Singularity categories, Leavitt path algebras and shift spaces

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SFB Workshop, 2012/Bielefeld

Plan

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- We assume for simplicity that for each vertex in Q, there exist at least one arrow starting at it, and one arrow ending at it.

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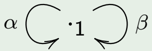
Let Q be the following *rose quiver* with two panels



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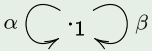

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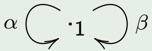

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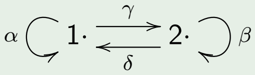

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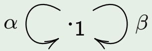

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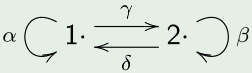

Then $Q'_0 = \{1, 2\}$, $Q'_1 = \{\alpha, \beta, \gamma, \delta\}$, $s(\gamma) = 1$ for example.

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  For example, $e_i e_j = \delta_{i,j} e_i$, $e_i p = \delta_{i,t(p)} p$, $p e_i = \delta_{s(p),i} p$.

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  $e_i e_j = \delta_{i,j} e_i$, $e_i \alpha = \delta_{i,t(\alpha)} \alpha$, $\beta e_j = \delta_{s(\beta),j} \beta$, $\alpha\beta = 0$.

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Let Q' be the quiver as above. Then $A_{Q'}$ is a six dimensional algebra with basis $\{e_1, e_2, \alpha, \beta, \gamma, \delta\}$, such that $1 = e_1 + e_2$ is the unit.

The A_Q-modules

- a (finite dimensional) left A_Q-*module* V = a family $\{V_i\}_{i \in Q_0}$ of (finite dimensional) vector spaces indexed by Q_0 together with a family $\{V_\alpha \colon V_{s(\alpha)} \to V_{t(\alpha)}\}_{\alpha \in Q_1}$ of linear maps indexed by Q_1 such that $V_\alpha \circ V_\beta = 0$.

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- This gives rise to the A_Q-module category, denoted by A_Q-mod.

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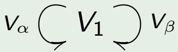

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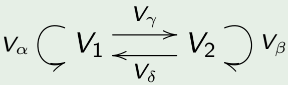

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Remarks: all indecomposable modules of A_Q and $A_{Q'}$ are known.

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  For example, for any $n \geq 1$, there are **nontrivial** exact sequence $0 \to V \to V_1 \to \cdots \to V_n \to W \to 0$ in A_Q-mod.

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- The category $\mathbf{D}^b(A\text{-mod})$ contains almost all information on homological properties of A-mod (or A): for example, a long exact sequence $0 \to V \to V_1 \to \cdots \to V_n \to W \to 0$ of modules corresponds to a morphism $W \to V[n]$ in $\mathbf{D}^b(A\text{-mod})$, or equivalently, an element in $\mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(W, V[n])$.

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Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the quotient category

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  More generally, a complex $X^\bullet$ is zero in $\mathbf{D}_{\rm sg}(A) \Longleftrightarrow X^\bullet$ is perfect.
- The category $\mathbf{D}_{\rm sg}(A)$ is a homological invariant of A, a measure on how far A is from having finite global dimension.

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- For a non-commutative algebra A, $\mathrm{gl.dim}\, A = \infty$ indicates that A has certain "**homological singularity**". This property is captured by the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$.

The completion of the singularity category

Definition (Krause 2005)

The *stable derived category* $\mathbf{S}(A)$ is by definition the homotopy category $\mathbf{K}_{\rm ac}(A\text{-Inj})$ of unbounded acyclic complexes of (not necessarily finite dimensional) injective A-modules.

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- a triangle equivalence $\mathbf{S}(A) \xrightarrow{\sim} \mathbf{S}(B) \Longrightarrow$ a triangle equivalence $\mathbf{D}_{\rm sg}(A) \xrightarrow{\sim} \mathbf{D}_{\rm sg}(B)$.

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- The structure of the singularity category $\mathbf{D}_{\rm sg}(A_Q)$ and its completion $\mathbf{S}(A_Q)$: morphisms, thick subcategories, ...
- The conditions for two quivers Q and Q' such that $\mathbf{D}_{\rm sg}(A_Q) \xrightarrow{\sim} \mathbf{D}_{\rm sg}(A_{Q'})$ or $\mathbf{S}(A_Q) \xrightarrow{\sim} \mathbf{S}(A_{Q'})$.

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Here, CK stands for Cuntz-Krieger.

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Let Q be the rose quiver with two panels. Then we have an isomorphism

$$L(Q) \simeq \frac{k\langle x_1, x_2, y_1, y_2\rangle}{\langle x_i y_j - \delta_{i,j}, y_1 x_1 + y_2 x_2 - 1\rangle}.$$

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- The algebra L_2 is non-noetherian and simple.

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  Sometimes true, not always!

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- Each component $L(Q)_n$ is linearly spanned by $\{p^*q \mid p, q \text{ are paths in } Q \text{ with } t(p) = t(q) \text{ and } \mathrm{len}(q) - \mathrm{len}(p) = n\}$.

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- The canonical map $\iota\colon kQ \to L(Q)$ is injective and a universal localization in the sense of Cohen-Schofield.

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- Main tools: von Neumann regular rings and their Grothendieck groups! Very recent work of [Hazrat, 2011/2012], [Ara-Pardo 2012].

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- Symbolic dynamics by [Hadamard 1898], [Morse-Hellund, 1938]

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- Two shift spaces X_Q and $X_{Q'}$ is *(topologically) conjugate* if there is a homeomorphism $\phi\colon X_Q \to X_{Q'}$ that commutes with the shift maps.
- Main concern: when two shift spaces are conjugate? Using (algebraical) invariants!

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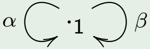
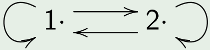
$Q =$ 

$Q' =$ 

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Then $M(Q) = (2) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and

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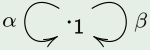
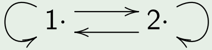
$Q =$ 

$Q' =$ 

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$M(Q') = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$. Then $M(Q)$ and $M(Q')$ are strong shift equivalent, and thus X_Q and $X_{Q'}$ are conjugate.

Shift equivalences

Definition (Williams 1973)

Two matrices M and N consisting of nonnegative integers is *shift equivalent* provided that there exist a pair (R, S) of rectangular matrices consisting of nonnegative integers and $r \geq 1$ such that

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- an algebraic invariant of [Krieger 1980]: M and N are shift equivalent if and only if their *dimension groups* are isomorphic.

Recent results

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Connections, in summary

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\begin{aligned}
X_Q \text{ and } X_{Q'} \text{ conjugate} &\iff M(Q) \text{ and } M(Q') \text{ strong shift equivalent}\\
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Expectation: based on Bratteli's classification theorem on the ultramatricial algebras $L(Q)_0$, the last "$\implies$" might be "$\iff$"!

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Consider the following two quivers.

$$Q = \alpha \circlearrowleft \cdot_1 \circlearrowright \beta \qquad\qquad Q' = \circlearrowleft 1\cdot \rightleftarrows 2\cdot \circlearrowright$$

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G. Abrams and G. Aranda Pino, *The Leavitt path algebra of a graph*, J. Algebra **293** (2) (2005), 319–334.

G. Abrams, A. Louly, E. Pardo and C. Smith, *Flow invariants in the classification of Leavitt path algebras*, J. Algebra **333** (2011), 202–231.

P. Ara, M.A. Moreno and E. Pardo, *Nonstable K-theory for graph algebras*, Algebr. Represent. Theory **10** (2) (2007), 157–178.

P. Ara and E. Pardo, *K-theoretic characterization of graded isomorphisms between Leavitt path algebras*, arXiv:1210.31287v1.

R.O. Buchweitz, Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings, unpublished manuscript, 1987.

X.W. Chen, *The singularity category of an algebra with radical square zero*, Doc. Math. **16** (2011), 921–936.

X.W. Chen, D. Yang, *Homotopy categories, Leavitt path algebras and Gorenstein projective modules,* preprint.

R. Hazrat, *The graded Grothendieck group and classification of Leavitt path algebras*, Math. Ann., to appear, arXiv: 1102.4088.

R. Hazrat, *The dynamics of Leavitt path algebras*, arXiv:1209.2908.

K.H. Kim and F.W. Roush, *Williams' conjecture is false for reducible subshifts*, J. Amer. Math. Soc. **5** (1992), 213–215.

H. Krause, *The stable derived category of a noetherian scheme,* Compositio Math. **141** (2005), 1128–1162.

- D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Trudy Steklov Math. Institute **204** (2004), 240–262.
- S.P. Smith, *Equivalence of categories involving graded modules over path algebras of quivers*, Adv. Math. **230** (2012), 1780–1810.
- S.P. Smith, *Shift equivalence and a category equivalence involving graded modules over path algebras of quivers*, arXiv: 1108.4994.
- R.F. Williams, *Classification of subshifts of finite type*, Ann. Math. **98** (1973), 120–153; *erratum*, Ann. Math. **99** (1974), 380–381.

Thank You!

http://home.ustc.edu.cn/~xwchen