ON TRIVIAL EXTENSIONS OF MONOMIAL ALGEBRAS

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Abstract. We analyze the quivers and relations of trivial extensions of monomial algebras.

Let R be a commutative artinian ring with unit, and A be an artin R-algebra. Denote by $D = \mathrm{Hom}_R(-, E)$ the Matlis duality functor, where E is the injective envelope of $R/\mathrm{rad}(R)$. We have the A-A-bimodule DA, whose actions are given by

$$(a.f.b)(x) = f(bxa)$$

for any $a, b, x \in A$ and $f \in DA$. We observe that R acts on DA centrally.

The *trivial extension* $TA = A \oplus DA$ of A is defined such that

$$(a, f)(b, g) = (ab, f.b + a.g).$$

It is a symmetric artin R-algebra. We observe that DA is a square-zero ideal of TA.

Lemma 1. *We have* $\mathrm{rad}(TA) = \mathrm{rad}(A) \oplus DA$ *and then* $\mathrm{rad}^2(TA) = \mathrm{rad}^2(A) \oplus \{\mathrm{rad}(A).(DA) + (DA).\mathrm{rad}(A)\}$. □

Consequently, we have the following two isomorphisms:

$$TA/\mathrm{rad}(TA) \simeq A/\mathrm{rad}(A);$$

$$\begin{aligned} \mathrm{rad}(TA)/\mathrm{rad}^2(TA) &\simeq \mathrm{rad}(A)/\mathrm{rad}^2(A) \oplus DA/\{\mathrm{rad}(A).(DA) + (DA).\mathrm{rad}(A)\} \\ &\simeq \mathrm{rad}(A)/\mathrm{rad}^2(A) \oplus D(\mathrm{soc}({}_AA) \cap \mathrm{soc}(A_A)). \end{aligned} \tag{0.1}$$

They are very useful in computing the Ext-quiver of TA.

In what follows, we concentrate on monomial algebras. We fix a field k. Let $A = kQ/I$ be a finite dimensional algebra given by a quiver Q and a monomial admissible ideal I.

A *nonzero path* in A means a path in Q which does not lie in I. These nonzero paths form a standard basis of A. Then $DA = \mathrm{Hom}_k(A, k)$ has the dual basis. The A-A-bimodule action on DA is given by truncations of paths. More precisely, for two nonzero paths a and p, we have

$$a.p^* = \begin{cases} b^*, \text{ if } p = ba \text{ for some nonzero path } b; \\ 0, \text{ otherwise.} \end{cases}$$

Similarly, we have

$$p^*.a = \begin{cases} c^*, \text{ if } p = ac \text{ for some nonzero path } c; \\ 0, \text{ otherwise.} \end{cases}$$

We say that a nonzero path p is *maximal*, if it is not a proper segment of any nonzero path in A. We observe that maximal paths form a basis for $\mathrm{soc}({}_AA) \cap$

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$\mathrm{soc}(A_A)$. In view of (0.1), we expect that maximal paths are related to the Gabriel quiver of TA.

We define a new quiver Q^{te}, containing Q, as follows: $Q_0^{\mathrm{te}} = Q_0$ and

$$Q_1^{\mathrm{te}} = Q_1 \cup \{p^* \mid p \text{ maximal paths in } A\};$$

we set $s(p^*) = t(p)$ and $t(p^*) = s(p)$.

We introduce three classes of new relations on Q^{te}:

(1) the *truncation relation* yp^*x, with x, y nonzero paths satisfying $t(x) = t(p)$ and $s(y) = s(p)$ and that $p \neq xzy$ for any nonzero path z;
(2) the *square-zero relation* p^*xq^*, with x any nonzero path satisfying $s(x) = s(q)$ and $t(x) = t(p)$; moreover, q starts with x and p terminates with x.
(3) the *overlap relation* $yp^*x - zq^*w$, with x, y, z, w nonzero paths satisfying $s(x) = s(w)$, $t(y) = t(z)$, $t(x) = t(p)$, $s(y) = s(p)$, $t(w) = t(q)$ and $s(z) = s(q)$; moreover, $p = xuy$ and $q = wuz$ for some nonzero path u.

We emphasize that the paths x, y, z, w and u above might be trivial paths. The new ideal I^{te} of kQ^{te} is defined to be generated by I and these new relations.

The following result is well known.

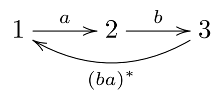
Proposition 2. *Let $A = kQ/I$ be a monomial algebra as above. Then the trivial extension TA is isomorphic to $kQ^{\mathrm{te}}/I^{\mathrm{te}}$.*

Proof. For the Gabriel quiver of TA, one needs to analyze the quotient space $\mathrm{rad}(TA)/\mathrm{rad}^2(TA)$. The proof of the precise relations of TA are more subtle, using the explicit basis and A-A-bimodule actions of DA. □

Example 3. *Let A be the path algebra of the following linear quiver.*

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

The only maximal path is ba. Then TA is given by the following quiver



with the relations given by all paths of length four.

Let us consider the special case where A is radical square zero, that is, the ideal I is generated by all paths in Q with length two. To avoid the trivial cases, we assume that Q has no isolated vertices. Then maximal paths are precisely arrows in Q. Consequently, $Q^{\mathrm{te}} = \overline{Q}$ coincides with the *double quiver* of Q; the new relations are as follows:

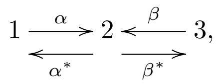
(1) the truncation relation $\beta\alpha^*$, with $\beta, \alpha \in Q_1$ satisfying $s(\beta) = s(\alpha)$ and $\beta \neq \alpha$;
the truncation relation $\alpha^*\gamma$, with $\alpha, \gamma \in Q_1$ satisfying $t(\alpha) = t(\gamma)$ and $\alpha \neq \gamma$;
the truncation relation $\alpha\alpha^*\alpha$, with $\alpha \in Q_1$;
(2) the square-zero relation $\alpha^*\beta^*$, with $\alpha, \beta \in Q_1$ satisfying $s(\beta) = t(\alpha)$;
the square-zero relation $\alpha^*\alpha\alpha^*$, with $\alpha \in Q_1$;
(3) the overlap relation $\alpha\alpha^* - \beta\beta^*$, with $\alpha, \beta \in Q_1$ satisfying $t(\alpha) = t(\beta)$ and $\alpha \neq \beta$;
the overlap relation $\alpha^*\alpha - \gamma^*\gamma$, with $\alpha, \gamma \in Q_1$ satisfying $s(\alpha) = s(\gamma)$ and $\alpha \neq \gamma$;
the overlap relation $\alpha^*\alpha - \eta\eta^*$, with $\alpha, \eta \in Q_1$ satisfying $s(\alpha) = t(\eta)$.

We observe that I^{te} contains all paths of length three in $\overline{Q}$.

Example 4. *Let A' be the path algebra of the following bipartite quiver.*

$$1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

Then TA' is given by the following quiver

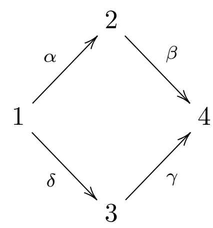


subject to the relations $\alpha^*\beta, \beta^*\alpha, \alpha\alpha^* - \beta\beta^*$. We observe that the set of the new relations listed above is usually not minimal.

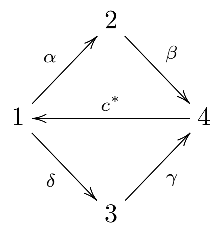
Remark 5. By [1, Theorem 3.1 and Corollary 3.2], the above trivial extensions TA and TA' in the examples are derived equivalent, and thus stably equivalent.

We end this note with the trivial extension of a non-monomial algebra. For a general result, we refer to [2, Theorem 3.9].

Example 6. *Let A be the algebra given by the following quiver*



with the relation $\beta\alpha - \gamma\delta$. Although A is not minimal, it still have a canonical basis given by paths. Denote by $c = \beta\alpha$ and by c^* the corresponding element in the dual basis. Then TA is given by the following quiver



subject to the relations $\{\beta\alpha - \gamma\delta\} \cup \{\delta c^*\beta, \alpha c^*\gamma, c^*\beta\alpha c^*\}$.

References

[1] J. Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), 303–317.

[2] E.A. Fernández, and M. Platzeck, *Presentations of trivial xxtensions of finite dimensional algebras and a theorem of Sheila Brenner*, J. Algebra **249** (2002), 326–344.

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