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SINGULAR EQUIVALENCES INDUCED BY HOMOLOGICAL EPIMORPHISMS

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Abstract. We prove that a certain homological epimorphism between two algebras induces a triangle equivalence between their singularity categories. Applying the result to a construction of matrix algebras, we describe the singularity categories of some non-Gorenstein algebras.

1. Introduction

Let A be a finite dimensional algebra over a field k. Denote by A-mod the category of finitely generated left A-modules and by $\mathbf{D}^b(A\text{-mod})$ the bounded derived category. Following [20], the *singularity category* $\mathbf{D}_{\mathrm{sg}}(A)$ of A is the Verdier quotient triangulated category of $\mathbf{D}^b(A\text{-mod})$ with respect to the full subcategory formed by perfect complexes; see also [4, 6, 8, 14, 16, 17, 23].

The singularity category measures the homological singularity of an algebra: the algebra A has finite global dimension if and only if its singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is trivial. Meanwhile, the singularity category captures the stable homological features of an algebra ([6]).

A fundamental result of Buchweitz and Happel states that for a Gorenstein algebra A, the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is triangle equivalent to the stable category of (maximal) Cohen-Macaulay A-modules ([6, 14]), where the latter category is related to Tate cohomology theory ([2, 6]). This result specializes Rickard's result ([23]) on self-injective algebras. For non-Gorenstein algebras, not much is known about their singularity categories ([7, 9]).

The following concepts might be useful in the study of singularity categories. Two algebras A and B are said to be *singularly equivalent* provided that there is a triangle equivalence between $\mathbf{D}_{\mathrm{sg}}(A)$ and $\mathbf{D}_{\mathrm{sg}}(B)$. Such an equivalence is called a *singular equivalence*; compare [21]. In this case, if A is non-Gorenstein and B is Gorenstein, then Buchweitz-Happel's theorem applies to give a description of $\mathbf{D}_{\mathrm{sg}}(A)$ in terms of (maximal) Cohen-Macaulay B-modules. We observe that a derived equivalence of two algebras, that is, a triangle equivalence between their bounded derived categories, naturally induces a singular equivalence. The converse is not true in general.

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Let A be an algebra and let $J \subseteq A$ be a two-sided ideal. Following [22], we call J a *homological ideal* provided that the canonical map $A \to A/J$ is a homological epimorphism ([12]), meaning that the naturally induced functor $\mathbf{D}^b(A/J\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ is fully faithful.

The main observation we make is as follows.

Theorem. *Let A be a finite dimensional k-algebra and let $J \subseteq A$ be a homological ideal which has finite projective dimension as an A-A-bimodule. Then there is a singular equivalence between A and A/J.*

This paper is structured as follows. In Section 2, we recall some ingredients and then prove the Theorem. In Section 3, we apply the Theorem to a construction of matrix algebras and then describe the singularity categories of some non-Gorenstein algebras. In particular, we give two examples which extend in different manners an example considered by Happel in [14].

2. Proof of the Theorem

We will present the proof of the Theorem in this section. Before that, we recall from [25] and [15] some results on triangulated categories and derived categories.

Let $\mathcal{T}$ be a triangulated category. We will denote its translation functor by [1]. For a triangulated subcategory $\mathcal{N}$, we denote by $\mathcal{T}/\mathcal{N}$ the Verdier quotient triangulated category. The quotient functor $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ has the property that $q(X) \simeq 0$ if and only if X is a direct summand of an object in $\mathcal{N}$. In particular, if $\mathcal{N}$ is a *thick* subcategory, that is, it is closed under direct summands, we have that $\operatorname{Ker} q = \mathcal{N}$. Here, for a triangle functor F, $\operatorname{Ker} F$ denotes its essential kernel, that is, the (thick) triangulated subcategory consisting of objects on which F vanishes.

The following result is well known.

Lemma 2.1. *Let $F\colon \mathcal{T} \to \mathcal{T}'$ be a triangle functor which allows a fully faithful right adjoint G. Then F induces uniquely a triangle equivalence $\mathcal{T}/\operatorname{Ker} F \simeq \mathcal{T}'$.*

Proof. The existence of the induced functor follows from the universal property of the quotient functor. The result is a triangulated version of [11, Proposition I. 1.3]. For details, see [5, Propositions 1.5 and 1.6]. □

Let $F\colon \mathcal{T} \to \mathcal{T}'$ be a triangle functor. Assume that $\mathcal{N} \subseteq \mathcal{T}$ and $\mathcal{N}' \subseteq \mathcal{T}'$ are triangulated subcategories satisfying $F\mathcal{N} \subseteq \mathcal{N}'$. Then there is a uniquely induced triangle functor $\bar{F}\colon \mathcal{T}/\mathcal{N} \to \mathcal{T}'/\mathcal{N}'$.

Lemma 2.2 ([20, Lemma 1.2])**.** *Let $F\colon \mathcal{T} \to \mathcal{T}'$ be a triangle functor which has a right adjoint G. Assume that $\mathcal{N} \subseteq \mathcal{T}$ and $\mathcal{N}' \subseteq \mathcal{T}'$ are triangulated subcategories satisfying the fact that $F\mathcal{N} \subseteq \mathcal{N}'$ and $G\mathcal{N}' \subseteq \mathcal{N}$. Then the induced functor $\bar{F}\colon \mathcal{T}/\mathcal{N} \to \mathcal{T}'/\mathcal{N}'$ has a right adjoint $\bar{G}$. Moreover, if G is fully faithful, so is $\bar{G}$.*

Proof. The unit and counit of (F, G) induce uniquely two natural transformations $\mathrm{Id}_{\mathcal{T}/\mathcal{N}} \to \bar{G}\bar{F}$ and $\bar{F}\bar{G} \to \mathrm{Id}_{\mathcal{T}'/\mathcal{N}'}$, which are the corresponding unit and counit of the adjoint pair $(\bar{F}, \bar{G})$; consult [19, Chapter IV, Section 1, Theorem 2(v)]. Note that the fully-faithfulness of G is equivalent to the fact that the counit of (F, G) is an isomorphism. It follows that the counit of $(\bar{F}, \bar{G})$ is also an isomorphism, which is equivalent to the fully-faithfulness of $\bar{G}$; consult [19, Chapter IV, Section 3, Theorem 1]. □

Let k be a field and let A be a finite dimensional k-algebra. Recall that A-mod is the category of finite dimensional left A-modules. We write ${}_AA$ for the regular left A-module. Denote by $\mathbf{D}(A\text{-mod})$ (*resp.* $\mathbf{D}^b(A\text{-mod})$) the (*resp.* bounded) derived category of A-mod. We identify A-mod as the full subcategory of $\mathbf{D}^b(A\text{-mod})$ consisting of stalk complexes concentrated at degree zero; see [15, Proposition I. 4.3].

A complex of A-modules is usually denoted by $X^\bullet = (X^n, d^n)_{n\in\mathbb{Z}}$, where X^n are A-modules and the differentials $d^n\colon X^n \to X^{n+1}$ are homomorphisms of modules satisfying $d^{n+1}\circ d^n = 0$. Recall that a complex in $\mathbf{D}^b(A\text{-mod})$ is *perfect* provided that it is isomorphic to a bounded complex consisting of projective modules. The full subcategory consisting of perfect complexes is denoted by $\operatorname{perf}(A)$. Recall from [6, Lemma 1.2.1] that a complex $X^\bullet$ in $\mathbf{D}^b(A\text{-mod})$ is perfect if and only if there is a natural number n_0 such that for each A-module M, $\operatorname{Hom}_{\mathbf{D}^b(A\text{-mod})}(X^\bullet, M[n]) = 0$ for all $n \geq n_0$. It follows that $\operatorname{perf}(A)$ is a thick subcategory of $\mathbf{D}^b(A\text{-mod})$. Indeed, it is the smallest thick subcategory of $\mathbf{D}^b(A\text{-mod})$ containing ${}_AA$.

Let $\pi\colon A \to B$ be a homomorphism of algebras. The functor of restricting of scalars $\pi^*\colon B\text{-mod} \to A\text{-mod}$ is exact, and it extends to a triangle functor $\mathbf{D}^b(B\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$, which will still be denoted by π^*. Following [12], we call the homomorphism π a *homological epimorphism* provided that $\pi^*\colon \mathbf{D}^b(B\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ is fully faithful. By [12, Theorem 4.1(1)] this is equivalent to the fact that $\pi \otimes^{\mathbf{L}}_A B\colon B \simeq A \otimes^{\mathbf{L}}_A B \to B \otimes^{\mathbf{L}}_A B$ is an isomorphism in $\mathbf{D}(A^e\text{-mod})$. Here, $A^e = A \otimes_k A^{\mathrm{op}}$ is the enveloping algebra of A, and we identify A^e-mod as the category of A-A-bimodules.

Lemma 2.3 ([22, Proposition 2.2(a)]). *Let $J \subseteq A$ be an ideal and let $\pi\colon A \to A/J$ be the canonical projection. Then π is a homological epimorphism if and only if $J^2 = J$ and $\operatorname{Tor}_i^A(J, A/J) = 0$ for all $i \geq 1$.*

In the situation of the lemma, the ideal J is called a *homological ideal* in [22]. As a special case, we call an ideal J a *hereditary ideal* provided that $J^2 = J$ and J is a projective A-A-bimodule; compare [22, Lemma 3.4].

Proof. The natural exact sequence $0 \to J \to A \xrightarrow{\pi} A/J \to 0$ of A-A-bimodules induces a triangle $J \to A \xrightarrow{\pi} A/J \to J[1]$ in $\mathbf{D}^b(A^e\text{-mod})$. Applying the functor $- \otimes^{\mathbf{L}}_A A/J$, we get a triangle $J \otimes^{\mathbf{L}}_A A/J \to A/J \to A/J \otimes^{\mathbf{L}}_A A/J \to J \otimes^{\mathbf{L}}_A A/J[1]$. Then π is a homological epimorphism or, equivalently, $\pi \otimes^{\mathbf{L}}_A A/J$ is an isomorphism if and only if $J \otimes^{\mathbf{L}}_A A/J = 0$; see [13, Lemma I.1.7]. This is equivalent to the fact that $\operatorname{Tor}_i^A(J, A/J) = 0$ for all $i \geq 0$. We note that $\operatorname{Tor}_0^A(J, A/J) \simeq J \otimes_A A/J \simeq J/J^2$. $\square$

Now we are in the position to prove the Theorem. Recall that for an algebra A, its singularity category $\mathbf{D}_{\mathrm{sg}}(A) = \mathbf{D}^b(A\text{-mod})/\operatorname{perf}(A)$. Moreover, a complex $X^\bullet$ becomes zero in $\mathbf{D}_{\mathrm{sg}}(A)$ if and only if it is perfect. Here, we use the fact that $\operatorname{perf}(A) \subseteq \mathbf{D}^b(A\text{-mod})$ is a thick subcategory.

Proof of the Theorem. Write $B = A/J$. Since J, as an A-A-bimodule, has finite projective dimension, so it has finite projective dimension both as a left and right A-module. Consider the natural exact sequence $0 \to J \to A \to B \to 0$. It follows that B, both as a left and right A-module, has finite projective dimension. Moreover, for a complex $X^\bullet$ in $\mathbf{D}^b(A\text{-mod})$, $J \otimes^{\mathbf{L}}_A X^\bullet$ is perfect. Indeed, take a bounded projective resolution $P^\bullet \to J$ as an A^e-module. Then $J \otimes^{\mathbf{L}}_A X^\bullet \simeq P^\bullet \otimes_A X^\bullet$. This is a perfect complex, since each left A-module $P^i \otimes_A X^j$ is projective.

Denote by $\pi\colon A \to B$ the canonical projection. By the assumption, the functor $\pi^*\colon \mathbf{D}^b(B\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ is fully faithful. Since $\pi^*(B) = {}_AB$ is perfect, the functor π^* sends perfect complexes to perfect complexes. Then it induces a triangle functor $\bar{\pi^*}\colon \mathbf{D}_{\mathrm{sg}}(B) \to \mathbf{D}_{\mathrm{sg}}(A)$. We will show that $\bar{\pi^*}$ is an equivalence.

The functor $\pi^*\colon \mathbf{D}^b(B\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ has a left adjoint $F = B \otimes^{\mathbf{L}}_A -\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$. Here we use the fact that the right A-module B_A has finite projective dimension. Since F sends perfect complexes to perfect complexes, we have the induced triangle functor $\bar{F}\colon \mathbf{D}_{\mathrm{sg}}(A) \to \mathbf{D}_{\mathrm{sg}}(B)$. By Lemma 2.2 we have the adjoint pair $(\bar{F}, \bar{\pi^*})$; moreover, the functor $\bar{\pi^*}$ is fully faithful. By Lemma 2.1 there is a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(A)/\mathrm{Ker}\ \bar{F} \simeq \mathbf{D}_{\mathrm{sg}}(B)$.

It remains to show that the essential kernel $\mathrm{Ker}\ \bar{F}$ is trivial. For this, we assume that a complex $X^\bullet$ lies in $\mathrm{Ker}\ \bar{F}$. This means that the complex $F(X^\bullet)$ in $\mathbf{D}^b(B\text{-mod})$ is perfect. Since π^* preserves perfect complexes, it follows that $\pi^*F(X^\bullet)$ is also perfect. The natural exact sequence $0 \to J \to A \to B \to 0$ induces a triangle $J \otimes^{\mathbf{L}}_A X^\bullet \to X^\bullet \to \pi^*F(X^\bullet) \to J \otimes^{\mathbf{L}}_A X^\bullet[1]$ in $\mathbf{D}^b(A\text{-mod})$. Recall that $J \otimes^{\mathbf{L}}_A X^\bullet$ is perfect. It follows that $X^\bullet$ is perfect, since $\mathrm{perf}(A) \subseteq \mathbf{D}^b(A\text{-mod})$ is a triangulated subcategory. The proves that $X^\bullet$ is zero in $\mathbf{D}_{\mathrm{sg}}(A)$. □

The following special case of the Theorem is of interest.

Corollary 2.4. *Let A be a finite dimensional algebra and $J \subseteq A$ a hereditary ideal. Then we have a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(A) \simeq \mathbf{D}_{\mathrm{sg}}(A/J)$.*

Proof. It suffices to observe by Lemma 2.3 that J is a homological ideal. □

3. Examples

In this section, we will describe a construction of matrix algebras to illustrate Corollary 2.4. In particular, the singularity categories of some non-Gorenstein algebras are described.

The following construction is similar to [18, Section 4]. Let A be a finite dimensional algebra over a field k. Let ${}_AM$ and N_A be a left and right A-module, respectively. Then $M \otimes_k N$ becomes an A-A-bimodule. Consider an A-A-bimodule monomorphism $\phi\colon M \otimes_k N \to A$. Then $\mathrm{Im}\ \phi$ is a two-sided ideal of A. We require further that $(\mathrm{Im}\ \phi)M = 0$ and $N(\mathrm{Im}\ \phi) = 0$. The matrix $\Gamma = \begin{pmatrix} A & M \\ N & k \end{pmatrix}$ becomes an associative algebra via the following multiplication:

$$\begin{pmatrix} a & m \\ n & \lambda \end{pmatrix} \begin{pmatrix} a' & m' \\ n' & \lambda' \end{pmatrix} = \begin{pmatrix} aa' + \phi(m \otimes n') & am' + \lambda' m \\ na' + \lambda n' & \lambda\lambda' \end{pmatrix}.$$

For the associativity, we need the above requirement on $\mathrm{Im}\ \phi$.

Proposition 3.1. *Keep the notation and assumption as above. Then there is a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(\Gamma) \simeq \mathbf{D}_{\mathrm{sg}}(A/\mathrm{Im}\ \phi)$.*

Proof. Set $J = \Gamma e \Gamma$ with $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Observe that $\Gamma/J = A/\mathrm{Im}\ \phi$. The ideal J is hereditary: $J^2 = J$ is clear, while the natural map $\Gamma e \otimes_k e\Gamma \to J$ is an isomorphism of Γ-Γ-bimodules and then J is a projective Γ-Γ-bimodule. The isomorphism uses the assumption that ϕ is mono. Then we apply Corollary 2.4. □

Remark 3.2. The above construction contains the one-point extension and coextension of algebras, where M or N is zero. Hence Proposition 3.1 contains the results in [9, Section 4].

We will illustrate Proposition 3.1 by three examples. Two of these examples extend an example considered by Happel in [14]. In particular, based on results in [9], we obtain descriptions of the singularity categories of some non-Gorenstein algebras.

Recall from [14] that an algebra A is *Gorenstein* provided that both as a left and right module, the regular module A has finite injective dimension. It follows from [6, Theorem 4.4.1] and [14, Theorem 4.6] that in the Gorenstein case, the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is *Hom-finite*. This means that all Hom spaces in $\mathbf{D}_{\mathrm{sg}}(A)$ are finite dimensional over k.

For algebras given by quivers and relations, we refer to [1, Chapter III].

Example 3.3. Let Γ be the k-algebra given by the following quiver Q with relations $\{x^2, \delta x, \beta x, x\gamma, x\alpha, \beta\gamma, \delta\alpha, \beta\alpha, \delta\gamma, \alpha\beta - \gamma\delta\}$. We write the concatenation of paths from right to left.

$$\begin{tikzcd} \cdot 1 \arrow[r, bend left, "\alpha"] & * \arrow[l, bend left, "\beta"] \arrow[loop above, "x"] \arrow[r, bend left, "\delta"] & \cdot 2 \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

We have in Γ that $1 = e_1 + e_* + e_2$, where the e's are the primitive idempotents corresponding to the vertices. Set $\Gamma' = \Gamma/\Gamma e_1 \Gamma$. It is an algebra with radical square zero, whose quiver is obtained from Q by removing the vertex 1 and the adjacent arrows.

We identify Γ with $\begin{pmatrix} A & k\alpha \\ k\beta & k \end{pmatrix}$, where the k in the southeast corner is identified with $e_1\Gamma e_1$, and $A = (1-e_1)\Gamma(1-e_1)$. The corresponding $\mathrm{Im}\ \phi$ equals $k\alpha\beta$, and we have $A/\mathrm{Im}\ \phi = \Gamma'$; consult the proof of Proposition 3.1. Then Proposition 3.1 yields a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(\Gamma) \simeq \mathbf{D}_{\mathrm{sg}}(\Gamma')$.

The triangulated category $\mathbf{D}_{\mathrm{sg}}(\Gamma')$ is completely described in [9] (see also [24]); in particular, it is not Hom-finite. More precisely, it is equivalent to the category of finitely generated projective modules on a von Neumann regular algebra. The algebra Γ', or rather its Koszul dual, is related to the noncommutative space of Penrose tilings via the work of Smith; see [24, Theorem 7.2 and Example]. We point out that the algebra Γ is non-Gorenstein, since $\mathbf{D}_{\mathrm{sg}}(\Gamma)$ is not Hom-finite.

Example 3.4. Let Γ be the k-algebra given by the following quiver Q with relations $\{x_1^2, x_2^2, x_1\alpha_1, x_2\alpha_1, \beta_2\alpha_1, \beta_2\alpha_1, x_1\alpha_2, x_2\alpha_2, \beta_1\alpha_2, \beta_2\alpha_2, \alpha_1\beta_1 - x_1x_2, \alpha_2\beta_2 - x_2x_1\}$:

$$\begin{tikzcd} \cdot 1 \arrow[r, bend left, "\alpha_1"] & * \arrow[l, bend left, "\beta_1"] \arrow[loop above, "x_1"] \arrow[loop below, "x_2"] \arrow[r, bend left, "\beta_2"] & \cdot 2 \arrow[l, bend left, "\alpha_2"] \end{tikzcd}$$

We claim that there is a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(\Gamma) \simeq \mathbf{D}_{\mathrm{sg}}(k\langle x_1, x_2\rangle/(x_1, x_2)^2)$. Here, $k\langle x_1, x_2\rangle$ is the free algebra with two variables.

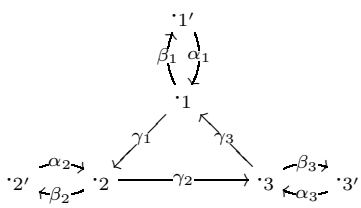
We point out that the triangulated category $\mathbf{D}_{\mathrm{sg}}(k\langle x_1, x_2\rangle/(x_1, x_2)^2)$ is described completely in [9, Example 3.11], where related results are contained in [3, Section 10]. Similar to the example above, this algebra Γ is non-Gorenstein.

To see the claim, we observe that the quiver Q has two loops and two 2-cycles. The proof is done by "removing the 2-cycles". We have a natural isomorphism $\Gamma = \begin{pmatrix} A & k\alpha_1 \\ k\beta_1 & k \end{pmatrix}$, where $k = e_1\Gamma e_1$ and $A = (1-e_1)\Gamma(1-e_1)$. We observe that Proposition 3.1 applies with the corresponding $\operatorname{Im}\phi = k\alpha_1\beta_1$. Set $A/\operatorname{Im}\phi = \Gamma'$. So $\mathbf{D}_{\mathrm{sg}}(\Gamma) \simeq \mathbf{D}_{\mathrm{sg}}(\Gamma')$. The quiver of Γ' is obtained from Q by removing the vertex 1 and the adjacent arrows, while its relations are obtained from the ones of Γ by replacing $\alpha_1\beta_1 - x_1x_2$ with x_1x_2. Similarly, $\Gamma' = \begin{pmatrix} A' & k\alpha_2 \\ k\beta_2 & k \end{pmatrix}$ with $k = e_2\Gamma' e_2$ and $A' = e_*\Gamma' e_*$. Then Proposition 3.1 applies again, and we get the equivalence $\mathbf{D}_{\mathrm{sg}}(\Gamma') \simeq \mathbf{D}_{\mathrm{sg}}(k\langle x_1, x_2\rangle/(x_1,x_2)^2)$.

This example generalizes directly to a quiver with n loops and n 2-cycles with similar relations. The corresponding statement for the case $n = 1$ is implicitly contained in [14, 2.3 and 4.8].

The last example is a Gorenstein algebra.

Example 3.5. Let $r \geq 2$. Consider the following quiver Q consisting of three 2-cycles and a central 3-cycle Z_3. We identify γ_3 with γ_0 and denote by p_i the path in the central cycle starting at vertex i of length 3.



Let Γ be the k-algebra given by the quiver Q with relations $\{\beta_i\alpha_i, \gamma_i\alpha_i, \beta_i\gamma_{i-1}, \alpha_i\beta_i - p_i^r \mid i = 1,2,3\}$. We point out that in Γ all paths in the central cycle of length strictly larger than $3r+1$ vanish.

Set $A = kZ_3/(\gamma_1,\gamma_2,\gamma_3)^{3r}$, where kZ_3 is the path algebra of the central 3-cycle Z_3. The algebra A is self-injective and Nakayama ([1, p.111]). Denote by A-$\underline{\mathrm{mod}}$ the stable category of A-modules; it is naturally a triangulated category (see [13, Theorem I.2.6]).

We claim that there is a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(\Gamma) \simeq A$-$\underline{\mathrm{mod}}$.

For the claim, we observe an isomorphism $A = \Gamma/\Gamma(e_{1'} + e_{2'} + e_{3'})\Gamma$. We argue as in Example 3.4 by removing the three 2-cycles and applying Proposition 3.1 repeatedly. Then we get a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(\Gamma) \simeq \mathbf{D}_{\mathrm{sg}}(A)$. Finally, by [23, Theorem 2.1] we have a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(A) \simeq A$-$\underline{\mathrm{mod}}$. Then we are done.

We point out that the algebra Γ is Gorenstein with self-injective dimension two. Hence by [6, Theorem 4.4.1] and [14, Theorem 4.6] there is a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(\Gamma) \simeq \underline{\mathrm{MCM}}(\Gamma)$, where $\underline{\mathrm{MCM}}(\Gamma)$ denotes the stable category of (maximal) Cohen-Macaulay Γ-modules. Then we have a triangle equivalence

$$\underline{\mathrm{MCM}}(\Gamma) \simeq A\text{-}\underline{\mathrm{mod}}.$$

We mention that Γ is a special biserial algebra of finite representation type (by [10, Lemma II.8.1]). It would be interesting to identify (maximal) Cohen-Macaulay Γ-modules in the Auslander-Reiten quiver of Γ.

This example generalizes directly to a quiver with n 2-cycles and a central n-cycle with similar relations. The case where $n = 1$ and $r = 2$ coincides with the examples considered in [14, 2.3 and 4.8].

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