 \otimes \alpha \otimes p_2 q$$

Here,  $p$  and  $q$  are paths satisfying  $s(p) = t(q)$ . If  $p$  is trivial,  $r(p \otimes q)$  is set to be zero. We mention that  $r$  is a homomorphism of right  $\mathbb{K}Q$ -modules.  $\square$

We mention the following *universal derivation* of the extension  $\mathbb{K}Q_0 \rightarrow \mathbb{K}Q$ .

$$\Delta: \mathbb{K}Q \longrightarrow \bigoplus_{\alpha \in Q_1} \mathbb{K}Q e_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)} \mathbb{K}Q$$

$$p \longmapsto \sum_{p=p_1 \alpha p_2} p_1 \otimes \alpha \otimes p_2$$

Here,  $p$  is a path of positive length. We set  $\Delta(e_i) = 0$  for each vertex  $i$ .

Let  $A = \mathbb{K}Q/I$  be a quotient algebra of  $\mathbb{K}Q$  with respect to a two-sided ideal  $I$ . Denote by  $\pi: \mathbb{K}Q \rightarrow A$  the canonical projection. By definition, we have a short exact sequence of  $\mathbb{K}Q$ - $\mathbb{K}Q$ -bimodules.

$$(0.3) \quad \xi: 0 \longrightarrow I \xrightarrow{\text{inc}} \mathbb{K}Q \xrightarrow{\pi} A \longrightarrow 0.$$

Consider the following map

$$\left( \bigoplus_{\alpha \in Q_1} \pi \otimes \mathbb{K}\alpha \otimes \pi \right) \circ \Delta: I \longrightarrow \bigoplus_{\alpha \in Q_1} A e_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)} A.$$

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It is a pleasant exercise to prove that it is a homomorphism of  $\mathbb{K}Q$ - $\mathbb{K}Q$ -bimodules. Since it vanishes on  $I^2$ , we have the following induced homomorphism of  $A$ - $A$ -bimodules.

$$c: I/I^2 \longrightarrow \bigoplus_{\alpha \in Q_1} Ae_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A$$

The following result is standard; see [2, (1.2)].

**Proposition 2.** *We have the following exact sequence of  $A$ - $A$ -bimodules.*

$$(0.4) \quad 0 \longrightarrow I/I^2 \xrightarrow{c} \bigoplus_{\alpha \in Q_1} Ae_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A \xrightarrow{d'} \bigoplus_{i \in Q_0} Ae_i \otimes e_iA \xrightarrow{\mu'} A \longrightarrow 0$$

Here,  $\mu'$  is given by the multiplication of  $A$  and  $d'$  is induced from  $d$ .

*Proof.* Applying  $-\otimes_{\mathbb{K}Q} A$  to (0.1), we obtain the following exact sequence of  $\mathbb{K}Q$ - $A$ -bimodules.

$$\eta: 0 \longrightarrow \bigoplus_{\alpha \in Q_1} \mathbb{K}Qe_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A \xrightarrow{d \otimes_{\mathbb{K}Q} A} \bigoplus_{i \in Q_0} \mathbb{K}Qe_i \otimes e_iA \xrightarrow{\mu \otimes_{\mathbb{K}Q} A} A \longrightarrow 0$$

Consider the tensor bicomplex  $\xi \otimes_{\mathbb{K}Q} \eta$ . We obtain the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \bigoplus_{\alpha \in Q_1} Ae_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A & \longrightarrow & \bigoplus_{i \in Q_0} Ae_i \otimes e_iA & \longrightarrow & I/I^2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow 0 \\ 0 & \longrightarrow & \bigoplus_{\alpha \in Q_1} \mathbb{K}Qe_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A & \xrightarrow{d \otimes_{\mathbb{K}Q} A} & \bigoplus_{i \in Q_0} \mathbb{K}Qe_i \otimes e_iA & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ & & \bigoplus_{\alpha \in Q_1} Ae_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A & \longrightarrow & \bigoplus_{i \in Q_0} Ae_i \otimes e_iA & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow 0 \\ & & 0 & & 0 & & 0 \end{array}$$

Here, we identify  $I \otimes_{\mathbb{K}Q} A$  with  $I/I^2$ . The desired sequence follows by applying the Snake Lemma to the upper part. To obtain the map  $c$ , we recall (0.2) and implicitly use the fact that  $r \otimes_{\mathbb{K}Q} A$  is a retract of  $d \otimes_{\mathbb{K}Q} A$ .  $\square$

In what follows, we apply the general consideration above to preprojective algebras.

Denote by  $\overline{Q}$  the *double quiver* of  $Q$ , which is obtained from  $Q$  by adding an arrow  $\alpha^*$  in the converse direction for each  $\alpha \in Q_1$ ; we will later identify  $\alpha$  with  $(\alpha^*)^*$ . For any arrow  $\beta \in \overline{Q}_1$ , we set  $\epsilon(\beta) = 1$  if  $\beta \in Q_1$ ; otherwise, we set  $\epsilon(\beta) = -1$ .

The *preprojective algebra* of  $Q$  is defined as follows.

$$\Pi(Q) = \mathbb{K}\overline{Q}/(\sum_{\alpha \in Q_1} [\alpha, \alpha^*]) = \mathbb{K}\overline{Q}/(\sum_{\beta \in \overline{Q}_1} \epsilon(\beta)\beta\beta^*)$$

In what follows, we write  $\Pi$  for  $\Pi(Q)$ .

Denote by  $I$  the defining ideal above. For each  $i \in Q_0$ , we set

$$\rho_i = \sum_{\{\beta \in \overline{Q}_1 \mid t(\beta)=i\}} \epsilon(\beta)\beta\beta^*.$$

Since the set  $\{\rho_i \mid i \in Q_0\}$  generates  $I$ , we have a surjective homomorphism of bimodules.

$$\bigoplus_{i \in Q_0} \Pi e_i \otimes e_i \Pi \longrightarrow I/I^2, \quad e_i \otimes e_i \longmapsto \rho_i + I^2$$

By connecting this homomorphism with the sequence (0.4), we obtain the following exact sequence of  $\Pi$ - $\Pi$ -bimodules.

(0.5)

$$\bigoplus_{i \in Q_0} \Pi e_i \otimes e_i \Pi \xrightarrow{c'} \bigoplus_{\beta \in \overline{Q}_1} \Pi e_{t(\beta)} \otimes \mathbb{K}\beta \otimes e_{s(\beta)} \Pi \xrightarrow{d'} \bigoplus_{i \in Q_0} \Pi e_i \otimes e_i \Pi \xrightarrow{\mu} \Pi \longrightarrow 0$$

Here,  $\mu$  is given by the multiplication of  $\Pi$ . The homomorphism  $c'$  is uniquely determined by the following identity.

$$c'(e_i \otimes e_i) = \sum_{\{\beta \in \overline{Q}_1 \mid s(\beta)=i\}} \epsilon(\beta^*)\beta^* \otimes \beta \otimes e_i + \sum_{\{\beta \in \overline{Q}_1 \mid t(\beta)=i\}} \epsilon(\beta)e_i \otimes \beta \otimes \beta^*$$

The following result is fundamental; compare [1, Propositions 3.3 and 3.1], [3, Lemma 1] and [4, Subsection 8.5]. Denote by  $D$  the usual  $\mathbb{K}$ -duality.

**Proposition 3.** *Let  $M$  and  $N$  be finite dimensional  $\Pi$ -modules. Then we have a functorial isomorphism*

$$\text{Ext}_{\Pi}^1(M, N) \simeq D\text{Ext}_{\Pi}^1(N, M).$$

*Proof.* By (0.5), we have the following exact sequence of left  $\Pi$ -modules.

$$\bigoplus_{i \in Q_0} \Pi e_i \otimes e_i M \xrightarrow{f} \bigoplus_{\beta \in \overline{Q}_1} \Pi e_{t(\beta)} \otimes \mathbb{K}\beta \otimes e_{s(\beta)} M \xrightarrow{g} \bigoplus_{i \in Q_0} \Pi e_i \otimes e_i M \longrightarrow M \longrightarrow 0$$

Here,  $f$  is induced by  $c'$  and  $g$  is induced by  $d'$ . Consequently,  $\text{Ext}_{\Pi}^1(M, N)$  is isomorphic to the middle cohomology of the following complex.

$$(0.6) \quad \bigoplus_{i \in Q_0} (e_i M, e_i N) \xrightarrow{g^*} \bigoplus_{\beta \in \overline{Q}_1} (\mathbb{K}\beta \otimes e_{s(\beta)} M, e_{t(\beta)} N) \xrightarrow{f^*} \bigoplus_{i \in Q_0} (e_i M, e_i N)$$

Here, we write  $(-, -)$  for  $\text{Hom}_{\mathbb{K}}(-, -)$ . In particular,  $\text{Ext}_{\Pi}^1(M, N)$  is finite dimensional. By interchanging  $M$  and  $N$  and taking duality, we infer that  $D\text{Ext}_{\Pi}^1(N, M)$  is isomorphic to the middle cohomology of the following complex.

(0.7)

$$\bigoplus_{i \in Q_0} D(e_i N, e_i M) \xrightarrow{D(f^*)} \bigoplus_{\beta \in \overline{Q}_1} D(\mathbb{K}\beta \otimes e_{s(\beta)} N, e_{t(\beta)} M) \xrightarrow{D(g^*)} \bigoplus_{i \in Q_0} D(e_i N, e_i M)$$

For the desired result, it suffices to show that the two complexes (0.6) and (0.7) are isomorphic. To this end, we recall that for each vertex  $i$ , we have the following canonical isomorphism

$$\phi_i: (e_i M, e_i N) \xrightarrow{\sim} D(e_i N, e_i M), \theta \longmapsto (\theta' \mapsto \text{Tr}_{e_i M}(\theta' \circ \theta)).$$

Here,  $\theta: e_i M \rightarrow e_i N$  and  $\theta': e_i N \rightarrow e_i M$  are linear maps, and  $\text{Tr}$  denotes the trace. Similarly, for each arrow  $\beta \in \overline{Q}_1$ , we have an isomorphism

$$\psi_{\beta}: (\mathbb{K}\beta \otimes e_{s(\beta)} M, e_{t(\beta)} N) \xrightarrow{\sim} D(\mathbb{K}\beta^* \otimes e_{s(\beta^*)} N, e_{t(\beta^*)} M),$$

which sends  $\theta: \mathbb{K}\beta \otimes e_{s(\beta)} M \rightarrow e_{t(\beta)} N$  to the following map

$$\theta' \longmapsto \text{Tr}_{e_{s(\beta)} M}(\theta'(\beta^* \otimes \theta(\beta \otimes -))).$$

Here, the blank takes values in  $e_{s(\beta)} M$ . We use the fact that  $t(\beta^*) = s(\beta)$ .

It remains to verify that the following two squares

$$\begin{array}{ccc} \bigoplus_{i \in Q_0} (e_i M, e_i N) & \xrightarrow{g^*} & \bigoplus_{\beta \in \overline{Q}_1} (\mathbb{K}\beta \otimes e_{s(\beta)} M, e_{t(\beta)} N) \\ \bigoplus_{i \in Q_0} \phi_i \downarrow & & \downarrow \bigoplus_{\beta \in \overline{Q}_1} \epsilon(\beta) \psi_\beta \\ \bigoplus_{i \in Q_0} D(e_i N, e_i M) & \xrightarrow{D(f^*)} & \bigoplus_{\beta \in \overline{Q}_1} D(\mathbb{K}\beta \otimes e_{s(\beta)} N, e_{t(\beta)} M) \end{array}$$

and

$$\begin{array}{ccc} \bigoplus_{\beta \in \overline{Q}_1} (\mathbb{K}\beta \otimes e_{s(\beta)} M, e_{t(\beta)} N) & \xrightarrow{f^*} & \bigoplus_{i \in Q_0} (e_i M, e_i N) \\ \bigoplus_{\beta \in \overline{Q}_1} \epsilon(\beta) \psi_\beta \downarrow & & \downarrow \bigoplus_{i \in Q_0} \phi_i \\ \bigoplus_{\beta \in \overline{Q}_1} D(\mathbb{K}\beta \otimes e_{s(\beta)} N, e_{t(\beta)} M) & \xrightarrow{D(g^*)} & \bigoplus_{i \in Q_0} D(e_i N, e_i M) \end{array}$$

commute.

We only verify the commutativity of the second square. For this, we fix  $\alpha \in \overline{Q}_1$ ,  $j \in Q_0$ , and two linear maps  $\theta: \mathbb{K}\alpha \otimes e_{s(\alpha)} M \rightarrow e_{t(\alpha)} N$  and  $h: e_j N \rightarrow e_j M$ . The linear function  $(\bigoplus_{i \in Q_0} \phi_i) \circ f^*(\theta)$  sends  $h$  to

$$\delta_{s(\alpha), j} \epsilon(\alpha^*) \text{Tr}_{e_j M}(h \circ \alpha^* \theta(\alpha \otimes -)) + \delta_{t(\alpha), j} \epsilon(\alpha) \text{Tr}_{e_j M}(h \circ \theta(\alpha \otimes \alpha^* -)).$$

Here, the blank takes values in  $e_j M$ . On the other hand, the linear function  $D(g^*) \circ (\bigoplus_{\beta \in \overline{Q}_1} \epsilon(\beta) \psi_\beta)$  sends  $h$  to

$$\delta_{s(\alpha^*), j} \epsilon(\alpha) \text{Tr}_{e_{s(\alpha)} M}(\alpha^* h \circ \theta(\alpha \otimes -)) - \delta_{t(\alpha^*), j} \epsilon(\alpha) \text{Tr}_{e_j M}(h \circ \alpha^* \theta(\alpha \otimes -)).$$

Using the well-known symmetric property of trace maps, we infer that the two elements above coincide.  $\square$

**Remark 4.** (1) Recall that the symmetric bilinear form

$$(-, -): \mathbb{Z}Q_0 \times \mathbb{Z}Q_0 \longrightarrow \mathbb{Z}$$

is defined such that  $(\mathbf{x}, \mathbf{y}) = \sum_{i \in Q_0} 2x_i y_i - \sum_{\beta \in \overline{Q}_1} x_{s(\beta)} y_{t(\beta)}$ . Consider the complex (0.6). The isomorphism between (0.6) and (0.7) above implies that the rightmost cohomology of (0.6) is isomorphic to  $D\text{Hom}_\Pi(N, M)$ . The leftmost cohomology is clearly isomorphic to  $\text{Hom}_\Pi(M, N)$ . Combining these facts, we infer the following identity in [3, Lemma 1].

$$(\mathbf{dim} M, \mathbf{dim} N) = \dim \text{Hom}_\Pi(M, N) + \dim \text{Hom}_\Pi(N, M) - \dim \text{Ext}_\Pi^1(M, N)$$

(2) Assume that  $Q$  is connected. It is well known that  $\Pi(Q)$  is finite dimensional if and only if  $Q$  is Dynkin. In this situation, the isomorphism in Proposition 3 implies that  $\Pi(Q)$  is selfinjective and stably 2-Calabi-Yau.

## REFERENCES

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