ON TRIVIAL EXTENSIONS OF MONOMIAL ALGEBRAS

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Abstract. We analyze the quivers and relations of trivial extensions of monomial algebras.

Let $R$ be a commutative artinian ring with unit, and $A$ be an artin $R$-algebra. Denote by $D = \text{Hom}_R(\cdot, E)$ the Matlis duality functor, where $E$ is the injective envelope of $R/\text{rad}(R)$. We have the $A$-$A$-bimodule $DA$, whose actions are given by

$$(a.f.b)(x) = f(bxa)$$

for any $a,b,x \in A$ and $f \in DA$. We observe that $R$ acts on $DA$ centrally.

The trivial extension $TA = A \oplus DA$ of $A$ is defined such that

$$(a,f)(b,g) = (ab, f.b + a.g).$$

It is a symmetric artin $R$-algebra. We observe that $DA$ is a square-zero ideal of $TA$.

**Lemma 1.** We have $\text{rad}(TA) = \text{rad}(A) \oplus DA$ and then $\text{rad}^2(TA) = \text{rad}^2(A) \oplus \{\text{rad}(A).(DA) + (DA).\text{rad}(A)\}$. \[\square\]

Consequently, we have the following two isomorphisms:

$$TA/\text{rad}(TA) \cong A/\text{rad}(A);$$

$$\text{rad}(TA)/\text{rad}^2(TA) \cong \text{rad}(A)/\text{rad}^2(A) \oplus DA/\{\text{rad}(A).(DA) + (DA).\text{rad}(A)\} \cong \text{rad}(A)/\text{rad}^2(A) \oplus D(\text{soc}(A^A) \cap \text{soc}(A_A)).$$

(0.1)

They are very useful in computing the Ext-quiver of $TA$.

In what follows, we concentrate on monomial algebras. We fix a field $k$. Let $A = kQ/I$ be a finite dimensional algebra given by a quiver $Q$ and a monomial admissible ideal $I$.

A nonzero path in $A$ means a path in $Q$ which does not lie in $I$. These nonzero paths form a standard basis of $A$. Then $DA = \text{Hom}_k(A,k)$ has the dual basis. The $A$-$A$-bimodule action on $DA$ is given by truncations of paths. More precisely, for two nonzero paths $a$ and $p$, we have

$$a.p^* = \begin{cases} b^*, & \text{if } p = ba \text{ for some nonzero path } b; \\ 0, & \text{otherwise}. \end{cases}$$

Similarly, we have

$$p^*.a = \begin{cases} c^*, & \text{if } p = ac \text{ for some nonzero path } c; \\ 0, & \text{otherwise}. \end{cases}$$

We say that a nonzero path $p$ is maximal, if it is not a proper segment of any nonzero path in $A$. We observe that maximal paths form a basis for $\text{soc}(A^A) \cap \text{soc}(A_A)$...
We emphasize that the paths new ideal $I$ are as follows: we set $Q = Q_0$ and 

$$Q^e_1 = Q_1 \cup \{ p^* | p \text{ maximal paths in } A \};$$

we set $s(p^*) = t(p)$ and $t(p^*) = s(p)$.

We introduce three classes of new relations on $Q^e$:

1. the truncation relation $yp^*x$, with $x, y$ nonzero paths satisfying $t(x) = t(p)$ and $s(y) = s(p)$ and that $p \neq xzy$ for any nonzero path $z$;
2. the square-zero relation $p^*xq^*$, with $x$ any nonzero path satisfying $s(x) = s(q)$ and $t(x) = t(p)$; moreover, $q$ starts with $x$ and $p$ terminates with $x$.
3. the overlap relation $yp^*x - qz^*$, with $x, y, z, w$ nonzero paths satisfying $s(x) = s(w), t(y) = t(z), t(x) = t(p), s(y) = s(p), t(w) = t(q)$ and $s(z) = s(q)$; moreover, $p = xuy$ and $q = wuz$ for some nonzero path $u$.

We emphasize that the paths $x, y, z, w$ and $u$ above might be trivial paths. The new ideal $I^e$ of $kQ^e$ is defined to be generated by $I$ and these new relations.

The following result is well known.

**Proposition 2.** Let $A = kQ/I$ be a monomial algebra as above. Then the trivial extension $TA$ is isomorphic to $kQ^e/I^e$.

**Proof.** For the Gabriel quiver of $TA$, one needs to analyze the quotient space $\text{rad}(TA)/\text{rad}^2(TA)$. The proof of the precise relations of $TA$ are more subtle, using the explicit basis and $A$-$A$-bimodule actions of $DA$.  

**Example 3.** Let $A$ be the path algebra of the following linear quiver.

![Linear quiver diagram]

The only maximal path is $ba$. Then $TA$ is given by the following quiver

![Modified linear quiver diagram]

with the relations given by all paths of length four.

Let us consider the special case where $A$ is radical square zero, that is, the ideal $I$ is generated by all paths in $Q$ with length two. To avoid the trivial cases, we assume that $Q$ has no isolated vertices. Then maximal paths are precisely arrows in $Q$. Consequently, $Q^e = Q$ coincides with the double quiver of $Q$; the new relations are as follows:

1. the truncation relation $\beta\alpha^*$, with $\beta, \alpha \in Q_1$ satisfying $s(\beta) = s(\alpha)$ and $\beta \neq \alpha$;
2. the truncation relation $\alpha^*\gamma$, with $\alpha, \gamma \in Q_1$ satisfying $t(\alpha) = t(\gamma)$ and $\alpha \neq \gamma$;
3. the truncation relation $\alpha^*\alpha^*$, with $\alpha \in Q_1$;
4. the square-zero relation $\alpha^*\beta^*$, with $\alpha, \beta \in Q_1$ satisfying $s(\beta) = t(\alpha)$;
5. the square-zero relation $\alpha^*\alpha^*$, with $\alpha \in Q_1$;
6. the overlap relation $\alpha^*\alpha - \beta^*$, with $\alpha, \beta \in Q_1$ satisfying $t(\alpha) = t(\beta)$ and $\alpha \neq \beta$;
7. the overlap relation $\alpha^*\alpha - \gamma^*\gamma$, with $\alpha, \gamma \in Q_1$ satisfying $s(\alpha) = s(\gamma)$ and $\alpha \neq \gamma$;
8. the overlap relation $\alpha^*\alpha - \eta \eta^*$, with $\alpha, \eta \in Q_1$ satisfying $s(\alpha) = t(\eta)$.

We observe that $I^e$ contains all paths of length three in $Q$.  


Example 4. Let $A'$ be the path algebra of the following bipartite quiver.

\[
\begin{array}{ccc}
1 & \alpha & 2 \\
\downarrow & \downarrow & \downarrow \\
\alpha^* & \beta & 3
\end{array}
\]

Then $TA'$ is given by the following quiver

\[
\begin{array}{ccc}
1 & \alpha & 2 \\
\downarrow & \downarrow & \downarrow \\
\alpha^* & \beta & 3
\end{array}
\]

subject to the relations $\alpha^* \beta, \beta^* \alpha, \alpha \alpha^* - \beta \beta^*$. We observe that the set of the new relations listed above is usually not minimal.

Remark 5. By [1, Theorem 3.1 and Corollary 3.2], the above trivial extensions $TA$ and $TA'$ in the examples are derived equivalent, and thus stably equivalent.

We end this note with the trivial extension of a non-monomial algebra. For a general result, we refer to [2, Theorem 3.9].

Example 6. Let $A$ be the algebra given by the following quiver

\[
\begin{array}{ccc}
1 & \alpha & 2 \\
\downarrow & \downarrow & \downarrow \\
\beta & \gamma & 4 \\
\downarrow & \downarrow & \downarrow \\
\delta & \gamma & 3
\end{array}
\]

with the relation $\beta \alpha - \gamma \delta$. Although $A$ is not minimal, it still have a canonical basis given by paths. Denote by $c = \beta \alpha$ and by $c^*$ the corresponding element in the dual basis. Then $TA$ is given by the following quiver

\[
\begin{array}{ccc}
1 & \alpha & 2 \\
\downarrow & \downarrow & \downarrow \\
\beta & \gamma & 4 \\
\downarrow & \downarrow & \downarrow \\
\delta & \gamma & 3
\end{array}
\]

subject to the relations \{ $\beta \alpha - \gamma \delta$ \} $\cup \{ \delta c^*, \alpha c^* \gamma, c^* \beta \alpha c^* \}$.  

References


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