COHOMOLOGICAL FUNCTORS

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Let $\mathcal{S}$ be a *right triangulated category*, whose translation functor is denoted by Σ. Recall that a right triangle in $\mathcal{S}$ is of the form $X \to Y \to Z \to \Sigma(X)$. We mention that a right triangulated category is called a suspended category in [4], and that a triangulated category is the same as a right triangulated category with the translation functor being an autoequivalence.

We will assume that $\mathcal{S}$ is skeletally small. Recall that any additive functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Ab}$ is also known as a right $\mathcal{S}$-module. An additive functor $F\colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Ab}$ is *cohomological* if it sends any right triangle $X \to Y \to Z \to \Sigma(X)$ to an exact sequence $F(Z) \to F(Y) \to F(X)$.

The following result is fundamental, which might be deduced from [3, Lemma 2 and (1.3) Theorem].

Proposition 1. *Let $F\colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Ab}$ be an additive functor. Then F is cohomological if and only if the right $\mathcal{S}$-module F is flat.*

Proof. We recall that any representable functor $\mathcal{S}(-, A)$ is cohomological, and that a flat module is a filtered colimit of finitely generated free modules, that is, representable functors [6, (4.34) Theorem]; moreover, filtered colimits are exact in Ab. Combining these facts, we infer the "if" part.

Conversely, we assume that F is cohomological. Using the argument in [1, the proof of Chapter II, Proposition 2.4], we will show that F is flat. Since any module is a filtered colimit of finitely presented modules and the Tor functors commute with filtered colimits, it suffices to prove that $\mathrm{Tor}_1^{\mathcal{S}}(F, G) = 0$ for any finitely presented covariant functor G. Take a finite projective presentation

$$\mathcal{S}(Y, -) \xrightarrow{\mathcal{S}(a,-)} \mathcal{S}(X, -) \longrightarrow G \longrightarrow 0.$$

The morphism a fits into a right triangle $X \xrightarrow{a} Y \to Z \to \Sigma(X)$. Then we have a longer projective presentation of G as follows.

$$\mathcal{S}(Z, -) \longrightarrow \mathcal{S}(Y, -) \xrightarrow{\mathcal{S}(a,-)} \mathcal{S}(X, -) \longrightarrow G \longrightarrow 0.$$

Recall the canonical isomorphism $F \otimes_{\mathcal{S}} \mathcal{S}(A, -) \simeq F(A)$ and that F is cohomological. Applying $F \otimes_{\mathcal{S}} -$ to the above presentation and using the canonical isomorphism, we infer that $\mathrm{Tor}_1^{\mathcal{S}}(F, G) = 0$. This completes the proof. □

Let $\mathcal{S}'$ be a Krull-Schmidt right triangulated category. Assume that $\mathcal{S} \subseteq \mathcal{S}'$ is a full right triangulated subcategory which is closed under direct summands. It follows that $\mathcal{S}$ is also Krull-Schmidt. The following result is esentially due to [5, 1.3 Proposition] and [1, Chapter II, Proposition 2.4].

Corollary 2. *Keep the assumptions above. Then the subcategory $\mathcal{S}$ is contravariantly finite in $\mathcal{S}'$ if and only if the inclusion $\mathcal{S} \hookrightarrow \mathcal{S}'$ admits a right adjoint.*

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Proof. The "if" part is trivial. For the "only if" part, we take an arbitrary object X in $\mathcal{S}'$. Then the restricted $\mathcal{S}$-module $\mathcal{S}'(-,X)|_{\mathcal{S}}$ is finitely generated. Since $\mathcal{S}$ is Krull-Schmidt, the module $\mathcal{S}'(-,X)|_{\mathcal{S}}$ has a projective cover; see [2]. Since $\mathcal{S}'(-,X)|_{\mathcal{S}}$ is cohomological, it is flat by Proposition 1. Recall from [6, Excercises for §4, no. 20] that a flat module with a projective cover is necessarily projective. We infer that the functor $\mathcal{S}'(-,X)|_{\mathcal{S}}$ is representable. Then the existence of the required right adjoint follows immediately. □

The following result is due to [7, Proposition 1.4]. Let $\mathcal{T}$ be a triangulated category and $\mathcal{S} \subseteq \mathcal{T}$ be a full additive subcategory which is closed under extensions and Σ. It follows that $\mathcal{S}$ becomes a right triangulated category. We assume further that $\mathcal{S}$ is closed under direct summands.

Corollary 3. *Keep the assumptions above. Then the subcategory $\mathcal{S}$ is contravariantly finite in $\mathcal{T}$ if and only if the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$ admits a right adjoint.*

Proof. We only need to prove the "only if" part. Take an arbitrary object X in $\mathcal{T}$ and a right $\mathcal{S}$-approximation $a\colon S_0 \to X$ of X. Form a triangle

$$X_1 \xrightarrow{b} S_1 \xrightarrow{a} X \longrightarrow \Sigma(X_1),$$

and take a right $\mathcal{S}$-approximation $c\colon S_1 \to X$ of X_1. We obtain a finite projective presentation as follows.

$$\mathcal{S}(-,S_1) \xrightarrow{\mathcal{S}(-,b\circ c)} \mathcal{S}(-,S_0) \longrightarrow \mathcal{T}(-,X)|_{\mathcal{S}} \longrightarrow 0$$

In other words, the restricted $\mathcal{S}$-module $\mathcal{T}(-,X)|_{\mathcal{S}}$ is finitely presented. Recall from [6, (4.30) Theorem] that a finitely presented flat module is projective. By Proposition 1, the $\mathcal{S}$-module $\mathcal{T}(-,X)|_{\mathcal{S}}$ is flat. It follows that it is representable. Then we infer the required existence. □

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