A NOTE ON TORSION PAIRS

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Abstract. For any given torsion pair in the module category over an artin algebra, we obtain an equation involving the cardinalities of indecomposable Ext-projectives andExt-injectives. The equation implies the following well-known result: the torsion class is functorially finite if and only if so is the torsionfree class.

Let $A$ be an artin algebra over a commutative artinian ring $k$. We denote by $A$-mod the category of finitely generated left $A$-modules. By subcategories of $A$-mod, we always mean full additive subcategories which are closed under direct summands.

We identify modules if they are isomorphic. We denote by $n(A)$ the number of simple $A$-modules. For a module $M$, we denote by $\mu(M)$ the number of indecomposable modules, which are isomorphic to a direct summand of $M$.

Recall that a torsion pair in $A$-mod consists of a pair $(\mathcal{T}, \mathcal{F})$ of subcategories such that $\mathcal{T} = \{X | \text{Hom}_A(X,F) = 0 \text{ for all } F \in \mathcal{F}\}$ and $\mathcal{F} = \{Y | \text{Hom}_A(T,Y) = 0 \text{ for all } T \in \mathcal{T}\}$, where we call $\mathcal{T}$ a torsion class and $\mathcal{F}$ a torsionfree class. In this case, for each module $M$ there is a canonical exact sequence

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0$$

with $t(M) \in \mathcal{T}$ and $M/t(M) \in \mathcal{F}$. We observe that a subcategory is a torsion class if and only if it is closed under extensions and factor modules.

Let $C$ be a subcategory and $M$ an $A$-module. By a right $C$-approximation of $M$, we mean a morphism $f : C \rightarrow M$ with $C \in C$ such that any morphism $t : C' \rightarrow M$ with $C' \in C$ admits a factorization $t = f \circ t'$ for some $t' : C' \rightarrow C$. The subcategory $C$ is contravariantly finite provided that each $A$-module has a right $C$-approximation.

Dually, one has the notions of a left $C$-approximation and a covariantly finite subcategory. A subcategory is functorially finite if it is both contravariantly finite and covariantly finite.

We observe that a torsion class $\mathcal{T}$ is contravariantly finite, since the monomorphism $t(M) \rightarrow M$ in the canonical sequence is a right $\mathcal{T}$-approximation of $M$. Similarly, a torsionfree class is covariantly finite.

For a subcategory $\mathcal{C}$ which is closed under extensions, an object $X \in \mathcal{C}$ is Ext-projective if $\text{Ext}^1_A(X, -)$ vanishes on $\mathcal{C}$. We denote by $\exp(\mathcal{C})$ the cardinality of indecomposable Ext-projective objects in $\mathcal{C}$. Dually, one has the notion of Ext-injective objects and the cardinality $\text{exi}(\mathcal{C})$.

The following result is implicitly contained in the proof of [8, Theorem], where the “if” part of the second statement is essentially due to [4, Theorem A.6].

Proposition 1. Let $\mathcal{T}$ be a torsion class in $A$-mod. Then we have

$$\exp(\mathcal{T}) \leq \text{exi}(\mathcal{T}) \leq n(A).$$
Moreover, \( \exp(\mathcal{T}) = \text{exi}(\mathcal{T}) \) if and only if \( \mathcal{T} \) is covariantly finite.

For the proof, we make some preparation. We emphasize that the classical tilting theory is essentially used in the following argument.

For a module \( M \), we denote by \( \text{fac}(M) \) the subcategory formed by modules generated by \( M \), that is, factor modules of finite direct sums of copies of \( M \). A module \( M \) is \( \tau \)-rigid if \( \text{Hom}_A(M, \tau(M)) = 0 \), where \( \tau \) denotes the Auslander-Reiten translation.

**Lemma 2.** Let \( \mathcal{T} \) be a torsion class. Then the following statements are equivalent.

1. The subcategory \( \mathcal{T} \) is covariantly finite.
2. \( \mathcal{T} = \text{fac}(M) \) for some \( \tau \)-rigid module \( M \).
3. \( \mathcal{T} = \text{fac}(M) \) for some module \( M \).

**Proof.** The equivalence between (1) and (3) is contained in [2, Proposition 4.6(c)]. To see “(3)⇒(2)”, we may assume that \( M \) is minimal in the sense of [1, 1.2] and then apply [3, Propositions 5.5 and 5.8].

The second statement of the following result is due to [7, Lemma 4]; compare [4, Corollary A.4].

**Lemma 3.** Let \( \mathcal{T} \) be a torsion class, which contains all injective \( A \)-modules. Then the following facts hold.

1. An object \( X \) in \( \mathcal{T} \) is Ext-injective if and only if it is an injective \( A \)-module.
2. Any Ext-projective object in \( \mathcal{T} \) has projective dimension at most one.

**Proof.** For (1), we only prove the “only if” part. Take an exact sequence \( 0 \to X \to I \to C \to 0 \) with \( I \) injective. Then \( C \) lies in \( \mathcal{T} \). Since \( X \) is Ext-injective, the sequence splits. It follows that \( X \) is injective.

Let \( Y \) be an Ext-projective object. For any module \( M \), we take an exact sequence \( 0 \to M \to I' \to C' \to 0 \) with \( I' \) injective. Then we have \( C' \in \mathcal{T} \). By a dimension-shift, we have \( \text{Ext}_A^1(Y, M) \simeq \text{Ext}_A^1(Y, C') \), which equals zero by the Ext-projectivity of \( Y \). Then we are done.

An \( A \)-module \( T \) is partial tilting if \( \text{Ext}_A^1(T, T) = 0 \) and its projective dimension is at most one. A partial tilting module \( T \) is tilting if \( \mu(T) = n(A) \). This is equivalent to the original definition in [6] by [5, 2.1]. For a module \( X \), we denote by \( \text{add} X \) the subcategory consisting of direct summands of finite direct sums of \( X \).

**Lemma 4.** The following statements hold.

1. For any partial tilting \( A \)-module \( T \), we have \( \mu(T) \leq n(A) \).
2. For a faithful partial tilting module \( T \), there is a tilting module \( X = T \oplus T' \) with \( \text{fac}(X) = \text{fac}(T) \).
3. For a tilting \( A \)-module \( T \), we have \( \mathcal{T} = \text{fac}(T) = \{ M \mid \text{Ext}_A^1(T, M) = 0 \} \) and \( \exp(T) = \text{exi}(T) = n(A) \).

**Proof.** (1) is contained in [5, 2.1], and (3) is contained in [1, Lemma 1.6 and Corollary 1.8]. For (2), we take a monomorphism \( f: A \to T'' \) which is a left (add \( T' \))-approximation of \( A \). Set \( T' \) to be the cokernel. Then \( X = T \oplus T' \) is a tilting module; compare the last paragraph in the proof of [1, Lemma 3.1].

**Proof of Proposition 1.** We denote by \( \text{Ann}(\mathcal{T}) \) the ideal of \( A \), which is the intersection of the annihilators of all modules in \( \mathcal{T} \). Let \( A' = A/\text{Ann}(\mathcal{T}) \). We view \( \mathcal{T} \) as a subcategory in \( A' \)-mod.

We observe that \( \mathcal{T} \) contains a faithful \( A' \)-module. Indeed, there is some module \( M \in \mathcal{T} \), whose annihilator coincides with \( \text{Ann}(\mathcal{T}) \). Then \( M \) is a faithful \( A' \)-module.
It follows that $M$ generates all injective $A'$-modules. In particular, $\mathcal{T}$ contains all injective $A'$-modules. By Lemma 3(1), we have $\text{exi}(\mathcal{T}) = n(A') \leq n(A)$. Moreover, any Ext-projective object $Y$ in $\mathcal{T}$ is a partial tilting $A'$-module, and thus $\mu(Y) \leq n(A')$ by Lemma 4(1). It implies that $\exp(\mathcal{T}) \leq n(A')$.

It remains to prove the second statement. For the “only if” part, we assume that $\exp(\mathcal{T}) = \text{exi}(\mathcal{T})$, which equals $n(A')$. Take the direct sum $Y = \bigoplus_{i=1}^{n(A')} T_i$ of all the indecomposable Ext-projectives. Then $Y$ is a tilting $A'$-module. We observe that $\text{fac}(Y) \subseteq \mathcal{T} \subseteq \{M \in A'\text{-mod} | \text{Ext}^1_{A'}(Y, M) = 0\}$. Then we have equalities by Lemma 4(3). Then $\mathcal{T}$ is covariantly finite by Lemma 2.

For the “if” part, we may assume that $\mathcal{T} = \text{fac}(M)$ for a faithful $A'$-module $M$ which is $\tau$-rigid; see Lemma 2. In particular, $M$ is a partial tilting $A'$-module; compare [1, Lemma 1.5]. By Lemma 4(2), we have $\mathcal{T} = \text{fac}(T)$ for some tilting $A'$-module $T$. Then we are done by Lemma 4(3). □

We have the following dual of Proposition 1.

**Proposition 5.** Let $\mathcal{F}$ be a torsionfree class in $A$-mod. Then we have
\[ \text{exi}(\mathcal{F}) \leq \exp(\mathcal{F}) \leq n(A). \]
Moreover, $\text{exi}(\mathcal{F}) = \exp(\mathcal{F})$ if and only if $\mathcal{F}$ is contravariantly finite. □

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. An almost split sequence $0 \to \tau(X) \to E \to X \to 0$ of $A$-modules is connecting provided that $X \in \mathcal{T}$ and $\tau(X) \in \mathcal{F}$.

We denote by $c$ the cardinality of connecting sequences. We denote by $p$ the number of indecomposable projective $A$-modules contained in $\mathcal{T}$, and by $i$ the number of indecomposable injective $A$-modules contained in $\mathcal{F}$.

We have the following main result, whose last statement is due to [8].

**Theorem 6.** Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $A$-mod. Then we have
\[ \text{exi}(\mathcal{T}) - \exp(\mathcal{T}) = \exp(\mathcal{F}) - \text{exi}(\mathcal{F}) = n(A) - p - i - c \geq 0. \]
In particular, $\mathcal{T}$ is covariantly finite if and only if $\mathcal{F}$ is contravariantly finite.

The essential argument is contained in the following result.

**Lemma 7.** Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. Then the following facts hold.

1. An indecomposable object $X$ in $\mathcal{T}$ is Ext-injective if and only if $X \simeq t(I)$ for an indecomposable injective $A$-module $I$, which is not contained in $\mathcal{F}$.

2. An indecomposable object $Y$ in $\mathcal{F}$ is Ext-projective if and only if $X \simeq P/\tau(P)$ for an indecomposable projective $A$-module $P$, which is not contained in $\mathcal{T}$.

3. There is a bijection between
\{indecomposable Ext-projectives in $\mathcal{T}$\} $\leftrightarrow$ \{indecomposable Ext-injectives in $\mathcal{F}$\}

sending $X$ to $\tau(X)$, which are in a bijection to the set of connecting sequences.

**Proof.** (1) and (2) are contained in [3, Propositions 3.1 and 3.2], and (3) follows from [3, Corollaries 3.4 and 3.7]; compare [1, Lemma 1.4]. □

The following counting argument resembles the ones in [7, Lemma 6] and [8, Theorem].

**Proof of Theorem 6.** By Lemma 7(1) and (2), we have $\text{exi}(\mathcal{T}) = n(A) - i$ and $\exp(\mathcal{F}) = n(A) - p$. By the bijections in Lemma 7(3), we have $\exp(\mathcal{T}) - p = \text{exi}(\mathcal{F}) - i = c$. Combining these equations, we deduce the required equation.

In view of Propositions 1 and 5, the last statement follows immediately. □
Acknowledgements. The result is obtained when the author was giving a series of lectures on the classical tilting theory in USTC, which is based on [1]. The author thanks Jie Li for pointing out the “only if” part of Proposition 1. The work is supported by National Natural Science Foundation of China (No. 11522113).

REFERENCES


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