ANTICHAIN-FINITENESS AND DICKSON'S LEMMA

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Let $X = (X, \preceq)$ be a partially ordered set, a poset for short. Recall that X is noetherian, provided that any ascending chain of elements in X stabilizes. Dually, it is artianin if any descending chain of elements stabilizes. Denote by $\max(X)$ the subset consisting of maximal elements in X, and by $\min(X)$ the subset consisting of minimal elements in X. A subset S of X is called an *antichain* if any distinct elements in S are incomparable. For example, both $\max(X)$ and $\min(X)$ are antichains.

The poset X is said to be *antichain-finite* provided that any antichain in X is finite.

Proposition 1. *Let X be an antichain-finite poset, which is both noetherian and artinian. Then X is finite.*

Proof. We assume on the contrary that X is infinite. Set $X_0 = \min(X)$ and $X_1 = \min(X \backslash X_0)$. Inductively, we set

$$X_n = \min(X \backslash (X_0 \cup X_1 \cup \cdots \cup X_{n-1})).$$

Each X_n is an antichain, and thus finite. Moreover, each X_n is nonempty, since otherwise $X = X_0 \cup X_1 \cup \cdots \cup X_{n-1}$, which contradicts to the infiniteness of X.

The following fact will be useful. For any $x \in X_n$ and $y \preceq x$, we have $y \in \bigcup_{i=0}^{n} X_n$. Otherwise, y belongs to $X \backslash (X_0 \cup X_1 \cup \cdots \cdots \cup X_{n-1})$. Since x is minimal in $X \backslash (X_0 \cup X_1 \cup \cdots \cup X_{n-1})$ and $y \preceq x$, we have $y = x$.

Write $X' = \bigcup_{n \geq 0} X_n$, which is a disjoint union of finite nonempty subsets. In particular, the set X' is infinite. The subset $\max(X')$ is an antichain, and thus finite. Set $\max(X') = \{y_1, \cdots, y_m\}$. Take n_0 sufficiently large such that each y_j belongs to $\bigcup_{0 \leq n \leq n_0} X_n$. For each element z in X', there exists some y_j satisfying $z \preceq y_j$. By the fact above, we infer that z belongs to $\bigcup_{0 \leq n \leq n_0} X_i$. This is impossible, since $\bigcup_{0 \leq n \leq n_0} X_i$ is finite. □

Let $Y = (Y, \preceq)$ be another poset. The *product poset* $X \times Y$ is defined such that $(x, y) \preceq (x', y')$ if and only if $x \preceq x'$ and $y \preceq y'$.

The following fact is immediate.

Lemma 2. *Assume that both X and Y are noetherian. Then so is the product poset $X \times Y$.* □

In contrast, we have the following fact.

Example 3. *The set $\mathbb{Z}$ of integers is certainly antichain-finite. However, the product $\mathbb{Z}^2$ is not antichain-finite.*

The following result can be found in [2, Chapter 2, Exercise 2.20].

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Proposition 4. *Let X and Y be two posets, which are both noetherian and antichain-finite. Then so is the product poset $X \times Y$.*

Proof. In view of Lemma 2, it suffices to prove the antichain-finiteness. Let $S \subseteq X \times Y$ be an antichain. Consider $X_S = \{x \in X \mid \text{there exists some } (x, y) \in S\}$. We claim that X_S is artinian.

For the claim, we assume on the contrary that there is a strictly descending chain

$$x_0 \succ x_1 \succ x_2 \succ \cdots$$

in X_S. For each $i \geq 0$, we take $y_i \in Y$ with $(x_i, y_i) \in S$. Since S is an antichain, we infer that these y_i's are pairwise distinct; moreover, whenever $i < j$, the inequality $y_j \preceq y_i$ does not hold. Consider the following set.

$$Y' = \{y_i \mid i \geq 0, \text{ there exists no such } j > i \text{ with } y_j \succ y_i\}$$

We infer that Y' is an antichain in Y, and thus finite. Take $m_0 = |Y'| + 1$. Then y_{m_0} does not belong to Y'. Therefore, we have some $m_1 > m_0$ with $y_{m_1} \succ y_{m_0}$. We iterate this process and obtain a strict ascending chain

$$y_{m_0} \prec y_{m_1} \prec y_{m_2} \prec \cdots$$

in Y, which leads to a contradiction.

We use the claim and Proposition 1 to infer that X_S is finite. Similarly, the set $Y_S = \{y \in Y \mid \text{there exists some } (x, y) \in S\}$ is also finite. It follows that S is finite, as required. $\square$

By duality, we have the following result.

Proposition 5. *Let X and Y be two posets, which are both artinian and antichain-finite. Then so is the product poset $X \times Y$.* $\square$

The following immediate consequence of Proposition 5 is due to [1, Lemma A].

Corollary 6. (Dickson's Lemma) *For each $m \geq 1$, the product poset $\mathbb{N}^m$ is antichain-finite.*

Remark 7. Let k be any field. We mention that Dickson's Lemma can be proved directly by the following well-known fact: any monomial ideal in the polynomial algebra $k[x_1, \cdots, x_m]$ is finitely generated.

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References

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