A NOTE ON TORSION PAIRS

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Abstract. For any given torsion pair in the module category over an artin algebra, we obtain an equation involving the cardinalities of indecomposable Ext-projectives and Ext-injectives. The equation implies the following well-known result: the torsion class is functorially finite if and only if so is the torsionfree class.

Let A be an artin algebra over a commutative artinian ring k. We denote by A-mod the category of finitely generated left A-modules. By subcategories of A-mod, we always mean full additive subcategories which are closed under direct summands.

We identify modules if they are isomorphic. We denote by $n(A)$ the number of simple A-modules. For a module M, we denote by $\mu(M)$ the number of indecomposable modules, which are isomorphic to a direct summand of M.

Recall that a *torsion pair* in A-mod consists of a pair $(\mathcal{T}, \mathcal{F})$ of subcategories such that $\mathcal{T} = \{X|\ \mathrm{Hom}_A(X, F) = 0 \text{ for all } F \in \mathcal{F}\}$ and $\mathcal{F} = \{Y|\ \mathrm{Hom}_A(T, Y) = 0 \text{ for all } T \in \mathcal{T}\}$, where we call $\mathcal{T}$ a *torsion class* and $\mathcal{F}$ a *torsionfree class.* In this case, for each module M there is a *canonical* exact sequence

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0$$

with $t(M) \in \mathcal{T}$ and $M/t(M) \in \mathcal{F}$. We observe that a subcategory is a torsion class if and only if it is closed under extensions and factor modules.

Let $\mathcal{C}$ be a subcategory and M an A-module. By a *right $\mathcal{C}$-approximation* of M, we mean a morphism $f\colon C \to M$ with $C \in \mathcal{C}$ such that any morphism $t\colon C' \to M$ with $C' \in \mathcal{C}$ admits a factorization $t = f \circ t'$ for some $t'\colon C' \to C$. The subcategory $\mathcal{C}$ is *contravariantly finite* provided that each A-module has a right $\mathcal{C}$-approximation.

Dually, one has the notions of a *left $\mathcal{C}$-approximation* and a *covariantly finite* subcategory. A subcategory is *functorially finite* if it is both contravariantly finite and covariantly finite.

We observe that a torsion class $\mathcal{T}$ is contravariantly finite, since the monomorphism $t(M) \to M$ in the canonical sequence is a right $\mathcal{T}$-approximation of M. Similarly, a torsionfree class is covariantly finite.

For a subcategory $\mathcal{C}$ which is closed under extensions, an object $X \in \mathcal{C}$ is *Ext-projective* if $\mathrm{Ext}^1_A(X, -)$ vanishes on $\mathcal{C}$. We denote by $\exp(\mathcal{C})$ the cardinality of indecomposable Ext-projective objects in $\mathcal{C}$. Dually, one has the notion of *Ext-injective* objects and the cardinality $\mathrm{exi}(\mathcal{C})$.

The following result is implicitly contained in the proof of [8, Theorem], where the "if" part of the second statement is essentially due to [4, Theorem A.6].

Proposition 1. *Let $\mathcal{T}$ be a torsion class in A-mod. Then we have*

$$\exp(\mathcal{T}) \leq \mathrm{exi}(\mathcal{T}) \leq n(A).$$

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Moreover, $\exp(\mathcal{T}) = \mathrm{exi}(\mathcal{T})$ *if and only if* $\mathcal{T}$ *is covariantly finite.*

For the proof, we make some preparation. We emphasize that the classical tilting theory is essentially used in the following argument.

For a module M, we denote by $\mathrm{fac}(M)$ the subcategory formed by modules generated by M, that is, factor modules of finite direct sums of copies of M. A module M is *τ-rigid* if $\mathrm{Hom}_A(M, \tau(M)) = 0$, where τ denotes the Auslander-Reiten translation.

Lemma 2. *Let $\mathcal{T}$ be a torsion class. Then the following statements are equivalent.*

(1) *The subcategory $\mathcal{T}$ is covariantly finite.*
(2) $\mathcal{T} = \mathrm{fac}(M)$ *for some τ-rigid module M.*
(3) $\mathcal{T} = \mathrm{fac}(M)$ *for some module M.*

Proof. The equivalence between (1) and (3) is contained in [2, Proposition 4.6(c)]. To see "(3)⇒(2)", we may assume that M is minimal in the sense of [1, 1.2] and then apply [3, Propositions 5.5 and 5.8]. □

The second statement of the following result is due to [7, Lemma 4]; compare [4, Corollary A.4].

Lemma 3. *Let $\mathcal{T}$ be a torsion class, which contains all injective A-modules. Then the following facts hold.*

(1) *An object X in $\mathcal{T}$ is Ext-injective if and only if it is an injective A-module.*
(2) *Any Ext-projective object in $\mathcal{T}$ has projective dimension at most one.*

Proof. For (1), we only prove the "only if" part. Take an exact sequence $0 \to X \to I \to C \to 0$ with I injective. Then C lies in $\mathcal{T}$. Since X is Ext-injective, the sequence splits. It follows that X is injective.

Let Y be an Ext-projective object. For any module M, we take an exact sequence $0 \to M \to I' \to C' \to 0$ with I' injective. Then we have $C' \in \mathcal{T}$. By a dimension-shift, we have $\mathrm{Ext}^2_A(Y, M) \simeq \mathrm{Ext}^1_A(Y, C')$, which equals zero by the Ext-projectivity of Y. Then we are done. □

An A-module T is *partial tilting* if $\mathrm{Ext}^1_A(T, T) = 0$ and its projective dimension is at most one. A partial tilting module T is *tilting* if $\mu(T) = n(A)$. This is equivalent to the original definition in [6] by [5, 2.1]. For a module X, we denote by $\mathrm{add}\, X$ the subcategory consisting of direct summands of finite direct sums of X.

Lemma 4. *The following statements hold.*

(1) *For any partial tilting A-module T, we have $\mu(T) \leq n(A)$.*
(2) *For a faithful partial tilting module T, there is a tilting module $X = T \oplus T'$ with $\mathrm{fac}(X) = \mathrm{fac}(T)$.*
(3) *For a tilting A-module T, we have $\mathcal{T} = \mathrm{fac}(T) = \{M \mid \mathrm{Ext}^1_A(T, M) = 0\}$ and $\exp(\mathcal{T}) = \mathrm{exi}(\mathcal{T}) = n(A)$.*

Proof. (1) is contained in [5, 2.1], and (3) is contained in [1, Lemma 1.6 and Corollary 1.8]. For (2), we take a monomorphism $f\colon A \to T^n$ which is a left $(\mathrm{add}\, T)$-approximation of A. Set T' to be the cokernel. Then $X = T \oplus T'$ is a tilting module; compare the last paragraph in the proof of [1, Lemma 3.1]. □

Proof of Proposition 1. We denote by $\mathrm{Ann}(\mathcal{T})$ the ideal of A, which is the intersection of the annihilators of all modules in $\mathcal{T}$. Let $A' = A/\mathrm{Ann}(\mathcal{T})$. We view $\mathcal{T}$ as a subcategory in A'-mod.

We observe that $\mathcal{T}$ contains a faithful A'-module. Indeed, there is some module $M \in \mathcal{T}$, whose annihilator coincides with $\mathrm{Ann}(\mathcal{T})$. Then M is a faithful A'-module.

It follows that M generates all injective A'-modules. In particular, $\mathcal{T}$ contains all injective A'-modules. By Lemma 3(1), we have $\mathrm{exi}(\mathcal{T}) = n(A') \leq n(A)$. Moreover, any Ext-projective object Y in $\mathcal{T}$ is a partial tilting A'-module, and thus $\mu(Y) \leq n(A')$ by Lemma 4(1). It implies that $\exp(\mathcal{T}) \leq n(A')$.

It remains to prove the second statement. For the "only if" part, we assume that $\exp(\mathcal{T}) = \mathrm{exi}(\mathcal{T})$, which equals $n(A')$. Take the direct sum $Y = \bigoplus_{i=1}^{n(A')} T_i$ of all the indecomposable Ext-projectives. Then Y is a tilting A'-module. We observe that $\mathrm{fac}(Y) \subseteq \mathcal{T} \subseteq \{M \in A'\text{-mod} \mid \mathrm{Ext}^1_{A'}(Y, M) = 0\}$. Then we have equalities by Lemma 4(3). Then $\mathcal{T}$ is covariantly finite by Lemma 2.

For the "if" part, we may assume that $\mathcal{T} = \mathrm{fac}(M)$ for a faithful A'-module M which is τ-rigid; see Lemma 2. In particular, M is a partial tilting A'-module; compare [1, Lemma 1.5]. By Lemma 4(2), we have $\mathcal{T} = \mathrm{fac}(T)$ for some tilting A'-module T. Then we are done by Lemma 4(3). □

We have the following dual of Proposition 1.

Proposition 5. *Let $\mathcal{F}$ be a torsionfree class in A-mod. Then we have*

$$\mathrm{exi}(\mathcal{F}) \leq \exp(\mathcal{F}) \leq n(A).$$

Moreover, $\mathrm{exi}(\mathcal{F}) = \exp(\mathcal{F})$ *if and only if $\mathcal{F}$ is contravariantly finite.* □

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. An almost split sequence $0 \to \tau(X) \to E \to X \to 0$ of A-modules is *connecting* provided that $X \in \mathcal{T}$ and $\tau(X) \in \mathcal{F}$.

We denote by c the cardinality of connecting sequences. We denote by p the number of indecomposable projective A-modules contained in $\mathcal{T}$, and by i the number of indecomposable injective A-modules contained in $\mathcal{F}$.

We have the following main result, whose last statement is due to [8].

Theorem 6. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in A-mod. Then we have*

$$\mathrm{exi}(\mathcal{T}) - \exp(\mathcal{T}) = \exp(\mathcal{F}) - \mathrm{exi}(\mathcal{F}) = n(A) - p - i - c \geq 0.$$

In particular, $\mathcal{T}$ is covariantly finite if and only if $\mathcal{F}$ is contravariantly finite.

The essential argument is contained in the following result.

Lemma 7. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. Then the following facts hold.*

(1) *An indecomposable object X in $\mathcal{T}$ is Ext-injective if and only if $X \simeq t(I)$ for an indecomposable injective A-module I, which is not contained in $\mathcal{F}$.*
(2) *An indecomposable object Y in $\mathcal{F}$ is Ext-projective if and only if $X \simeq P/t(P)$ for an indecomposable projective A-module P, which is not contained in $\mathcal{T}$.*
(3) *There is a bijection between*

$$\{\text{indec. non-proj. Ext-projectives in } \mathcal{T}\} \leftrightarrow \{\text{indec. non-inj. Ext-injectives in } \mathcal{F}\}$$

sending X to $\tau(X)$, which are in a bijection to the set of connecting sequences.

Proof. (1) and (2) are contained in [3, Propositions 3.1 and 3.2], and (3) follows from [3, Corollaries 3.4 and 3.7]; compare [1, Lemma 1.4]. □

The following counting argument resembles the ones in [7, Lemma 6] and [8, Theorem].

Proof of Theorem 6. By Lemma 7(1) and(2), we have $\mathrm{exi}(\mathcal{T}) = n(A) - i$ and $\exp(\mathcal{F}) = n(A) - p$. By the bijections in Lemma 7(3), we have $\exp(\mathcal{T}) - p = \mathrm{exi}(\mathcal{F}) - i = c$. Combining these equations, we deduce the required equation.

In view of Propositions 1 and 5, the last statement follows immediately. □

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