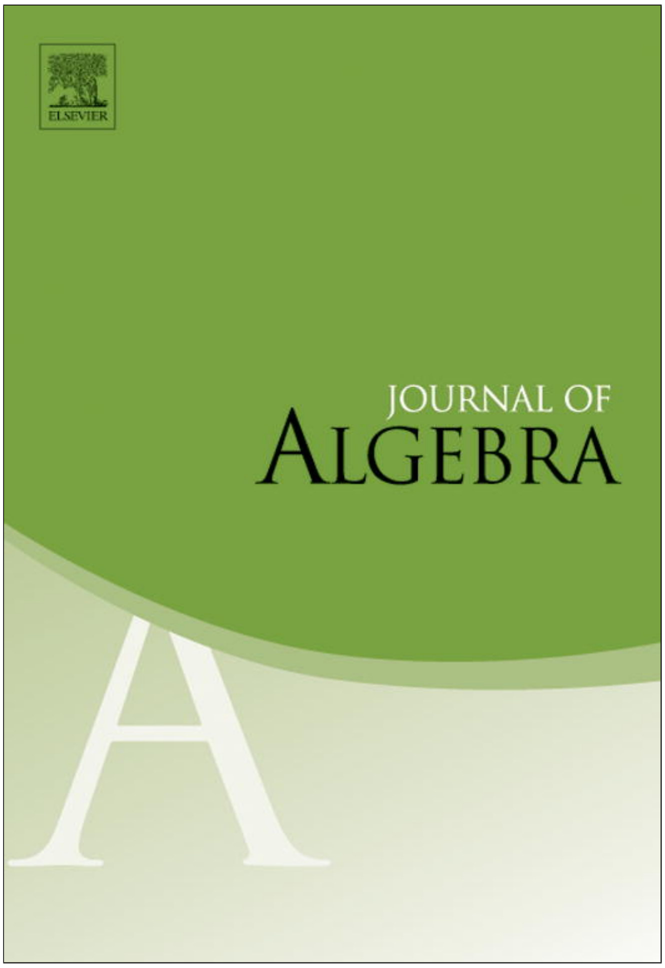
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Totally reflexive extensions and modules

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ABSTRACT

We introduce the notion of totally reflexive extension of rings. It unifies Gorenstein orders and Frobenius extensions. We prove that for a totally reflexive extension, a module over the extension ring is totally reflexive if and only if its underlying module over the base ring is totally reflexive.

1. Introduction

The study of totally reflexive modules goes back to Auslander [2,4], and is highlighted by the work of Buchweitz on Tate cohomology of Gorenstein rings [7]. Totally reflexive modules play an important role in singularity theory [7,19], cohomology theory of commutative rings [6,10] and representation theory of Artin algebras [5,18]. There are several different terminologies in the literature for these modules, such as modules of G-dimension zero [4], maximal Cohen–Macaulay modules [7] and (finitely generated) Gorenstein-projective modules [11,9].

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Recall that a ring extension means a homomorphism $\theta : S \to R$ between two rings; thus S is the base ring and R is the extension ring. We are interested in the following question.

What kind of ring extensions $\theta : S \to R$ satisfy the following condition: any R-module X is totally reflexive if and only if the underlying S-module X is totally reflexive?

We recall two well-known examples. Some results in [3, Chapter III] and [7, Section 7] suggest that a ring extension $\theta : S \to R$ satisfies the above condition, provided that S is a commutative Gorenstein ring and R is a Gorenstein S-order. This generalizes the following classical example. Let $S = \mathbb{Z}$, and let $R = \mathbb{Z}G$ be the integral group ring of a finite group G. Then a $\mathbb{Z}G$-module X is totally reflexive if and only if the underlying $\mathbb{Z}$-module X is totally reflexive, or equivalently, the abelian group X is free of finite rank; compare [7, Section 8].

For another example, let $S = kQ$ be the path algebra of a finite acyclic quiver Q over a field k and A a finite dimensional Frobenius algebra. Set $R = S \otimes_k A$ to be the tensor product. Then the natural embedding $S \to R$ satisfies the above condition. Consequently, an R-module X is totally reflexive if and only if the underlying S-module X is totally reflexive, or equivalently, the corresponding representation X of the quiver Q is projective; compare [8,18,20]. Here, we recall the fact that $S = kQ$ is hereditary. We observe that this natural embedding $S \to R$ is a Frobenius extension [17,14]. Note that the above classical example $\mathbb{Z} \to \mathbb{Z}G$ is also a Frobenius extension. So one expects that a Frobenius extension $\theta : S \to R$ might satisfy the above condition.

We introduce the notion of totally reflexive extension. It is a ring extension $\theta : S \to R$ subject to two conditions: (1) the natural S-module R is totally reflexive on both sides; (2) there is an R-S-bimodule isomorphism $\mathrm{Hom}_S(R, P) \simeq W$ for some invertible S-bimodule P and invertible R-bimodule W. Here, $\mathrm{Hom}_S(R, P)$ denotes the abelian group consisting of left S-module homomorphisms, which carries a natural R-S-bimodule structure.

The new notion above unifies Gorenstein orders [3] and Frobenius extensions [17]. Indeed, it unifies their generalizations such as Gorenstein algebras [4] and Frobenius extensions of the third kind [15].

The main result gives a partial answer to the above question.

Theorem. *Let $\theta : S \to R$ be a totally reflexive extension with respect to an invertible S-bimodule P and an invertible R-bimodule W. Then an R-module X is totally reflexive if and only if the underlying S-module X is totally reflexive.*

We mention that some related problems are considered in [13] for (weak) excellent extensions of rings and in [16] for change of rings.

The paper is organized as follows. In Section 2, we introduce the notion of reflexive extension and show that it is left–right symmetric; see Definition 2.3 and Proposition 2.4. This notion is justified by a similar result as Theorem, where one replaces “totally reflexive” by “reflexive”; see Proposition 2.7. In Section 3, we introduce the notion of totally reflexive extension as a special kind of reflexive extension, which is also left–right symmetric; see Definition 3.3. We point out that this notion unifies Gorenstein algebras and Frobenius extensions of the third kind; see Example 3.5. Then we prove the main result; see Theorem 3.6.

All rings in the paper are associative rings with a unit. Homomorphisms of rings are required to send the unit to the unit.

2. Reflexive extensions

In this section, we introduce the notion of reflexive extension of rings. This notion is justified by the following result: in a reflexive extension, any module over the extension ring is reflexive if and only the underlying module over the base ring is reflexive. The left–right symmetric property of reflexive extensions is proved.

2.1. Reflexive modules

Let S be a ring. A left S-module X is sometimes written as ${}_SX$. We denote by S-Mod the category of left S-modules. For two left S-modules ${}_SX$ and ${}_SY$, denote by $\mathrm{Hom}_S(X,Y)$ the abelian group consisting of left S-homomorphisms between them. A right S-module M is written as M_S. We identify right S-modules as left S^{op}-modules, where S^{op} is the opposite ring of S. Hence, for two right S-modules M_S and N_S, we denote by $\mathrm{Hom}_{S^{\mathrm{op}}}(M,N)$ the abelian group consisting of right S-homomorphisms between them. The S-action on modules is usually denoted by ".".

Let R and T be another two rings. Assume that ${}_SX_R$ and ${}_SY_T$ are an S-R-bimodule and an S-T-bimodule, respectively. Then the abelian group $\mathrm{Hom}_S(X,Y)$ is naturally an R-T-bimodule in the following way: for $r\in R$, $f\in \mathrm{Hom}_S(X,Y)$ and $t\in T$, $(r.f.t)(x)=f(x.r).t$ for all $x\in X$.

Let P be an S-bimodule. For a left S-module X, the *evaluation map* with values in P

$$\mathrm{ev}_X^P : X \to \mathrm{Hom}_{S^{\mathrm{op}}}\big(\mathrm{Hom}_S(X,P),P\big),$$

defined as $\mathrm{ev}_X^P(x)(f)=f(x)$, is a left S-homomorphism. The module ${}_SX$ is *P-reflexive* (resp. *P-torsionless*), provided that ev_X^P is an isomorphism (resp. a monomorphism). In case that $P={}_SS_S$ is the regular bimodule, ev_X^P is abbreviated as ev_X, and P-reflexive (resp. P-torsionless) modules are called *reflexive modules* (resp. *torsionless modules*). For example, a finitely generated projective S-module is reflexive. We have similar notions for right modules.

We recall that an S-bimodule P is *invertible* provided that the endofunctor $\mathrm{Hom}_S(P,-)$ on the category S-Mod of left S-modules is an equivalence. For example, the regular bimodule ${}_SS_S$ is invertible.

The following fact is taken from [12, Chapter 12].

Lemma 2.1. *Let P be an invertible S-bimodule. The following statements hold*:

(1) *both ${}_SP$ and P_S are finitely generated projective generators*;
(2) *the natural ring homomorphisms $S\to \mathrm{End}_{S^{\mathrm{op}}}(P)$ and $S^{\mathrm{op}}\to \mathrm{End}_S(P)$, induced by the left S-action and right S-action, respectively, are isomorphisms*;
(3) *both the endofunctors $-\otimes_S P$ and $\mathrm{Hom}_{S^{\mathrm{op}}}(P,-)$ on the category S^{op}-Mod of right S-modules are equivalences.* $\square$

The following result is well known.

Lemma 2.2. *Let P be an invertible S-bimodule. Then a left S-module X is reflexive* (*resp. torsionless*) *if and only if it is P-reflexive* (*P-torsionless*).

Proof. Recall that the left S-module ${}_SP$ is finitely generated projective. Hence, the natural homomorphism

$$t:\mathrm{Hom}_S(X,S)\otimes_S P\to \mathrm{Hom}_S(X,P), \tag{2.1}$$

defined by $t(f\otimes p)(x)=f(x).p$, is an isomorphism. The following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\mathrm{ev}_X"] \arrow[dd, equal] & \mathrm{Hom}_{S^{\mathrm{op}}}(\mathrm{Hom}_S(X,S),S) \arrow[d, "-\otimes_S P"] \\
& \mathrm{Hom}_{S^{\mathrm{op}}}(\mathrm{Hom}_S(X,S)\otimes_S P, P) \arrow[d, "\mathrm{Hom}_{S^{\mathrm{op}}}(t,P)"] \\
X \arrow[r, "\mathrm{ev}_X^P"] & \mathrm{Hom}_{S^{\mathrm{op}}}(\mathrm{Hom}_S(X,P),P)
\end{tikzcd}$$

Here, we identify P with $S \otimes_S P$. Recall that the endofunctor $-\otimes_S P$ on S^{op}-Mod is an equivalence. Then in the diagram above, the map ev_X is an isomorphism (resp. a monomorphism) if and only if so is ev_X^P. □

2.2. Reflexive extensions

Recall that a ring extension is a ring homomorphism $\theta : S \to R$. Thus S is the base ring and R is the extension ring. We have that via θ, R becomes an S-bimodule. We sometimes consider R as an R-S-bimodule or an S-R-bimodule. Similarly, an R-module X is naturally an S-module, which is referred as the *underlying* S-module.

Definition 2.3. Let $\theta : S \to R$ be a ring extension. Let P be an invertible S-bimodule and W an invertible R-bimodule. The ring extension θ is *P-W-reflexive*, provided that the following two conditions are satisfied:

(1) the left S-module ${}_SR$ is reflexive;
(2) there is an isomorphism $\mathrm{Hom}_S(R,P) \simeq W$ of R-S-bimodules.

In case that $P = {}_SS_S$ and $W = {}_RR_R$ are the corresponding regular bimodules, a P-W-reflexive extension is called a *reflexive extension*. □

The following result implies that the above notion is left–right symmetric: a ring extension $\theta : S \to R$ is P-W-reflexive if and only if the corresponding extension $\theta^{\mathrm{op}} : S^{\mathrm{op}} \to R^{\mathrm{op}}$ is P-W-reflexive. It is analogous to a classical result [14, Theorem 1.2] of Frobenius extensions.

Proposition 2.4. *Let $\theta : S \to R$ be a ring extension. Assume that P is an invertible S-bimodule and that W is an invertible R-bimodule. Then the following statements are equivalent*:

(1) *the ring extension θ is P-W-reflexive*;
(2) *there is an S-bimodule homomorphism $\phi : W \to P$ such that the induced maps*

$$l_\phi : W \to \mathrm{Hom}_S(R,P) \quad \textit{and} \quad r_\phi : W \to \mathrm{Hom}_{S^{\mathrm{op}}}(R,P),$$

defined by $l_\phi(w) = \phi(-.w)$ and $r_\phi(w) = \phi(w.-)$, are bijective;
(3) *the right S-module R_S is reflexive, and there is an isomorphism of S-R-bimodules* $\mathrm{Hom}_{S^{\mathrm{op}}}(R,P) \simeq W$.

For each element $w \in W$, the left S-homomorphism $\phi(-.w) : R \to P$ is defined as $\phi(-.w)(r) = \phi(r.w)$. Similarly, we have the right S-homomorphism $\phi(w.-) : R \to P$.

The S-bimodule homomorphism $\phi : W \to P$ is called the *reflexive homomorphism* of the ring extension θ. We remark that the induced maps l_ϕ and r_ϕ are always an R-S-homomorphism and an S-R-homomorphism, respectively.

Proof. "(1) $\Rightarrow$ (2)" Denote by $\Phi : W \to \mathrm{Hom}_S(R, P)$ the isomorphism of R-S-bimodules in the definition. We define $\phi : W \to P$ by $\phi(w) = \Phi(w)(1)$. Observe that

$$\phi(r.w) = \Phi(r.w)(1) = \big(r.\Phi(w)\big)(1) = \Phi(w)(r), \tag{2.2}$$

where the second equality uses that Φ is a left R-homomorphism. This shows that $\Phi = l_\phi$; in particular, l_ϕ is bijective.

We claim that ϕ is an S-bimodule homomorphism. Applying (2.2), we have $\phi(s.w) = \Phi(w)(\theta(s)) = s.(\Phi(w)(1))$, since $\Phi(w) : R \to P$ is a left S-homomorphism. This proves that ϕ is a left S-homomorphism. It remains to show $\phi(w.s) = \phi(w).s$. Recall that $\Phi(w.s) = \Phi(w).s$ for $s \in S$, since Φ is a right S-homomorphism. Then we have the following equalities

$$\phi(w.s) = \Phi(w.s)(1) = \big(\Phi(w).s\big)(1) = \Phi(w)(1).s = \phi(w).s.$$

It remains to show that r_ϕ is bijective. Observe that r_ϕ is a right R-homomorphism. Recall that ${}_SR$ is reflexive, and thus by Lemma 2.2 the evaluation map ev_R^P is an isomorphism. Consider the following composite of isomorphisms

$$\begin{aligned}\Theta : R \xrightarrow{\mathrm{ev}_R^P} \mathrm{Hom}_{S^{\mathrm{op}}}\big(\mathrm{Hom}_S(R, P), P\big) &\xrightarrow{\mathrm{Hom}_{S^{\mathrm{op}}}(\Phi, P)} \mathrm{Hom}_{S^{\mathrm{op}}}(W, P)\\ &\simeq \mathrm{Hom}_{S^{\mathrm{op}}}(W \otimes_R R, P) \simeq \mathrm{Hom}_{R^{\mathrm{op}}}\big(W, \mathrm{Hom}_{S^{\mathrm{op}}}(R, P)\big),\end{aligned}$$

where we identify W with $W \otimes_R R$, and the last isomorphism is the adjoint isomorphism. One computes that $(\Theta(r)(w))(r') = \phi(r.w.r')$. It follows that the following diagram commutes

$$\begin{tikzcd} R \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{R^{\mathrm{op}}}(W, W) \arrow[d, "\mathrm{Hom}_{R^{\mathrm{op}}}(W, r_\phi)"] \\ R \arrow[r, "\Theta"] & \mathrm{Hom}_{R^{\mathrm{op}}}(W, \mathrm{Hom}_{S^{\mathrm{op}}}(R, P)) \end{tikzcd}$$

Here, the upper row is the isomorphism induced by the left R-action on the invertible R-bimodule W; see Lemma 2.1(2). It follows that $\mathrm{Hom}_{R^{\mathrm{op}}}(W, r_\phi)$ is an isomorphism. Recall that $\mathrm{Hom}_{R^{\mathrm{op}}}(W, -)$ is an auto-equivalence on R^{op}-Mod; see Lemma 2.1(3). Then the homomorphism r_ϕ is an isomorphism.

"(2) $\Rightarrow$ (3)" Recall that the map r_ϕ is a homomorphism of S-R-bimodules. Then we have the required isomorphism. It remains to show that the right S-module R_S is reflexive.

Consider the composite of natural isomorphisms

$$\begin{aligned}\Xi : \mathrm{Hom}_S\big(\mathrm{Hom}_{S^{\mathrm{op}}}(R, P), P\big) \xrightarrow{\mathrm{Hom}_S(r_\phi, P)} &\mathrm{Hom}_S(W, P) \simeq \mathrm{Hom}_S(R \otimes_R W, P)\\ \simeq\ &\mathrm{Hom}_R\big(W, \mathrm{Hom}_S(R, P)\big),\end{aligned}$$

where we identify W with $R \otimes_R W$, and the last isomorphism is the adjoint isomorphism. One computes that $(\Xi(f)(w))(r') = f(\phi(r'.w.-))$.

The following diagram commutes by direct calculation

$$\begin{tikzcd} R \arrow[r, "\mathrm{ev}_R^P"] \arrow[d] & \mathrm{Hom}_S(\mathrm{Hom}_{S^{\mathrm{op}}}(R, P), P) \arrow[d, "\Xi"] \\ \mathrm{Hom}_R(W, W) \arrow[r, "\mathrm{Hom}_R(W, l_\phi)"] & \mathrm{Hom}_R(W, \mathrm{Hom}_S(R, P)) \end{tikzcd}$$

where the left vertical map is the isomorphism induced by the right R-action on the invertible bimodule W; see Lemma 2.1(2). Observe that l_ϕ is a left R-homomorphism, and thus an isomorphism. It follows that $\mathrm{Hom}_R(W, l_\phi)$ is an isomorphism. Then the above commutative diagram implies that the evaluation map ev_R^P is an isomorphism. By Lemma 2.2, the right S-module R_S is reflexive.

"(3) ⇒ (1)" Observe that (3) means exactly that the ring extension $\theta^{\mathrm{op}} : S^{\mathrm{op}} \to R^{\mathrm{op}}$ is P-W-reflexive. Then we obtain this implication by symmetry from "(1) ⇒ (3)". □

We consider algebras over commutative rings. Let S be a commutative ring. An S-*algebra* is a ring extension $\theta : S \to R$ such that the image lies in the center of R. In this case, $\mathrm{Hom}_S(R, S)$ is naturally an R-bimodule. We sometimes suppress θ and say that R is an S-algebra.

We call an S-algebra R is *quasi-reflexive* provided that the S-module ${}_SR$ is reflexive and the R-bimodule $\mathrm{Hom}_S(R, S)$ is invertible; it is called *reflexive*, provided that in addition there is an R-bimodule isomorphism $\mathrm{Hom}_S(R, S) \simeq R$.

Lemma 2.5. *Let $\theta : S \to R$ be an algebra such that ${}_SR$ is reflexive. Then the following statements are equivalent*:

(1) *the ring extension θ is S-W-reflexive for some invertible R-module W*;
(2) *the left R-module $\mathrm{Hom}_S(R, S)$ is a finitely generated projective generator*;
(3) *the right R-module $\mathrm{Hom}_S(R, S)$ is a finitely generated projective generator.*

In this case, the R-bimodule $\mathrm{Hom}_S(R, S)$ is invertible.

Proof. We only show the equivalence (1) ⇔ (2). (1) implies (2), since an invertible bimodule is a finitely generated projective generator on each side. On the other hand, observe that $W' = \mathrm{Hom}_S(R, S)$ is an R-R-bimodule. Consider the ring homomorphism $\psi : R^{\mathrm{op}} \to \mathrm{End}_R(W')$ induced by the right R-action on W'. It suffices to show that it is an isomorphism. Then it follows from Morita theory that the R-module W' is invertible, and thus θ is S-W'-reflexive.

Consider the adjoint isomorphism $\gamma : \mathrm{Hom}_R(W', W') \xrightarrow{\sim} \mathrm{Hom}_S(\mathrm{Hom}_S(R, S), S)$. The following diagram commutes

$$\begin{tikzcd} R \arrow[r, "\psi"] \arrow[d, equal] & \mathrm{Hom}_R(W', W') \arrow[d, "\gamma"] \\ R \arrow[r, "\mathrm{ev}_R"] & \mathrm{Hom}_S(\mathrm{Hom}_S(R, S), S) \end{tikzcd}$$

By the assumption, ev_R is bijective. It follows that ψ is bijective. We are done. □

2.3. *Preserving reflexivity*

We will show that a reflexive extension $\theta : S \to R$ "preserves reflexivity": an R-module X is reflexive if and only if the underlying S-module X is reflexive.

We need the following observation.

Lemma 2.6. *Let $\theta : S \to R$ be a P-W-reflexive extension with $\phi : W \to P$ the reflexive homomorphism. Let ${}_RX$ and Y_R be a left R-module and a right R-module, respectively. Then we have the following*:

(1) *there is an isomorphism of right S-modules*

$$t_X : \mathrm{Hom}_R(X, W) \xrightarrow{\sim} \mathrm{Hom}_S(X, P),$$

sending f to $\phi \circ f$;

(2) *there is an isomorphism of left S-modules*

$$t_Y : \mathrm{Hom}_{R^{\mathrm{op}}}(Y, W) \xrightarrow{\sim} \mathrm{Hom}_{S^{\mathrm{op}}}(Y, P),$$

sending g to $\phi \circ g$.

Proof. (1) Recall that $l_\phi : W \to \mathrm{Hom}_S(R, P)$ is an isomorphism of R-S-bimodules. So we have the isomorphism $\mathrm{Hom}_R(X, W) \simeq \mathrm{Hom}_R(X, \mathrm{Hom}_S(R, P))$. Composing with an adjoint isomorphism, we have the required isomorphism t_X. A similar argument proves (2). □

The following result justifies the terminology "reflexive extension".

Proposition 2.7. *Let $\theta : S \to R$ be a P-W-reflexive extension for an invertible S-bimodule P and an invertible R-module W. Then a left R-module ${}_RX$ is reflexive (resp. torsionless) if and only if the underlying S-module ${}_SX$ is reflexive (resp. torsionless).*

Proof. Denote by ϕ the reflexive homomorphism of the ring extension. Recall from Lemma 2.6 the isomorphisms t_X and $t_{\mathrm{Hom}_R(X,W)}$. The following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\mathrm{ev}_X^W"] \arrow[dd, equal] & \mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(X, W), W) \arrow[d, "t_{\mathrm{Hom}_R(X,W)}"] \\
 & \mathrm{Hom}_{S^{\mathrm{op}}}(\mathrm{Hom}_R(X, W), P) \\
X \arrow[r, "\mathrm{ev}_X^P"] & \mathrm{Hom}_{S^{\mathrm{op}}}(\mathrm{Hom}_S(X, P), P) \arrow[u, "\mathrm{Hom}_{S^{\mathrm{op}}}(t_X, P)"']
\end{tikzcd}$$

Indeed, both composite maps send $x \in X$ to an element in $\mathrm{Hom}_{S^{\mathrm{op}}}(\mathrm{Hom}_R(X, W), P)$, which maps $g \in \mathrm{Hom}_R(X, W)$ to $\phi(g(x))$. It follows that the map ev_X^W is an isomorphism (resp. a monomorphism) if and only if so is ev_X^P. Applying Lemma 2.2 twice, we are done with the proof. □

3. Totally reflexive extensions

In this section, we introduce the notion of totally reflexive extension and then prove Theorem. Before that, we recall the notion of totally reflexive modules [4,6] and modules having finitely generated projective resolutions. We point out that totally reflexive extensions unify Gorenstein algebras [1] and Frobenius extensions of the third kind [15].

3.1. Totally reflexive modules

Let S a ring. Following [6], an unbounded acyclic complex $P^\bullet = \cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$ of S-modules is *totally acyclic*, if each P^i is finitely generated projective and the dual complex $\mathrm{Hom}_S(P^\bullet, S)$ is also acyclic. A left S-module X is *totally reflexive* provided that there is a totally acyclic complex $P^\bullet$ such that X is isomorphic to its first cocycle $Z^1(P^\bullet)$. It is well known that a totally reflexive module is reflexive; compare Lemma 3.1(2). As is pointed out in the introduction, there are different terminologies for totally reflexive modules in the literature.

We denote by S-tref the full subcategory of S-Mod consisting of totally reflexive S-modules. Observe that finitely generated projective modules are totally reflexive. By [5, Proposition 5.1] and its dual, the full subcategory S-tref is closed under finite direct sums, direct summands and extensions.

Recall that a left S-module X is said to have a *finitely generated projective resolution* provided that there is a projective resolution $\cdots \to P^{-2} \to P^{-1} \to P^0 \to X \to 0$ with each P^{-i} finitely generated. We denote by S-fgpr the full subcategory of S-Mod consisting of such modules. Observe that $S\text{-tref} \subseteq S\text{-fgpr}$.

The following result is well known; compare [4, (3.8)] and [11]. In particular, the notion of totally reflexive module is "coordinate-free" [7, 4.2].

Lemma 3.1. *Let P be an invertible S-bimodule. Then a left S-module X is totally reflexive if and only if the following conditions are satisfied*:

(1) *both ${}_SX$ and $\mathrm{Hom}_S(X,P)$ have finitely generated projective resolutions*;
(2) *${}_SX$ is P-reflexive*;
(3) $\mathrm{Ext}^n_S(X,P)=0$ *for* $n \geqslant 1$;
(4) $\mathrm{Ext}^n_{S^{\mathrm{op}}}(\mathrm{Hom}_S(X,P),P)=0$ *for* $n \geqslant 1$.

In particular, for a totally reflexive S-module ${}_SX$, the right S-module $\mathrm{Hom}_S(X,P)$ is also totally reflexive.

Proof. Let $P^\bullet$ be an acyclic complex of finitely generated projective left S-modules. The isomorphism (2.1) induces an isomorphism $\mathrm{Hom}_S(P^\bullet,P) \simeq \mathrm{Hom}_S(P^\bullet,S) \otimes_S P$ of complexes of right S-modules. Recall from Lemma 2.1(3) that the functor $- \otimes_S P$ is an auto-equivalence on S^{op}-Mod. Then $P^\bullet$ is totally acyclic if and only if the complex $\mathrm{Hom}_S(P^\bullet,P)$ is acyclic.

For the "only if" part, assume that $P^\bullet$ is a totally acyclic complex with $Z^1(P^\bullet) \simeq X$. Then the truncated complex $\cdots \to P^{-2} \to P^{-1} \to P^0 \to 0 \to \cdots$ is a projective resolution of X. Hence the acyclicity of $\mathrm{Hom}_S(P^\bullet,P)$ implies (3). It further implies that $\cdots \to \mathrm{Hom}_S(P^3,P) \to \mathrm{Hom}_S(P^2,P) \to \mathrm{Hom}_S(P^1,P) \to 0$ is a projective resolution of the right S-module $\mathrm{Hom}_S(X,P)$. Since each right S-module $\mathrm{Hom}_S(P^i,P)$ is finitely generated projective, we have (1). Applying $\mathrm{Hom}_{S^{\mathrm{op}}}(-,P)$ to this resolution, we deduce (2) and (4) from the acyclicity of $P^\bullet$; here, we use the isomorphism $P^\bullet \simeq \mathrm{Hom}_{S^{\mathrm{op}}}(\mathrm{Hom}_S(P^\bullet,P),P)$ of complexes, that is induced by the evaluation maps $\mathrm{ev}^P_{P^i}$.

For the "if" part, take projective resolutions $\cdots \to P^{-2} \to P^{-1} \to P^0 \to X \to 0$ and $\cdots \to Q^{-2} \to Q^{-1} \to Q^0 \to \mathrm{Hom}_S(X,P) \to 0$, where each P^{-i} and Q^{-j} are finitely generated. By (3), the complex $0 \to \mathrm{Hom}_{S^{\mathrm{op}}}(\mathrm{Hom}_S(X,P),P) \to P^1 \to P^2 \to P^3 \to \cdots$ is acyclic, where $P^i = \mathrm{Hom}_{S^{\mathrm{op}}}(Q^{-(i-1)},P)$ are finitely generated projective for $i \geqslant 1$. Since ${}_SX$ is P-reflexive, we may splice the two complexes of left S-modules into an acyclic complex $P^\bullet$ of finitely generated projective left S-modules. Observe that $X \simeq Z^1(P^\bullet)$. By the discussion in the first paragraph, the complex $P^\bullet$ is totally acyclic. $\square$

The following observation will be used in the next subsection.

Lemma 3.2. *Let S be a ring. Then the following statements hold*:

(1) *the subcategory S-fgpr of S-Mod is closed under finite direct sums, direct summands and extensions*;
(2) *an S-module ${}_SX$ lies in S-fgpr if and only if there is an acyclic complex $\cdots \to G^{-2} \to G^{-1} \to G^0 \to X \to 0$ of S-modules such that each G^{-i} lies in S-fgpr.*

Proof. (1) is a consequence of the dual of [5, Proposition 5.1]. The "only if" part of (2) is trivial.

For the "if" part, denote by $\mathcal{G}$ the class of S-modules X, which fit into some acyclic complex $\cdots \to G^{-2} \to G^{-1} \to G^0 \to X \to 0$ with each $G^{-i} \in S$-fgpr. It suffices to show that for each $X \in \mathcal{G}$, there is an exact sequence $0 \to X' \to P^0 \to X \to 0$ with ${}_SP^0$ finitely generated projective and ${}_SX' \in \mathcal{G}$.

Take an exact sequence $0 \to X'' \to G^{-1} \xrightarrow{f} G^0 \to X \to 0$ with $G^{-i} \in S$-fgpr and $X'' \in \mathcal{G}$. Since G^0 lies in S-fgpr, we may take an exact sequence $0 \to G' \to P^0 \xrightarrow{g} G^0 \to 0$ with ${}_SP^0$ finitely generated projective and $G' \in S$-fgpr. Take the pullback of f and g. We have the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & X'' \arrow[r] \arrow[d, equal] & E \arrow[r, dotted] \arrow[d, dotted] & P^0 \arrow[r, "h"] \arrow[d, "g"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X'' \arrow[r] & G^{-1} \arrow[r, "f"] & G^0 \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

Observe the exact sequence $0 \to G' \to E \to G^{-1} \to 0$ induced by the pullback. Applying (1), we have that E lies in S-fgpr. It follows that $X' = \operatorname{Ker} h$ lies in $\mathcal{G}$. This gives us the required exact sequence. $\square$

3.2. *Totally reflexive extensions*

We introduce the notion of totally reflexive extension which is a special kind of reflexive extension. The main result claims that in a totally reflexive extension, a module over the extension ring is totally reflexive if and only if so is the underlying module over the base ring; see Theorem 3.6.

Definition 3.3. Let $\theta : S \to R$ be a P-W-reflexive extension for an invertible S-bimodule P and an invertible R-bimodule W. The ring extension $\theta : S \to R$ is called *totally P-W-reflexive* provided in addition that the left S-module ${}_SR$ is totally reflexive.

In case that $P = {}_SS_S$ and $W = {}_RR_R$ are the regular bimodules, a totally P-W-reflexive extension is called a *totally reflexive extension*. $\square$

The following observation implies that the notion of totally reflexive extension is left–right symmetric; compare Proposition 2.4.

Lemma 3.4. *Let $\theta : S \to R$ be a totally P-W-reflexive extension. Then the right S-module R_S is totally reflexive.*

Proof. Recall the isomorphism $\operatorname{Hom}_S(R, P) \simeq W$ of R-S-bimodules. By the assumption, ${}_SR$ is totally reflexive, and thus by Lemma 3.1 so is the right S-module W. Recall that the subcategory S^{op}-tref is closed under finite direct sums and direct summands, and that the right R-module W_R is a projective generator; see Lemma 2.1(1). The right S-module R_S is a direct summand of a finite direct sum of copies of W_S. It follows that R_S is totally reflexive. $\square$

Example 3.5. (1) Let S be a commutative ring and $\theta : S \to R$ be an S-algebra. We call the S-algebra R *totally quasi-reflexive* provided that the ring extension θ is totally S-W-reflexive for an R-bimodule W; it is *totally reflexive*, provided in addition that there is an isomorphism $\operatorname{Hom}_S(R, S) \simeq R$ of R-bimodules.

Let S be a Gorenstein ring of finite Krull dimension. Then by Lemma 2.5, the algebra R is totally quasi-reflexive if and only if it is a Gorenstein algebra in the sense of [1, Definition 3.1]; the algebra R is totally reflexive if and only if it is a Gorenstein order in the sense of Auslander [3, Chapter III].

(2) Let $\theta : S \to R$ be a P-W-reflexive extension. It is called *P-W-Frobenius*, provided that the S-module ${}_SR$ is finitely generated projective, or equivalently, the right S-module R_S is finitely generated projective. Observe that a P-W-Frobenius extension is totally P-W-reflexive.

A P-W-Frobenius extension is called a *Frobenius extension*, if $P = {}_SS_S$ and $W = {}_RR_R$ are the regular bimodules. This classical notion goes to back to [17]. For example, the natural embedding $S \to SG$ of a ring S into the group ring SG of a finite group G is a Frobenius extension.

We point out that a P-R-Frobenius extension is exactly the P-Frobenius extension, or the Frobenius extension of the third kind in the sense of [15, Definition 7.2]. $\square$

We now have the main result of the paper.

Theorem 3.6. *Let $\theta : S \to R$ be a totally P-W-reflexive extension for an invertible S-bimodule P and an invertible R-module W. Then a left R-module ${}_RX$ is totally reflexive if and only if the underlying S-module ${}_SX$ is totally reflexive.*

For the proof, we need the following general fact.

Lemma 3.7. *Let $\theta : S \to R$ be a ring extension with the left S-module ${}_SR \in S$-fgpr. Then a left R-module ${}_RX$ lies in R-fgpr if and only if the underlying S-module ${}_SX$ lies in S-fgpr.*

Proof. For the "only if" part, take a projective R-resolution $\cdots \to P^{-2} \to P^{-1} \to P^0 \to X \to 0$ with each P^{-i} finitely generated. Recall that the subcategory S-fgpr is closed under finite direct sums and direct summands. From the assumption, we infer that each S-module P^{-i} lies in S-fgpr. By Lemma 3.2(2), we have that ${}_SX$ lies in S-fgpr.

For the "if" part, we assume that ${}_SX$ lies in S-fgpr. In particular, the R-module X is finitely generated. Take an exact sequence $0 \to X' \to P^0 \xrightarrow{f} X \to 0$ of R-modules with P^0 finitely generated projective. Using induction, it suffices to show that the underlying S-module of X' lies in S-fgpr.

Take an exact sequence $0 \to Y \to Q \xrightarrow{g} X \to 0$ of S-modules with ${}_SQ$ finitely generated projective and ${}_SY \in S$-fgpr. Consider the following commutative exact diagram of S-modules, that is induced by the pullback of f and g

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d, equal] & E \arrow[r, dotted] \arrow[d, dotted] & Q \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & X' \arrow[r] & P^0 \arrow[r, "f"] & X \arrow[r] & 0
\end{tikzcd}$$

The upper row splits, and thus $E \simeq X' \oplus Q$. Consider the exact sequence $0 \to Y \to E \to P^0 \to 0$ induced by the pullback. From the assumption, we have that ${}_SP^0$ lies in S-fgpr. Recall from Lemma 3.2(1) that the subcategory S-fgpr is closed under extensions and direct summands. We have that ${}_SE$ lies in S-fgpr, and thus ${}_SX'$ also lies in S-fgpr. $\square$

We are in a position to prove Theorem 3.6.

Proof of Theorem 3.6. Recall from Lemma 3.1 the conditions (1)–(4) for totally reflexive modules. Indeed, we will show that, for each $1 \leqslant i \leqslant 4$, the R-module ${}_RX$, with respect to the invertible R-bimodule W, satisfies the condition (i) if and only if so does the underlying S-module ${}_SX$, with respect to the invertible S-bimodule P.

For (1), recall the isomorphism in Lemma 2.6(1). Applying Lemma 3.7 and its counterpart for right modules, we have (1).

For (2), apply Lemma 2.2 and Proposition 2.7.

For (3), it suffices to show that there is an isomorphism

$$\mathrm{Ext}^n_R(X, W) \simeq \mathrm{Ext}^n_S(X, P)$$

of right S-modules, for each $n \geqslant 1$. Take a projective R-resolution $P^\bullet = \cdots \to P^{-2} \to P^{-1} \to P^0 \to 0 \to \cdots$ of X. The isomorphism in Lemma 2.6(1) induces an isomorphism $\mathrm{Hom}_R(P^\bullet, W) \simeq \mathrm{Hom}_S(P^\bullet, P)$ of complexes. Hence, we have $\mathrm{Ext}^n_R(X, W) \simeq H^n(\mathrm{Hom}_S(P^\bullet, P))$; here, $H^n(-)$ denotes the n-th cohomology of a complex. By assumption, the left S-module ${}_SR$ is totally reflexive. In particular, by Lemma 3.1(3) we have $\mathrm{Ext}^k_S(R, P) = 0$ for $k \geqslant 1$. It follows that each projective R-module P^{-i} satisfies that $\mathrm{Ext}^k_S(P^{-i}, P) = 0$ for $k \geqslant 1$. By [21, 2.4.3], this implies that $H^n(\mathrm{Hom}_S(P^\bullet, P)) \simeq \mathrm{Ext}^n_S(X, P)$.

For (4), it suffices to show that there is an isomorphism

$$\mathrm{Ext}^n_{R^{\mathrm{op}}}\big(\mathrm{Hom}_R(X, W), W\big) \simeq \mathrm{Ext}^n_{S^{\mathrm{op}}}\big(\mathrm{Hom}_S(X, P), P\big)$$

for each $n \geqslant 1$. A similar argument as above yields an isomorphism

$$\mathrm{Ext}^n_{R^{\mathrm{op}}}\big(\mathrm{Hom}_R(X, W), W\big) \simeq \mathrm{Ext}^n_{S^{\mathrm{op}}}\big(\mathrm{Hom}_R(X, W), P\big).$$

Applying the isomorphism in Lemma 2.6(1), we are done. $\square$

Recall that the notion of Gorenstein-projective module is a natural extension of totally reflexive module to unnecessarily finitely generated modules; see [11]. We expect that the following result holds: for a totally P-W-reflexive extension $\theta : S \to R$, an R-module X is Gorenstein-projective if and only if the underlying S-module X is Gorenstein-projective. It seems that a different argument is needed, since the present one cannot carry over to Gorenstein-projective modules.

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