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Representability and autoequivalence groups

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Abstract

For a finite dimensional algebra A, the bounded homotopy category of projective A-modules and the bounded derived category of A-modules are dual to each other via certain categories of locally-finite cohomological functors. We prove that the duality gives rise to a 2-categorical duality between certain strict 2-categories involving bounded homotopy categories and bounded derived categories, respectively. We apply the 2-categorical duality to the study of triangle autoequivalence groups.

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1. *Introduction*

Let k be field. It is well known that the homological behavior of a finite dimensional k-algebra A with infinite global dimension is similar to that of a singular projective scheme $\mathbb{X}$. For example, the difference between the category $\mathrm{perf}(\mathbb{X})$ of perfect complexes and the bounded derived category $\mathbf{D}^b(\mathrm{coh}\text{-}\mathbb{X})$ measures the singularity of $\mathbb{X}$; see [**17**]. In the same manner, the difference between the bounded homotopy category $\mathbf{K}^b(A\text{-proj})$ of projective A-modules and the bounded derived category $\mathbf{D}^b(A\text{-mod})$ measures the homological singularity of A, or more precisely, the stable properties of the module category A-mod; see [**4**, **7**, **11**].

The following remarkable result is obtained in [**2**]: for such a scheme $\mathbb{X}$, there is a duality of linear categories between $\mathrm{perf}(\mathbb{X})$ and $\mathbf{D}^b(\mathrm{coh}\text{-}\mathbb{X})$ via the categories of cohomological functors. This duality is applied to the study of triangle autoequivalence groups and the reconstruction of $\mathbb{X}$ from these triangulated categories.

The above duality is vastly extended in [**14**, **15**] to certain proper schemes over noetherian rings. We mention that the duality is essentially related to the representability of certain cohomological functors. We refer to [**3**, **6**, **20**] for relevant representability theorems.

The duality in [**14**, **15**] is very general. In particular, it implies that, very similar to [**2**], there is a duality of linear categories between $\mathbf{K}^b(A\text{-proj})$ and $\mathbf{D}^b(A\text{-mod})$; see Theorem 2·8. Here, we provide a slightly different proof to Theorem 2·8, which is based on a representability lemma in [**2**].

Following [2, section 4], we use the pseudo-adjunctions and the duality in Theorem 2·8 to obtain a 2-categorical duality, which involves these triangulated categories. For a more precise statement of the following result, we refer to Theorem 3·2.

THEOREM. *Let $\mathbb{K}^b$ be the strict 2-category with objects being all finite dimensional algebras A, 1-morphisms being triangle functors between $\mathbf{K}^b(A\text{-proj})$, and 2-morphisms being natural transformations. Let $\mathbb{D}^b$ be the analogous 2-category replacing $\mathbf{K}^b(A\text{-proj})$ by $\mathbf{D}^b(A\text{-mod})$. Then there is a 2-categorical duality*

$$\mathbb{K}^b \xrightarrow{\sim} \mathbb{D}^b,$$

which acts on objects by the identity.

We mention that an analogue of the above theorem for projective schemes is also true by the results in [2, section 4].

The above 2-categorical duality is applied to the study of triangle autoequivalence groups. For a triangulated category $\mathcal{T}$, we denote by $\mathrm{Aut}_\Delta(\mathcal{T})$ its triangle autoequivalence groups, whose elements are the isomorphism classes of triangle autoequivalences on $\mathcal{T}$. The derived Picard group $\mathrm{DPic}(A)$ is an important invariant of an algebra A, whose elements are the isomorphism classes of two-sided tilting complexes over A; see [22, 21].

The following group homomorphisms are well known

$$\mathrm{DPic}(A) \xrightarrow{\text{ev}} \mathrm{Aut}_\Delta(\mathbf{D}^b(A\text{-mod})) \xrightarrow{\text{res}} \mathrm{Aut}_\Delta(\mathbf{K}^b(A\text{-proj})).$$

Here, the evaluation homomorphism "ev" sends a two-sided tilting complex X to the derived tensor functor $X \otimes^{\mathbb{L}}_A -$, and "res" denotes the restriction of autoequivalences. Moreover, the evaluation homomorphism "ev" is injective. By the proof of [5, theorem 6·1], the homomorphism "res" is also injective.

The fundamental open question in [19, section 3] asks whether any derived equivalence is standard, or equivalently, whether "ev" is surjective. The following result implies that the open question is equivalent to the surjectivity of the composition "res $\circ$ ev"; see Corollary 3·6.

PROPOSITION. *Let A be a finite dimensional k-algebra. Then the restriction homomorphism*

$$\mathrm{Aut}_\Delta(\mathbf{D}^b(A\text{-mod})) \xrightarrow{\text{res}} \mathrm{Aut}_\Delta(\mathbf{K}^b(A\text{-proj}))$$

is an isomorphism.

The surjectivity of "res" is equivalent to the fact that any triangle autoequivalence on $\mathbf{K}^b(A\text{-proj})$ extends to a triangle autoequivalence on $\mathbf{D}^b(A\text{-mod})$. More generally, the extension of triangle functors is studied in Proposition 3·4. The relation between the isomorphism "res" and the work [5] is discussed at the end of this paper; see Corollary 3·9.

The structure of this paper is straightforward. Throughout, we require that all the algebras, categories and functors are k-linear over a fixed field k.

2. *Cohomological functors and representability*

In this section, we recall from [**14**, **15**] that there is a duality between the bounded homotopy category of projective modules and the bounded derived category of modules; compare [**2**]. The duality is realised via the categories of locally-finite cohomological functors.

2·1. *A representability lemma*

Let k be a field. Denote by k-Mod the category of k-vector spaces and by k-mod the full subcategory of finite dimensional vector spaces.

Let $\mathcal{C}$ be a skeletally small triangulated category, which is k-linear and Hom-finite. Here, the Hom-finiteness means that each Hom space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is finite dimensional. A k-linear functor $F: \mathcal{C} \to k$-Mod is *cohomological* provided that F sends exact triangles to long exact sequences of vector spaces. The cohomological functor F is *locally-finite* provided that the vector space $\bigoplus_{n\in\mathbb{Z}} F(\Sigma^n X)$ is finite dimensional for each object $X \in \mathcal{C}$. Denote by $\mathrm{coho}(\mathcal{C})$ the category of locally-finite cohomological functors.

Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. Recall that an object X is compact provided that the following canonical injection

$$\bigoplus_{i\in\Lambda} \mathrm{Hom}_{\mathcal{T}}(X, Y_i) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, \bigoplus_{i\in\Lambda} Y_i)$$

is surjective for any objects Y_i indexed by any set Λ. Denote by $\mathcal{T}^c$ the full subcategory of $\mathcal{T}$ formed by compact objects; it is a thick triangulated subcategory. The triangulated category $\mathcal{T}$ is said to be *compactly generated* provided that $\mathcal{T}^c$ is skeletally small and that for each nonzero object $Y \in \mathcal{T}$, there is a nonzero morphism $X \to Y$ with X compact.

We assume further that $\mathcal{T}$ is k-linear. An object X is *locally-finite* provided that the restricted Hom functor

$$\mathrm{Hom}_{\mathcal{T}}(-, X)|_{\mathcal{T}^c} : (\mathcal{T}^c)^{\mathrm{op}} \longrightarrow k\text{-Mod}$$

is locally-finite. Denote by $\mathcal{T}_{\mathrm{lf}}$ the full subcategory of $\mathcal{T}$ consisting of locally-finite objects, which is a thick triangulated subcategory.

The following fundamental result [**2**, lemma 3·3] is a finite version of the Brown representability theorem. Its first part is due to [**6**, lemma 2·14], and its second part relies on [**10**, section 2].

LEMMA 2·1. *Let $\mathcal{T}$ be a k-linear triangulated category which is compactly generated. Then any cohomological functor $F: (\mathcal{T}^c)^{\mathrm{op}} \longrightarrow k$-mod is represented by some object $X \in \mathcal{T}$, that is, isomorphic to $\mathrm{Hom}_{\mathcal{T}}(-, X)|_{\mathcal{T}^c}$. Moreover, any natural transformation between such cohomological functors is induced by some morphism between the representing objects.*

Recall that a morphism $f: X \to Y$ in $\mathcal{T}_{\mathrm{lf}}$ is *phantom* provided that each composition $f \circ g$ is zero for any morphism $g: C \to X$ with C compact. These phantom morphisms form a two-sided ideal **ph** of $\mathcal{T}_{\mathrm{lf}}$. Denote by $\mathcal{T}_{\mathrm{lf}}/\mathbf{ph}$ the factor category of $\mathcal{T}_{\mathrm{lf}}$ by the phantom ideal.

COROLLARY 2·2. *Let $\mathcal{T}$ be a k-linear triangulated category which is compactly generated. Then the restricted Yoneda functor*

$$\mathcal{T}_{\mathrm{lf}} \longrightarrow \mathrm{coho}((\mathcal{T}^c)^{\mathrm{op}}), \quad X \mapsto \mathrm{Hom}_{\mathcal{T}}(-, X)|_{\mathcal{T}^c}$$

is full and dense. In particular, it induces an equivalence of categories

$$\mathcal{T}_{\mathrm{lf}}/\mathbf{ph} \overset{\sim}{\longrightarrow} \mathrm{coho}((\mathcal{T}^c)^{\mathrm{op}}).$$

Proof. The first assertion follows from Lemma 2·1. It suffices to observe by definition that a morphism f is phantom if and only if $\mathrm{Hom}_{\mathcal{T}}(-, f)|_{\mathcal{T}^c} = 0$.

2·2. *Duality via cohomological functors*

Let A be a finite dimensional k-algebra. Denote by A-Mod the category of left A-modules. We denote by A-mod and A-proj the full subcategories consisting of finitely generated A-modules and finitely generated projective A-modules, respectively.

We use the cohomological notation. Denote a complex of A-modules by $X = (X^n, d_X^n)_{n\in\mathbb{Z}}$. The n-th cohomology of X is denoted by $H^n(X)$. For each n, we denote by $\sigma_{\geq n}(X)$ the brutal truncation of X, which is the subcomplex of X consisting of components with degree at least n. Denote by $\mathbf{K}(A\text{-Mod})$ and $\mathbf{D}(A\text{-Mod})$ the homotopy category and derived category of A-Mod, respectively. The translation of complexes is denoted by Σ.

We collect some well-known facts for later use. The following observation is contained in [**13**, lemma 2·6].

LEMMA 2·3. *Let $f: P \to X$ be a chain morphism such that P consists of projective modules and $H^n(X) = 0$ for $n < 0$. Assume that the restriction $f|_{\sigma_{\geq 0}(P)}: \sigma_{\geq 0}(P) \to X$ is homotopic to zero. Then f is homotopic to zero.*

Proof. We apply the cohomological functor $\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(-, X)$ to the canonical triangle

$$\sigma_{\geq 0}(P) \longrightarrow P \longrightarrow P/\sigma_{\geq 0}(P) \longrightarrow \Sigma\sigma_{\geq 0}(P).$$

By vanishing assumption on the cohomology of X, we observe that

$$\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(P/\sigma_{\geq 0}(P), X) = 0.$$

We deduce that the restriction map

$$\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(P, X) \longrightarrow \mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(\sigma_{\geq 0}(P), X)$$

is injective. Then the result follows.

For an A-module M, we denote by $\mathbf{i}(M)$ the injective resolution of M. Then we have a quasi-isomorphism $a_M: M \to \mathbf{i}(M)$, where M is viewed as a stalk complex concentrated in degree zero.

LEMMA 2·4. *Let X be a complex consisting of injective A-modules. Then there is an isomorphism*

$$\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(\mathbf{i}(M), X) \longrightarrow \mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(M, X), \quad f \longmapsto f \circ a_M. \tag{2·1}$$

In particular, we have an isomorphism

$$\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(\mathbf{i}(A), \Sigma^n(X)) \overset{\sim}{\longrightarrow} H^n(X) \tag{2·2}$$

for each integer n.

Proof. The first isomorphism is due to [**11**, lemma 2·1]. For the second, we just use the canonical isomorphism $\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(A, \Sigma^n(X)) \simeq H^n(X)$.

We denote by $\mathrm{rad}(A)$ the Jacobson radical of A and set $A_0 = A/\mathrm{rad}(A)$. For a complex X, we denote by $\tau_{<n}(X)$ and $\tau_{>n}(X)$ the good truncations. More precisely, we have $\tau_{<n}(X) = \cdots \to X^{n-3} \to X^{n-2} \to \mathrm{Ker}d_X^{n-1} \to 0$ and $\tau_{>n}(X) = 0 \to \mathrm{Cok}d_X^n \to X^{n+2} \to X^{n+3} \to \cdots$.

LEMMA 2·5. *Let X be a complex consisting of injective A-modules. Then the following statements are equivalent*:

(i) $\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(\mathbf{i}(A_0), X) = 0$*;*
(ii) $H^0(X) = 0$ *and* $\mathrm{Ker}d_X^{-1}$ *is an injective A-module;*
(iii) *The complex X is homotopic to $\tau_{<0}(X) \oplus \tau_{>0}(X)$.*

In this situation, the complex $\tau_{<0}(X) \oplus \tau_{>0}(X)$ also consists of injective A-modules.

Proof. For "(i) $\Rightarrow$ (ii)", we observe that $\mathbf{i}(A)$ is an iterated extension of direct summands of $\mathbf{i}(A_0)$ in $\mathbf{K}(A\text{-Mod})$. It follows that $\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(\mathbf{i}(A), X) = 0$. By (2·2) we have $H^0(X) = 0$. We observe an isomorphism

$$\mathrm{Ext}_A^1(A_0, \mathrm{Ker}d_X^{-1}) \simeq \mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(A_0, X),$$

since $0 \to \mathrm{Ker}d_X^{-1} \to X^{-1} \to X^0 \to X^1$ is a part of an injective resolution of $\mathrm{Ker}d_X^{-1}$. Applying (2·1) for $M = A_0$ and using this isomorphism, we deduce $\mathrm{Ext}_A^1(A_0, \mathrm{Ker}d_X^{-1}) = 0$, which implies that $\mathrm{Ker}d_X^{-1}$ is an injective A-module.

For "(ii) $\Rightarrow$ (iii)", we observe that the A-modules $\mathrm{Im}d_X^{-1}$, $\mathrm{Im}d_X^0$ and $\mathrm{Cok}d_X^0$ are all injective. It follows that as a complex, X is isomorphic to $\tau_{<0}(X) \oplus \tau_{>0}(X) \oplus E$, where $E = \cdots \to 0 \to \mathrm{Im}d_X^{-1} \to X^0 \to \mathrm{Im}d_X^0 \to 0 \to \cdots$ is homotopic to zero.

In view of (2·1) for $M = A_0$, the remaining implication "(iii) $\Rightarrow$ (i)" is trivial.

COROLLARY 2·6. *Let X be a complex consisting of injective A-modules and $n_0 > 0$. Assume that $\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(\mathbf{i}(A_0), \Sigma^n(X)) = 0$ whenever $|n| \geq n_0$. Then X is homotopic to $\tau_{<n_0}\tau_{>-n_0}(X)$, which is also consisting of injective A-modules.*

Proof. We apply Lemma 2·5 first to $\Sigma^n(X)$ for each $n \leq -n_0$. Then we have an isomorphism

$$X \simeq \tau_{<-n_0}(X) \oplus \tau_{>-n_0}(X)$$

in $\mathbf{K}(A\text{-Mod})$, where $\tau_{<-n_0}(X)$ is acyclic with injective cocycles. It follows that $\tau_{<-n_0}(X)$ is homotopic to zero. Hence X is homotopic to $\tau_{>-n_0}(X)$. Then we apply Lemma 2·5 to $\Sigma^n(\tau_{>-n_0}(X)) \simeq \Sigma^n(X)$ for each $n \geq n_0$. By a similar reasoning, we obtain the required isomorphism in $\mathbf{K}(A\text{-Mod})$.

The main concerns are the bounded homotopy category $\mathbf{K}^b(A\text{-proj})$ and the bounded derived category $\mathbf{D}^b(A\text{-mod})$. It is natural to view $\mathbf{K}^b(A\text{-proj})$ as a full triangulated subcategory of $\mathbf{D}^b(A\text{-mod})$; moreover, they are equal if and only if the algebra A has finite global dimension.

The following intrinsic description of the subcategory $\mathbf{K}^b(A\text{-proj})$ in $\mathbf{D}^b(A\text{-mod})$ is standard; see [**4**, lemma 1·2·1], or compare [**18**, the proof of proposition 6·2].

LEMMA 2·7. *Let $Y \in \mathbf{D}^b(A\text{-mod})$. Then Y lies in $\mathbf{K}^b(A\text{-proj})$ if and only if $\mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(Y, \Sigma^i(X)) = 0$ for each $X \in \mathbf{D}^b(A\text{-mod})$ and $i \gg 0$.*

The following result establishes a duality between $\mathbf{K}^b(A\text{-proj})$ and $\mathbf{D}^b(A\text{-mod})$. It is analogous to [2, theorem 3·2 and proposition 3·12].

We emphasise that the result is not new, as more general duality results are achieved in [14, 15]; see also [16]. Moreover, we mention that the first equivalence is implicit in [12, theorem 6·2], and the second equivalence might be deduced from [20, corollaries 4·17 and 4·29]. Here, using the representability lemma in the previous subsection, we prove the two equivalences in a unified manner.

THEOREM 2·8. *Let A be a finite dimensional k-algebra. Then we have equivalences of categories*

$$\mathbf{D}^b(A\text{-mod}) \xrightarrow{\sim} \mathrm{coho}(\mathbf{K}^b(A\text{-proj})^{\mathrm{op}}), \quad X \mapsto \mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(-, X)|_{\mathbf{K}^b(A\text{-proj})}$$

and

$$\mathbf{K}^b(A\text{-proj}) \xrightarrow{\sim} \mathrm{coho}(\mathbf{D}^b(A\text{-mod})), \quad P \longmapsto \mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(P, -).$$

Proof. For the first equivalence, we set $\mathcal{T} = \mathbf{D}(A\text{-Mod})$. It is well known that $\mathcal{T}$ is compactly generated and that there is a natural identification between $\mathcal{T}^c$ and $\mathbf{K}^b(A\text{-proj})$. Since $\mathbf{K}^b(A\text{-proj})$ is generated by A, an object $X \in \mathcal{T}$ is locally-finite if and only if $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(A, \Sigma^n(X))$ is finite dimensional. We recall the canonical isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(A, \Sigma^n(X)) \xrightarrow{\sim} H^n(X).$$

It follows that a complex $X \in \mathcal{T}$ is locally-finite if and only if the total cohomological space $\bigoplus_{n\in\mathbb{Z}} H^n(X)$ is finite dimensional, in other words, X lies in $\mathbf{D}^b(A\text{-mod})$. Hence, we identify $\mathcal{T}_{\mathrm{lf}}$ with $\mathbf{D}^b(A\text{-mod})$.

We observe that there is no non-zero phantom morphism $f : X \to Y$ in $\mathbf{D}^b(A\text{-mod})$. Indeed, we may assume that X is a bounded-above complex of projective modules and that f is a chain map. The phantom property implies that $f|_{\sigma_{\geq n}(X)}$ is homotopic to zero for any integer n. By Lemma 2·3, we infer that f is homotopic to zero. By combining these facts, the first equivalence follows from Corollary 2·2.

For the second equivalence, let A^{op} be the opposite algebra of A. We consider $\mathcal{T}' = \mathbf{K}(A^{\mathrm{op}}\text{-Inj})$, the homotopy category of injective A^{op}-modules. By [11, proposition 2·3], $\mathcal{T}'$ is compactly generated and there is a natural identification between $\mathcal{T}'^c$ and $\mathbf{D}^b(A^{\mathrm{op}}\text{-mod})$. Recall that we identify a complex Y in $\mathbf{D}^b(A^{\mathrm{op}}\text{-mod})$ with its injective resolution $\mathbf{i}(Y)$ in $\mathcal{T}'$.

We claim that an object I in $\mathcal{T}'$ is locally-finite if and only if it lies in $\mathbf{K}^b(A^{\mathrm{op}}\text{-inj})$, the bounded homotopy category of finitely generated injective A^{op}-modules. The "if" part is clear. Conversely, we assume that I is locally-finite. Then there is some $n_0 > 0$ such that

$$\mathrm{Hom}_{\mathbf{K}(A^{\mathrm{op}}\text{-Mod})}(\mathbf{i}(A_0), \Sigma^n(I)) = 0$$

whenever $|n| \geq n_0$. By Corollary 2·6, we may assume that I is a bounded complex of injective A-modules. By (2·2) we infer that the total cohomology space $\bigoplus_{n\in\mathbb{Z}} H^n(I)$ is finite dimensional. It follows that the bounded complex I is an injective resolution of a bounded complex of finitely generated A-modules. In other words, we have that up to isomorphism, I lies in $\mathbf{K}^b(A^{\mathrm{op}}\text{-inj})$. This proves the claim.

We now apply Corollary 2·2 to $\mathcal{T}'$. We identify $\mathcal{T}'^c$ with $\mathbf{D}^b(A^{\mathrm{op}}\text{-mod})$, and $\mathcal{T}'_{\mathrm{lf}}$ with $\mathbf{K}^b(A^{\mathrm{op}}\text{-inj})$. Since $\mathcal{T}'_{\mathrm{lf}} \subseteq \mathcal{T}'^c$, the phantom ideal vanishes. Hence, we have an equivalence

$$\mathbf{K}^b(A^{\mathrm{op}}\text{-inj}) \xrightarrow{\sim} \mathrm{coho}(\mathbf{D}^b(A^{\mathrm{op}}\text{-mod})^{\mathrm{op}}), \quad I \longmapsto \mathrm{Hom}_{\mathbf{D}^b(A^{\mathrm{op}}\text{-mod})}(-, I).$$

Using the duality functor $D = \mathrm{Hom}_k(-, k)$ on modules, we identify $\mathbf{K}^b(A\text{-proj})$ with $\mathbf{K}^b(A^{\mathrm{op}}\text{-inj})^{\mathrm{op}}$, and $\mathbf{D}^b(A\text{-mod})$ with $\mathbf{D}^b(A^{\mathrm{op}}\text{-mod})^{\mathrm{op}}$. Then the required equivalence follows immediately.

3. *Pseudo-adjunctions and triangle autoequivalences*

In this section, we apply Theorem 2·8 to obtain a 2-categorical duality between two strict 2-categories involving the bounded homotopy categories of projective modules and the bounded derived categories of module categories, respectively.

We mention that the triangulated structures are not properly captured by the duality in Theorem 2·8. We use the pseudo-adjunctions in [**2**] to obtain the required assignment between triangle functors.

Throughout this section, A and B will be two finite dimensional k-algebras. For 2-categories, we refer to [**8**, **9**].

3·1. *Pseudo-adjunctions and a 2-categorical duality*

Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories, with translation functors Σ and Σ', respectively. Recall that a triangle functor $(F, \omega)\colon \mathcal{T} \to \mathcal{T}'$ consists of an additive functor $F\colon \mathcal{T} \to \mathcal{T}'$ and a natural isomorphism $\omega\colon F\Sigma \to \Sigma' F$, called the *connecting isomorphism* of F, such that it respects exact triangles; more precisely, any exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ in $\mathcal{T}$ is sent to an exact triangle $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\omega_X \circ F(h)} \Sigma' F(X)$ in $\mathcal{T}'$. We will later suppress ω and denote (F, ω) simply by F. We emphasize that natural transformations between triangle functors are required to respect the connecting isomorphisms.

Let $F = (F, \omega)\colon \mathbf{K}^b(A\text{-proj}) \to \mathbf{K}^b(B\text{-proj})$ be a triangle functor. For each complex $X \in \mathbf{D}^b(B\text{-mod})$, the following cohomological functor

$$\mathrm{Hom}_{\mathbf{D}^b(B\text{-mod})}(F(-), X)\colon \mathbf{K}^b(A\text{-proj})^{\mathrm{op}} \longrightarrow k\text{-mod}$$

is locally-finite. By Theorem 2·8, there is a unique complex $F^\vee(X) \in \mathbf{D}^b(A\text{-mod})$ with a natural isomorphism

$$\mathrm{Hom}_{\mathbf{D}^b(B\text{-mod})}(F(-), X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(-, F^\vee(X))|_{\mathbf{K}^b(A\text{-proj})}.$$

Moreover, this defines a k-linear functor $F^\vee\colon \mathbf{D}^b(B\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ such that there is a k-linear bifunctorial isomorphism

$$\Phi_{P,X}\colon \mathrm{Hom}_{\mathbf{D}^b(B\text{-mod})}(F(P), X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(P, F^\vee(X))$$

for all $P \in \mathbf{K}^b(A\text{-proj})$ and $X \in \mathbf{D}^b(B\text{-mod})$. The connecting isomorphism ω yields a natural isomorphism $\omega^\vee\colon F^\vee\Sigma \to \Sigma F^\vee$ by the following commutative diagram,

$$\begin{tikzcd}
(\Sigma^{-1}F(P),X) \arrow[r, "\Sigma"] \arrow[d, "(\Sigma^{-1}\omega_{\Sigma^{-1}(P)},X)"] & (F(P),\Sigma(X)) \arrow[r, "\Phi_{P,\Sigma(X)}"] & (P,F^{\vee}\Sigma(X)) \arrow[d, "(P,\omega_X^{\vee})"] \\
(F\Sigma^{-1}(P),X) \arrow[r, "\Phi_{\Sigma^{-1}(P),X}"] & (\Sigma^{-1}(P),F^{\vee}(X)) \arrow[r, "\Sigma"] & (P,\Sigma F^{\vee}(X)),
\end{tikzcd}$$

where we omit the notation Hom in the Hom spaces.

LEMMA 3·1. *Keep the notation as above. Then* $F^{\vee}=(F^{\vee},\omega^{\vee})\colon \mathbf{D}^b(B\text{-mod})\rightarrow \mathbf{D}^b(A\text{-mod})$ *is a triangle functor.*

Following [2, section 4], we call $F^{\vee}$ the *right pseudo-adjoint* of F.

Proof. This is due to [2, lemma 4·11], where we replace the locally-free resolutions of complexes of sheaves in the proof by the projective resolutions of complexes of modules.

Conversely, for a triangle functor $G\colon \mathbf{D}^b(B\text{-mod})\rightarrow \mathbf{D}^b(A\text{-mod})$ and a complex $P\in \mathbf{K}^b(A\text{-proj})$, the following cohomological functor

$$\mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(P,G(-))\colon \mathbf{D}^b(B\text{-mod})\longrightarrow k\text{-mod}$$

is locally-finite. By Theorem 2·8 and a similar argument as above, we obtain a k-linear functor ${}^{\vee}G\colon \mathbf{K}^b(A\text{-proj})\rightarrow \mathbf{K}^b(B\text{-proj})$ and a bifunctorial isomorphism

$$\Psi_{P,X}\colon \mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(P,G(X))\overset{\sim}{\longrightarrow}\mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}({}^{\vee}G(P),X)$$

for all $P\in \mathbf{K}^b(A\text{-proj})$ and $X\in \mathbf{D}^b(B\text{-mod})$. Moreover, by [2, lemma 4·13], the functor ${}^{\vee}G$ is a triangle functor, called the *left pseudo-adjoint* of G. We call the above isomorphisms Φ and Ψ *pseudo-adjunctions*.

We denote by $\mathbb{K}^b$ the strict 2-category, whose objects are all the finite dimensional k-algebras A such that 1-morphisms are triangle functors between their bounded homotopy categories $\mathbf{K}^b(A\text{-proj})$ of projective modules and that 2-morphisms are natural transformations between these triangle functors. Similarly, we have the strict 2-category $\mathbb{D}^b$ by replacing $\mathbf{K}^b(A\text{-proj})$ with $\mathbf{D}^b(A\text{-mod})$. Denote by $(\mathbb{D}^b)^{\mathrm{coop}}$ the *bidual* of $\mathbb{D}^b$, where both the 1-morphisms and 2-morphisms are reversed.

The analogue of the following result for projective schemes is essentially proved in [2, section 4].

THEOREM 3·2. *The assignment* $F\mapsto F^{\vee}$ *gives rise to a 2-equivalence*

$$\mathbb{K}^b\overset{\sim}{\longrightarrow}(\mathbb{D}^b)^{\mathrm{coop}},$$

which acts on objects by the identity and whose inverse is given by the assignment $G\mapsto {}^{\vee}G$.

Proof. Using the pseudo-adjunctions, the assignment $F\mapsto F^{\vee}$ defines a (non-strict) 2-functor $\mathbb{K}^b\longrightarrow(\mathbb{D}^b)^{\mathrm{coop}}$, whose action on objects is the identity. By the following bifunctorial isomorphisms

$$(F(P),X)\overset{\Phi_{P,X}}{\longrightarrow}(P,F^{\vee}(X))\overset{\Psi_{P,X}}{\longrightarrow}({}^{\vee}(F^{\vee})(P),X)$$

we obtain an isomorphism $F \to {}^{\vee}(F^{\vee})$. Similarly, we obtain an isomorphism $G \to ({}^{\vee}G)^{\vee}$ for each 1-morphism G in $\mathbb{D}^b$. Then it is routine to verify that we have the required mutually inverse 2-equivalences.

3·2. *Extending functors and equivalences*

We will extract useful information from the 2-equivalence in Theorem 3·2. The treatment here is inspired by [**2**, lemmas 4·5 and 4·6] with substantial difference.

LEMMA 3·3. *Let* $F : \mathbf{K}^b(A\text{-proj}) \to \mathbf{K}^b(B\text{-proj})$ *and* $F_1 : \mathbf{K}^b(B\text{-proj}) \to \mathbf{K}^b(A\text{-proj})$ *be two triangle functors. Then the following statements hold:*

(i) the pair (F, F_1) *is adjoint if and only if* $(F^{\vee}, F_1^{\vee})$ *is adjoint;*
(ii) the functor F *is an equivalence if and only if so is* $F^{\vee}$.

Proof. We observe that adjoint 1-morphisms in $\mathbb{K}^b$ are just usual adjoint pairs of triangle functors between the relevant bounded homotopy categories, and internal equivalences corresponds to triangle equivalences. Similar remarks hold for $\mathbb{D}^b$. It is well known that that any 2-equivalence preserves adjoint 1-morphisms and internal equivalences; see [**8**, propositions 6·1·7 and 6·2·3]. Then the required results follow immediately from Theorem 3·2.

Let $F : \mathbf{K}^b(A\text{-proj}) \to \mathbf{K}^b(B\text{-proj})$ be a triangle functor. We say that a triangle functor $\tilde{F} : \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$ *extends* F, provided that $\tilde{F}(\mathbf{K}^b(A\text{-proj})) \subseteq \mathbf{K}^b(B\text{-proj})$ and that F is isomorphic to the restriction $\tilde{F}|_{\mathbf{K}^b(A\text{-proj})}$ as triangle functors.

We mention that the following is proved in [**1**, lemma 2·8] under the additional assumption that F is given by the tensor product of a certain bounded complex of bimodules.

PROPOSITION 3·4. *Let* $F : \mathbf{K}^b(A\text{-proj}) \to \mathbf{K}^b(B\text{-proj})$ *be a triangle functor. Then* F *admits an extension* $\tilde{F} : \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$ *if and only if* F *has a left adjoint.*

In this situation, the extension $\tilde{F}$ *of* F *is unique up to isomorphism, which necessarily has a right adjoint. Moreover,* F *is an equivalence if and only if so is its extension* $\tilde{F}$.

Proof. For the "only if" part of the first statement, we assume that $\tilde{F}$ extends F. For each $Q \in \mathbf{K}^b(B\text{-proj})$ and $P \in \mathbf{K}^b(A\text{-proj})$, we have bifunctorial isomorphisms

$$(Q, F(P)) \xrightarrow{\sim} (Q, \tilde{F}(P)) \xrightarrow{\Psi_{Q,P}} ({}^{\vee}\tilde{F}(Q), P).$$

This yields the required adjunction.

For the "if" part, we assume that (F_1, F) is an adjoint pair. Then by Lemma 3·3(1), we have an adjoint pair $(F_1^{\vee}, F^{\vee})$. For each $P \in \mathbf{K}^b(A\text{-proj})$ and $X \in \mathbf{D}^b(B\text{-mod})$, we have bifunctorial isomorphisms

$$(F(P), X) \xrightarrow{\Phi_{P,X}} (P, F^{\vee}(X)) \xrightarrow{\sim} (F_1^{\vee}(P), X).$$

By Yoneda's Lemma, we have a natural isomorphism $F(P) \simeq F_1^{\vee}(P)$, that is, $F_1^{\vee}$ extends F.

For the uniqueness of $\tilde{F}$, we observe that ${}^{\vee}(\tilde{F})$ is isomorphic to the left adjoint F_1 of F. It follows that $\tilde{F} \simeq F_1^{\vee}$, in particular, it admits a right adjoint $F^{\vee}$. If F is an equivalence, then F_1 and thus $F_1^{\vee}$ are equivalences. This proves the "only if" part of the last statement. The "if" part is well known; see Lemma 2·7.

An analogue of the following result for projective schemes is mentioned in [2, remark 4·6] with a different argument.

COROLLARY 3·5. *Let* $G\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$ *be a triangle functor. Then* G *has a right adjoint if and only if* $G(\mathbf{K}^b(A\text{-proj})) \subseteq \mathbf{K}^b(B\text{-proj})$.

Proof. The "only if" part is well known. Assume that G has a right adjoint G_1. Let $P \in \mathbf{K}^b(A\text{-proj})$. The adjunction

$$\mathrm{Hom}_{\mathbf{D}^b(B\text{-mod})}(G(P), \Sigma^i(X)) \simeq \mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(P, \Sigma^i G_1(X))$$

implies that $\mathrm{Hom}_{\mathbf{D}^b(B\text{-mod})}(G(P), \Sigma^i(X)) = 0$ for each $X \in \mathbf{D}^b(B\text{-mod})$ and $i \gg 0$. In view of Lemma 2·7, the complex $G(P)$ lies in $\mathbf{K}^b(B\text{-proj})$.

For the "if" part, we denote by $F = G|_{\mathbf{K}^b(A\text{-proj})}\colon \mathbf{K}^b(A\text{-proj}) \to \mathbf{K}^b(B\text{-proj})$ the restriction of G. In particular, G extends F. Then the existence of the right adjoint is proved in the second statement of Proposition 3·4.

For a triangulated category $\mathcal{T}$, $\mathrm{Aut}_\Delta(\mathcal{T})$ denotes its triangle autoequivalence group, which consists of the isomorphism classes of triangle autoequivalences on $\mathcal{T}$ and whose multiplication is induced by the composition of autoequivalences.

COROLLARY 3·6. *The restriction homomorphism between triangle autoequivalence groups*

$$\mathrm{Aut}_\Delta(\mathbf{D}^b(A\text{-mod})) \xrightarrow{\text{res}} \mathrm{Aut}_\Delta(\mathbf{K}^b(A\text{-proj})), \quad G \longmapsto G|_{\mathbf{K}^b(A\text{-proj})}$$

is an isomorphism.

Proof. Since G extends $G|_{\mathbf{K}^b(A\text{-proj})}$, the required injectivity follows from the uniqueness of the extension functor in Proposition 3·4. On the other hand, we infer from Proposition 3·4 that each triangle autoequivalence on $\mathbf{K}^b(A\text{-proj})$ extends to a triangle autoequivalence on $\mathbf{D}^b(A\text{-mod})$. This implies the required surjectivity.

We mention that Corollary 3·6 is related to the work [5]. Indeed, this is the starting point of this paper.

Recall from [21, 22] that $\mathrm{DPic}(A)$ is the *derived Picard group* of A, whose elements are the isomorphism classes of two-sided tilting complexes of A-modules and whose multiplication is given by the derived tensor product over A. The evaluation homomorphism

$$\mathrm{ev}\colon \mathrm{DPic}(A) \longrightarrow \mathrm{Aut}_\Delta(\mathbf{D}^b(A\text{-mod}))$$

sends a two-sided tilting complex X to the derived tensor functor $X \otimes^{\mathbb{L}}_A -$; compare [21, 2·2·1].

The following notions are taken from [5, definitions 4·1 and 5·1].

Definition 3·7. (1) An additive category $\mathcal{P}$ is $\mathbf{K}$-*standard* if the following condition is satisfied: any triangle autoequivalence F on $\mathbf{K}^b(\mathcal{P})$ is isomorphic to the identity functor as triangle functors, provided that it satisfies $F(\mathcal{P}) \subseteq \mathcal{P}$ and that $F|_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{P}$ is isomorphic to the identity functor.

(2) An abelian category $\mathcal{A}$ is **D**-*standard* if the following condition is satisfied: any triangle autoequivalence F on $\mathbf{D}^b(\mathcal{A})$ is isomorphic to the identity functor as triangle functors, provided that it satisfies $F(\mathcal{A}) \subseteq \mathcal{A}$ and that $F|_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}$ is isomorphic to the identity functor.

A triangle endofunctor F on $\mathbf{K}^b(\mathcal{P})$ satisfying the conditions in Definition 3·7(1) is called a *pseudo-identity* in [**5**, section 3]. The subtle difference between a pseudo-identity and the genuine identity lies in their actions on morphisms between complexes. In general, it seems very hard to verify whether a pseudo-identity is isomorphic to the genuine identity. Similar remarks hold for Definition 3·7(2).

The main motivation of these notions is [**5**, conjecture 5·11], which conjectures that any module category A-mod is **D**-standard. This conjecture goes back to the fundamental open question in [**19**, section 3], which asks whether any derived equivalence between algebras is *standard*, that is, isomorphic to the derived tensor functor by a two-sided tilting complex.

LEMMA 3·8. *The following statements hold:*

(i) the module category A-mod *is* **D**-*standard if and only if the evaluation homomorphism* "ev" *is surjective;*
(ii) the category A-proj *is* **K**-*standard if and only if the composition* "res $\circ$ ev" *is surjective.*

Proof. Recall that the surjectivity of "ev" is equivalent to the condition that every derived autoequivalence on $\mathbf{D}^b(A\text{-mod})$ is standard. Then (i) is contained in [**5**, theorem 5·10]. By a similar argument for $\mathbf{K}^b(A\text{-proj})$, one proves (ii).

Combining Corollary 3·6 and Lemma 3·8, we have the following immediate consequence.

COROLLARY 3·9. *Let A be a finite dimensional k-algebra. Then* A-proj *is* **K**-*standard if and only if* A-mod *is* **D**-*standard.*

We mention that the "only if" part is already known. Indeed, let $\mathcal{A}$ be an abelian category with enough projective objects. Denote by $\mathcal{P}$ its full subcategory formed by projective objects. By [**5**, theorem 6·1], the **K**-standardness of $\mathcal{P}$ implies the **D**-standardness of $\mathcal{A}$.

In view of Corollary 3·9, we expect that the inverse implication is true. More precisely, we expect an affirmative answer to the following question: if $\mathcal{A}$ is **D**-standard, is $\mathcal{P}$ necessarily **K**-standard? This question is related to the next one: does any triangle autoequivalence on $\mathbf{K}^b(\mathcal{P})$ extend to a triangle autoequivalence on $\mathbf{D}^b(\mathcal{A})$?

By the work [**16**] and [**12**], we suspect that the answer to the latter question is affirmative for the abelian category of finitely generated modules over a noetherian ring.

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