THE 2-CALABI-YAU COMPLETION VIA AUSLANDER-REITEN THEORY

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Abstract. The 2-Calabi-Yau completion of a finite dimensional algebra with finite global dimension appears naturally in the Auslander-Reiten theory of the bounded derived category.

Let A be an algebra over a field k. The enveloping algebra is defined to be $A^e = A \otimes_k A^{\mathrm{op}}$. We identify A-A-bimodules with left A^e-modules.

The *inverse dualizing complex* of A is defined to be

$$\mathbb{R}\mathrm{Hom}_{A^e}(A, A^e)$$

which is viewed as an object in $\mathbf{D}(A^e\text{-Mod})$. The 2-*Calabi-Yau completion* of A [3] is defined to be the tensor dg algebra

$$\Pi_2(A) = T_A(\Theta)$$

where $\Theta = \Sigma\ \mathbf{p}\mathbb{R}\mathrm{Hom}_{A^e}(A, A^e)$ with $\mathbf{p}$ the dg-projective resolution functor. The terminology is justified by the fundamental theorem: $\Pi_2(A)$ is 2-Calabi-Yau provided that A is homologically smooth; see [3, Theorem 4.8].

In this note, we will analyze the definition of $\Pi_2(A)$ via the Auslander-Reiten theory. In what follows, we assume that A is a finite dimensional algebra with finite global dimension. Then the bounded derived category $\mathbf{D}^b(A\text{-mod})$ has Serre duality. Its Auslander-Reiten translation is given by

$$\tau_{\mathbf{D}}(M) = \Sigma^{-1}\ DA \otimes^{\mathbb{L}}_A M.$$

Here, D denotes the k-duality functor.

Lemma 1. *There is an isomorphism in* $\mathbf{D}(A^e\text{-Mod})$

$$\mathbb{R}\mathrm{Hom}_{A^e}(A, A^e) \simeq \mathbb{R}\mathrm{Hom}_A(DA, A).$$

Proof. We identify A^e with $\mathrm{Hom}_k(DA, A)$. Then we have

$$\mathbb{R}\mathrm{Hom}_{A^e}(A, A^e) \simeq \mathbb{R}\mathrm{Hom}_{A^e}(A, \mathrm{Hom}_k(DA, A)) \simeq \mathbb{R}\mathrm{Hom}_A(DA, A).$$

Here, the right isomorphism reminds the classical isomorphism

$$\mathrm{Ext}^i_{A^e}(A, \mathrm{Hom}_k(X, Y)) \simeq \mathrm{Ext}^i_A(X, Y)$$

for any left A-modules X and Y. □

Lemma 2. *For each bounded complex M of A-modules, we have*

$$\tau_{\mathbf{D}}^{-1}(M) \simeq \Theta \otimes_A M$$

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Proof. Recall that $\mathbb{R}\mathrm{Hom}_A(DA, -)$ is quasi-inverse to $DA \otimes_A^{\mathbb{L}} -$. Therefore, we have the following first isomorphism.

$$\begin{aligned}\tau_{\mathbf{D}}^{-1}(M) &\simeq \Sigma\, \mathbb{R}\mathrm{Hom}_A(DA, M)\\ &\simeq \Sigma\, \mathbb{R}\mathrm{Hom}_A(DA, A) \otimes_A^{\mathbb{L}} M\\ &= \Theta \otimes_A M.\end{aligned}$$

Here, the second isomorphism uses the fact that M is perfect, and the last one uses the previous lemma. □

The lemma above implies the following observation.

Proposition 3. *There is an isomorphism in* $\mathbf{D}(A\text{-Mod})$

$$\Pi_2(A) \simeq \bigoplus_{n\geq 0} \tau_{\mathbf{D}}^{-n}(A).$$

In general, $\tau_{\mathbf{D}}$ is not easy to handle. However, it behaves nicely in the hereditary case.

Lemma 4. *Assume that A is hereditary and that M is a finite dimensional indecomposable A-module. Then the following statements hold.*

(1) *If M is non-injective, then $\tau_{\mathbf{D}}^{-1}(M) \simeq \tau^{-}(M)$.*
(2) *If M is injective, then $\tau_{\mathbf{D}}^{-1}(M) \simeq \Sigma\, \nu^{-}(M)$.*

Here, ν^- and τ^- denote the inverse Nakayama functor and inverse Auslander-Reiten translation on the module category.

By induction, we deduce the following result from the above lemma.

Lemma 5. *Assume that A is hereditary and that M is a finite dimensional A-module. Then*

$$H^0(\tau_{\mathbf{D}}^{-n}(M)) \simeq \tau^{-n}(M)$$

for each $n \geq 1$.

Let Q be a finite quiver which is connected and acyclic. Then the path algebra kQ is finite dimensional and hereditary. Denote by $\Pi(Q)$ the preprojective algebra of Q.

Proposition 6. *The 2-Calabi-Yau completion $\Pi_2(kQ)$ is negatively graded with $H^0(\Pi_2(kQ)) \simeq \Pi(Q)$. More over, we have the following two cases.*

(1) *Assume that Q is of Dynkin type. Then there are infinitely many positive integers n with $H^{-n}(\Pi_2(kQ)) \neq 0$.*
(2) *Assume that Q is non-Dynkin. Then $H^{-n}(\Pi_2(kQ)) = 0$ for any $n \geq 1$.*

Proof. The first statement follows by combining Lemma 4, Lemma 5 and the standard fact that $\Pi(Q) \simeq \bigoplus_{n\geq 0} \tau^{-n}(kQ)$. The Dynkin case uses the fractionally Calabi-Yau property of $\mathbf{D}^b(kQ\text{-mod})$. □

Remark 7. (1) The idea to introduce an algebra structure on the direct sum of preprojective modules went back to [2], while the definition of $\Pi(Q)$ was due to [1].

(2) The 2-Calabi-Yau completion $\Pi_2(A)$ is also called the *derived 2-preprojective dg algebra* of A; this terminology is quite reasonable in view of Propositions 3 and 6.

References

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