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Preprojective algebras, skew group algebras and Morita equivalences

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ABSTRACT

Let $\mathbb{K}$ be a field of characteristic p and G be a cyclic p-group which acts on a finite acyclic quiver Q. The folding process associates a Cartan triple to the action. We establish a Morita equivalence between the skew group algebra of the preprojective algebra of Q and the generalized preprojective algebra associated to the Cartan triple in the sense of Geiss, Leclerc and Schröer. The Morita equivalence induces an isomorphism between certain ideal monoids of these preprojective algebras, which is compatible with the embedding of Weyl groups appearing in the folding process.

1. Introduction

Weyl groups appear in the representation theory of quivers with different incarnations [2,22,19,31]. There is a remarkable bijection [3,29] between the Weyl group of a finite acyclic quiver and a certain ideal monoid of the preprojective algebra. Similarly, for a symmetrizable generalized Cartan matrix, there is a bijection [10] between the Weyl group and a certain ideal monoid of the generalized preprojective algebra in the sense of [14]. The motivation is to compare these two bijections via the folding process.

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The generalized preprojective algebras are used to study nilpotent varieties and the geometric construction of the crystal graph in the symmetrizable cases [15]. We mention related algebras in [5,38].

Let $\mathbb{K}$ be a field. Let Q be a finite acyclic quiver. Denote by $W(Q)$ its Weyl group and by $\Pi(Q)$ its preprojective algebra [12,33]. Each vertex $i \in Q_0$ gives rise to an idempotent e_i in $\Pi(Q)$. Denote by I_i the two-sided ideal of $\Pi(Q)$ generated by $1 - e_i$, and by $\langle I_i \mid i \in Q_0 \rangle$ the monoid generated by these ideals. The bijection established in [3,29]

$$\Theta_Q \colon W(Q) \longrightarrow \langle I_i \mid i \in Q_0 \rangle$$

sends the simple reflection s_i to I_i for each $i \in Q_0$.

Let (C, D, Ω) be a Cartan triple, that is, C is a symmetrizable generalized Cartan matrix [23], D is the symmetrizer and Ω is an acyclic orientation of C. Assume that the rows and columns of C are indexed by a set Λ. Denote by $W(C)$ the Weyl group and by $\Pi(C, D, \Omega)$ the generalized preprojective algebra in the sense of [14]. For each $\mathbf{j} \in \Lambda$, we denote by $e_{\mathbf{j}}$ the corresponding idempotent in $\Pi(C, D, \Omega)$. Denote by $L_{\mathbf{j}}$ the two-sided ideal of $\Pi(C, D, \Omega)$ generated by $1 - e_{\mathbf{j}}$. The bijection established in [10]

$$\Theta_C \colon W(C) \longrightarrow \langle L_{\mathbf{j}} \mid \mathbf{j} \in \Lambda \rangle$$

sends the simple reflection $r_{\mathbf{j}}$ to $L_{\mathbf{j}}$ for each $\mathbf{j} \in \Lambda$.

The consideration above is closely related to tilting theory. In non-Dynkin cases, the monoids $\langle I_i \mid i \in Q_0 \rangle$ and $\langle L_{\mathbf{j}} \mid \mathbf{j} \in \Lambda \rangle$ above coincide with the sets of cofinite tilting ideals in $\Pi(Q)$ and $\Pi(C, D, \Omega)$, respectively [21,3,10]. In Dynkin cases, they are bijective to the corresponding sets of isomorphic classes of basic support τ-tilting modules [29,10].

The folding process is classic in Lie theory [34] and plays a role in the representation theory of quivers [35,18]. Let G be a finite group which acts on Q by quiver automorphisms. One associates a Cartan triple (C, D, Ω) to the G-action. The rows and columns of both C and D are indexed by the orbit set Q_0/G of vertices in Q, and the entries of the diagonal matrix D are the cardinalities of certain stabilizers. By [34,17], there is a well-known isomorphism

$$\psi \colon W(C) \longrightarrow W(Q)^G,$$

which sends simple reflections $r_{\mathbf{j}}$ to $\prod_{i \in \mathbf{j}} s_i$. Here, $W(Q)^G$ denotes the subgroup formed by G-invariant elements in $W(Q)$.

In view of the bijections Θ_Q, Θ_C and the isomorphism ψ, it is natural to expect that the ideal monoids $\langle I_i \mid i \in Q_0 \rangle^G$ and $\langle L_{\mathbf{j}} \mid \mathbf{j} \in Q_0/G \rangle$ are isomorphic. We confirm this expectation; see Proposition 6.5.

Proposition A. *Let G be a finite group which acts on a finite acyclic quiver Q. Consider the associated Cartan triple (C, D, Ω). Then there is a unique isomorphism Ψ between monoids making the following diagram commute.*

$$\begin{tikzcd}
W(C) \arrow[r, "\psi"] \arrow[d, "\Theta_C"'] & W(Q)^G \arrow[d, "\Theta_Q^G"] \\
\langle L_{\mathbf{j}} \mid \mathbf{j} \in Q_0/G \rangle \arrow[r, "\Psi"] & \langle I_i \mid i \in Q_0 \rangle^G
\end{tikzcd}$$

Here, Θ_Q^G denotes the restriction of Θ_Q on $W(Q)^G$.

We mention a similar commutative diagram in Remark 5.7, where we replace $\langle L_{\mathbf{j}} \mid \mathbf{j} \in Q_0/G\rangle$ by a certain ideal monoid $\langle I_{\mathbf{j}}\#G \mid \mathbf{j} \in Q_0/G\rangle$ of the skew group algebra $\Pi(Q)\#G$.

The isomorphism Ψ suggests that the two preprojective algebras $\Pi(Q)$ and $\Pi(C, D, \Omega)$ might be closely related. The aim of this work is to relate these two preprojective algebras in a specific situation; see Theorem 7.2.

Theorem B. *Assume that* $\mathrm{char}(\mathbb{K}) = p > 0$ *and that* G *is a cyclic* p*-group. Assume that the* G*-action on* Q *satisfies* $G_\alpha = G_{s(\alpha)} \cap G_{t(\alpha)}$ *for any arrow* α *in* Q*. Then there is a Morita equivalence*

$$F\colon \Pi(Q)\#G\text{-Mod} \longrightarrow \Pi(C, D, \Omega)\text{-Mod}$$

such that

$$\Psi^{-1}(I) = \Phi_F(I\#G) \tag{1.1}$$

for each $I \in \langle I_i \mid i \in Q_0\rangle^G$.

Here, Φ_F denotes the isomorphism between the ideal monoids induced by the Morita equivalence F; see Proposition 4.3. For each arrow α with the starting vertex $s(\alpha)$ and terminating vertex $t(\alpha)$, we denote by G_α, $G_{s(\alpha)}$ and $G_{t(\alpha)}$ their stabilizers.

The identity (1.1) indicates that, in a certain sense, the isomorphism Ψ^{-1} is *categorified* by the Morita equivalence F and the induction functor $-\#G$.

The proof of Proposition A relies on the fact that these ideal monoids are isomorphic to the corresponding Weyl monoids [37]. Moreover, we establish an analogue of the isomorphism ψ for the Weyl monoids in Proposition 5.4. The proof of Theorem B relies on the Morita equivalence in [8] between the skew group algebra of $\mathbb{K}Q$ and the algebra $H(C, D, \Omega)$ in [14]. We also use general results on 2-preprojective algebras of arbitrary algebras, which are implicit in [26]. When Q is of type A and G is of order 2, such a Morita equivalence F is also established in [25].

The paper is structured as follows. We recall the definition of n-preprojective algebras of an arbitrary algebra in Section 2. We study skew group algebras, compatible bimodules and their induced bimodules in Section 3. We prove that the skew group algebra of a preprojective algebra is isomorphic to the preprojective algebra of the skew group algebra in Theorem 3.13. In Section 4, we prove that any Morita equivalence between two algebras extends naturally to a Morita equivalence between their preprojective algebras; see Proposition 4.7. We study Weyl groups and Weyl monoids [37] associated to quivers and Cartan matrices in Section 5.

We recall from [14] the generalized preprojective algebra and prove Proposition A in Section 6. In final section, we prove Theorem B, which is illustrated with an explicit example.

Throughout this paper, we work over a field $\mathbb{K}$. Unadorned Hom and tensor functors are all over $\mathbb{K}$. By default, a module means a left unital module. For an algebra A, we denote by A-Mod the category of all A-modules.

2. Tensor algebras and preprojective algebras

In this section, we fix notation and recall some well-known facts on tensor algebras and preprojective algebras.

Let A be an algebra, and V an A-A-bimodule on which $\mathbb{K}$ acts centrally. The associated *tensor algebra* $T_A(V)$ is given by

$$T_A(V) = A \oplus V \oplus V^{\otimes_A 2} \oplus \cdots \oplus V^{\otimes_A n} \oplus \cdots$$

where $V^{\otimes_A n}$ is the n-fold tensor product of V.

Let $\mathcal{C}$ be any category with an endofunctor $E\colon \mathcal{C} \to \mathcal{C}$. By a *representation* of E, we mean a pair (X, α) consisting of an object X in $\mathcal{C}$ and a morphism $\alpha\colon E(X) \to X$. A morphism $f\colon (X,\alpha) \to (Y,\beta)$ between representations is given by a morphism $f\colon X \to Y$ in $\mathcal{C}$ satisfying

$$f \circ \alpha = \beta \circ E(f).$$

These data form the category of representations of E, denoted by $\mathrm{rep}(E)$; see [33, p. 469] and [6, Subsection 2.1].

The following fact is standard.

Lemma 2.1. *Let $\mathcal{C}'$ be another category with an endofunctor E'. Assume that there is an equivalence $F\colon \mathcal{C} \to \mathcal{C}'$ of categories such that FE is isomorphic to $E'F$. Then we have an induced equivalence $\widetilde{F}\colon \mathrm{rep}(E) \to \mathrm{rep}(E')$.*

Proof. Assume that we are given a natural isomorphism $\eta\colon E'F \to FE$. The induced functor $\widetilde{F}$ sends a representation (X,α) of E to $(F(X), F(\alpha)\circ \eta_X)$ of E'. Similarly, one constructs a quasi-inverse of $\widetilde{F}$. □

Let A be an algebra, and V an A-A-bimodule. Consider the endofunctor

$$V \otimes_A -\colon A\text{-Mod} \longrightarrow A\text{-Mod}.$$

There is an isomorphism of categories

$$T_A(V)\text{-Mod} \longrightarrow \mathrm{rep}(V \otimes_A -), \tag{2.1}$$

which sends a $T_A(V)$-module X to the representation (X,α), where X is the underlying A-module and $\alpha(v \otimes_A x) = vx$ for $v \in V$ and $x \in X$; compare [33, Lemma 2].

Denote by $A^e = A \otimes A^{\mathrm{op}}$ the *enveloping algebra* of A. We identify A-A-bimodules with left A^e-modules, which are also identified with right A^e-modules.

Let e and f be two idempotents of A. Then $Ae \otimes fA$ is naturally an A-A-bimodule, which is cyclic and projective. We have a canonical isomorphism of A-A-bimodules

$$\mathrm{Hom}_{A^e}(Ae \otimes fA, A^e) \longrightarrow Af \otimes eA,\ \theta \mapsto \mathrm{swap}(\theta(e \otimes f)). \tag{2.2}$$

Here, $\mathrm{swap}(a \otimes b) = b \otimes a$, and the A-A-bimodule structure of $\mathrm{Hom}_{A^e}(Ae \otimes fA, A^e)$ is induced by the inner A-A-bimodule structure on A^e. Similarly, for each $n \geq 1$, the n-th extension space $\mathrm{Ext}^n_{A^e}(A, A^e)$ is naturally a right A^e-module, and thus an A-A-bimodule.

The following fact is well known.

Lemma 2.2. *Assume that the algebra A is finite dimensional. Then we have an isomorphism of A-A-bimodules*

$$\mathrm{Ext}^n_{A^e}(A, A^e) \simeq \mathrm{Ext}^n_A(DA, A)$$

for each $n \geq 1$.

Proof. The canonical map $A^e \to \mathrm{Hom}(DA, A)$, sending $a \otimes b$ to $(\theta \mapsto \theta(b)a)$, is an isomorphism of left A^e-modules. Therefore, we identify $\mathrm{Ext}^n_{A^e}(A, A^e)$ with $\mathrm{Ext}^n_{A^e}(A, \mathrm{Hom}(DA, A))$, which is canonically isomorphic to $\mathrm{Ext}^n_A(DA, A)$. □

In view of [33, Theorem A], [20, Definition 2.11] and Lemma 2.2, the following definition is natural; compare [14]. We mention that its derived analogue is due to [24, Subsection 4.1].

Definition 2.3. Let A be an algebra and $n \geq 2$. Then *n-preprojective algebra* of A is defined to be the tensor algebra $\Pi_n(A) = T_A(\mathrm{Ext}^{n-1}_{A^e}(A, A^e))$.

Let $Q = (Q_0, Q_1; s, t)$ be a finite quiver, where Q_0 is the finite set of vertices, Q_1 is the finite set of arrows and the two maps $s, t\colon Q_1 \to Q_0$ assign to each arrow α its starting vertex $s(\alpha)$ and its terminating vertex $t(\alpha)$. A path $p = \alpha_n \cdots \alpha_2\alpha_1$ of length n consists of n consecutive arrows α_i's, that is, $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq n-1$. Here, we write the concatenation from right to left. We observe that a path of length 1 is just an arrow. For each vertex i, we associate a trivial path e_i of length 0. Denote by $\mathbb{K}Q$ the *path algebra*, which has a basis given by all paths in Q and whose multiplication is given by the concatenation of paths.

Denote by $\overline{Q}$ the *double quiver* of Q, which is obtained from Q by adding for each arrow $\alpha \in Q_1$ an inverse arrow α^*. Following [33], the *preprojective algebra* of Q is defined by

$$\Pi(Q) = \mathbb{K}\overline{Q}/(\sum_{\alpha \in Q_1} \alpha\alpha^* - \alpha^*\alpha).$$

We mention the implicit appearance of the preprojective algebra in [27, Section 12].

The following result is well known; compare [33, Theorem A].

Lemma 2.4. *Let Q be any finite quiver. Then there is an isomorphism $\Pi_2(\mathbb{K}Q) \simeq \Pi(Q)$ of algebras.*

Proof. Write $A = \mathbb{K}Q$. We have a canonical bimodule projective resolution of A.

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} Ae_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A \xrightarrow{\partial} \bigoplus_{i \in Q_0} Ae_i \otimes e_iA \longrightarrow A \longrightarrow 0 \tag{2.3}$$

Here, $\partial(e_{t(\alpha)} \otimes \alpha \otimes e_{s(\alpha)}) = \alpha \otimes e_{s(\alpha)} - e_{t(\alpha)} \otimes \alpha$ and the unnamed arrow on the right is given by the multiplication in A. Applying $\mathrm{Hom}_{A^e}(-, A^e)$ to this sequence and using the isomorphism (2.2), we infer that $\mathrm{Ext}^1_{A^e}(A, A^e)$ is isomorphic to the cokernel of the following morphism

$$\bigoplus_{i \in Q_0} Ae_i \otimes e_iA \xrightarrow{\partial'} \bigoplus_{\alpha \in Q_1} Ae_{s(\alpha)} \otimes \mathbb{K}\alpha^* \otimes e_{t(\alpha)}A, \tag{2.4}$$

which is given by

$$\partial'(e_i \otimes e_i) = \sum_{\{\alpha \in Q_1 \mid s(\alpha) = i\}} e_i \otimes \alpha^* \otimes \alpha - \sum_{\{\beta \in Q_1 \mid t(\beta) = i\}} \beta \otimes \beta^* \otimes e_i.$$

Here for each $\alpha \in Q_1$, we identify $Ae_{t(\alpha)} \otimes \mathbb{K}\alpha \otimes e_{s(\alpha)}A$ with $Ae_{t(\alpha)} \otimes e_{s(\alpha)}A$, and $Ae_{s(\alpha)} \otimes \mathbb{K}\alpha^* \otimes e_{t(\alpha)}A$ with $Ae_{s(\alpha)} \otimes e_{t(\alpha)}A$.

We observe that the tensor algebra

$$T_A(\bigoplus_{\alpha \in Q_1} Ae_{s(\alpha)} \otimes \mathbb{K}\alpha^* \otimes e_{t(\alpha)}A)$$

is naturally isomorphic to the path algebra $\mathbb{K}\overline{Q}$ of the double quiver $\overline{Q}$. Then the required isomorphism follows immediately. $\square$

3. Skew group algebras and induced bimodules

In this section, we will recall basic facts on skew group algebras, and study their induced bimodules. We prove that the n-preprojective algebra of a skew group algebra is isomorphic to the skew group algebra of the n-preprojective algebra; see Theorem 3.13.

We emphasize that many results in this section are implicitly due to [26]. We provide full proofs for completeness, since the setting there is very different. We mention related work on skew group algebras of quiver algebras [9,16,36].

3.1. Compatible bimodules and induced bimodules

Let A be an algebra. Denote by $\mathbf{I}(A)$ the *ideal monoid* of A, which consists of two-sided ideals of A and whose multiplication is given by the multiplication between ideals. Let G be a finite group, which is written multiplicatively and whose identity is denoted by 1_G.

We fix a G-action $\rho\colon G \to \mathrm{Aut}(A)$ on A by algebra automorphisms. Write $\rho(g)(a) = g(a)$ for any $g \in G$ and $a \in A$. The *skew group algebra* is given by $A\#G = A \otimes \mathbb{K}G$, whose typical element is denoted by $a\#g$ and whose multiplication is defined by

$$(a\#g)(b\#h) = ag(b)\#gh.$$

The following identity

$$(1_A\#g)(a\#1_G)(1_A\#g^{-1}) = g(a)\#1_G \tag{3.1}$$

will be used. We view A as a subalgebra of $A\#G$ by identifying a with $a\#1_G$. Consider the projection $\mathrm{pr}\colon A\#G \to A$, which sends $a\#g$ to $\delta_{g,1_G}a$.

Lemma 3.1. *Let M be a left $A\#G$-module. The following two statements hold.*

(1) The space $\mathrm{Hom}_A(M, A)$ becomes a right $A\#G$-module in the following manner: for any $\theta \in \mathrm{Hom}_A(M, A)$, the element $\theta(a\#g) \in \mathrm{Hom}_A(M, A)$ sends each $m \in M$ to $g^{-1}(\theta((1_A\#g)m)a)$.

(2) The projection above induces an isomorphism of right $A\#G$-modules

$$\mathrm{Hom}_{A\#G}(M, A\#G) \longrightarrow \mathrm{Hom}_A(M, A).$$

Proof. The proof of (1) is routine. For (2), we refer to [32, Subsection 1.1 (B)]. The inverse map sends $\theta \in \mathrm{Hom}_A(M, A)$ to $\theta'\colon M \to A\#G$, which is given by $(m \mapsto \sum_{g\in G} g\theta((1\#g^{-1})m)\#g)$; compare [32, Lemma 1.2]. □

A two-sided ideal I of A is called *G-invariant* if $g(I) = I$ for all $g \in G$. Denote by $\mathbf{I}(A)^G$ the sub monoid of $\mathbf{I}(A)$ formed by G-invariant ideals.

We observe that the decomposition $A\#G = \oplus_{g\in G}(A\#g)$ makes $A\#G$ a G-graded algebra. A two-sided ideal J of $A\#G$ is *G-graded* if $J = \oplus_{g\in G}(J \cap (A\#g))$. Denote by $\mathbf{I}(A\#G)_G$ the sub monoid of $\mathbf{I}(A\#G)$ formed by G-graded ideals.

The following results are elementary.

Proposition 3.2. *Let I be a two-sided ideal of A and J a two-sided ideal of $A\#G$. Then the following results hold.*

(1) The subspace $\{a \in A \mid a\#1_G \in J\}$ *is a* G*-invariant two-sided ideal of* A*.*
(2) The subspace $I\#G = \{\sum_{g\in G} a_g\#g \mid a_g \in I\}$ *is a two-sided ideal of* $A\#G$ *if and only if the ideal* I *is* G*-invariant.*
(3) There is an isomorphism $\mathbf{I}(A)^G \to \mathbf{I}(A\#G)_G$ *of monoids, sending any* G*-invariant ideal* I' *to* $I'\#G$*.*

Proof. We use (3.1) to verify (1). The "only if" part of (2) follows from (1), and the "if" part is straightforward.

The isomorphism in (3) follows immediately from (1) and (2), whose inverse map sends any G-graded ideal J' of $A\#G$ to $\{a \in A \mid a\#1_G \in J'\}$, which belongs to $\mathbf{I}(A)^G$ by (1). □

The group G acts on the enveloping algebra A^e by algebra automorphisms, that is, $g(a\otimes b) = g(a)\otimes g(b)$. Denote by $\Delta = A^e\#G$ the corresponding skew group algebra. We observe an algebra embedding

$$\Delta \hookrightarrow (A\#G)^e, (a\otimes b)\#g \mapsto (a\#g)\otimes (g^{-1}(b)\#g^{-1}). \tag{3.2}$$

In view of this embedding, we might call Δ the *diagonal subalgebra* of $(A\#G)^e$; compare [26, Subsection 3.1]. We observe that $(A\otimes G)^e$ is free both as a left Δ-module and a right Δ-module. We have an isomorphism of algebra

$$\Delta \longrightarrow \Delta^{\mathrm{op}},\ (a\otimes b)\#g \mapsto (g^{-1}(b)\otimes g^{-1}(a))\#g^{-1}. \tag{3.3}$$

The following notion is implicitly due to [26, Subsection 3.1].

Definition 3.3. By a *G-compatible A-A-bimodule*, we mean an A-A-bimodule M with a $\mathbb{K}$-linear G-action satisfying that

$$g(amb) = g(a)g(m)g(b)$$

for any $a, b \in A$ and $m \in M$. Here, we denote by $g(amb)$ and $g(m)$ the g-actions on the elements amb and m, respectively.

Remark 3.4. We observe that a G-compatible A-A-bimodule structure is equivalent to a left Δ-module structure, and is also equivalent to a right Δ-module structure; compare (3.3). More precisely, let M be a G-compatible A-A-bimodule. Then M is naturally a left Δ-module with the following action

$$((a\otimes b)\#g)m = ag(m)b.$$

Similarly, M is naturally a right Δ-module given by

$$m((a\otimes b)\#g) = g^{-1}(bma).$$

Definition 3.5. Let M be a G-compatible A-A-bimodule. The *induced bimodule* $M\#G = M\otimes \mathbb{K}G$ is a bimodule over $A\#G$ defined as follows

$$(a\#g)(m\#h)(a'\#g') = ag(m)gh(a')\#ghg'$$

for any $a, a' \in A$, $m \in M$ and $g, h, g' \in G$.

The following remark justifies the terminology above.

Remark 3.6. Let M be a G-compatible A-A-bimodule. Then there is an isomorphism

$$(A\#G) \otimes_A M \longrightarrow M\#G, \quad (a\#g) \otimes_A m \mapsto ag(m)\#g$$

of left $A\#G$-modules. Therefore, the left $A\#G$-module structure on $M\#G$ is induced from the left A-module structure on M. A similar remark works on the right side.

Example 3.7. Let I be a G-invariant two-sided ideal of A. Then G acts on I. Therefore, as an A-A-bimodule, I is G-compatible. We observe that the corresponding induced bimodule structure on $I\#G$ is the same as the one inherited from the two-sided ideal $I\#G$ of $A\#G$.

The following results are essentially due to [26, Lemma 3.1.1].

Lemma 3.8. *Let M be a G-compatible A-A-bimodule. Then the following statements hold.*

(1) There is an isomorphism $(A\#G)^e \otimes_\Delta M \simeq M\#G$ of left $(A\#G)^e$-modules.
(2) There is an isomorphism $M \otimes_\Delta (A\#G)^e \simeq M\#G$ of right $(A\#G)^e$-modules.

In (1), we view M as a left Δ-module. Then using the algebra embedding (3.2), we have the induced left module $(A\#G)^e \otimes_\Delta M$. A similar remark holds for (2).

Proof. The isomorphism in (1) sends $((a\#g) \otimes (b\#h)) \otimes_\Delta m$ to $ag(mb)\#gh$. The isomorphism in (2) sends $m \otimes_\Delta ((a\#g) \otimes (b\#h))$ to $bh(ma)\#hg$. We omit the details. $\square$

Let M and N be two G-compatible A-A-bimodules. Then the A-A-bimodule $M \otimes_A N$ is G-compatible with the diagonal G-action. The proof of the following result is routine.

Lemma 3.9. *Let M and N be two G-compatible A-A-bimodules. Then there is an isomorphism of $(A\#G)$-$(A\#G)$-bimodules*

$$(M \otimes_A N)\#G \longrightarrow (M\#G) \otimes_{A\#G} (N\#G),$$

which sends $(m \otimes_A n)\#g$ to $(m\#1_G) \otimes_{A\#G} (n\#g)$. $\square$

Let V be a G-compatible A-A-bimodule. Then G acts naturally on the tensor algebra $T_A(V)$ by algebra automorphisms in the following manner: for $g \in G$ and $v_1 \otimes_A v_2 \otimes_A \cdots \otimes_A v_n \in V^{\otimes_A n}$, we define

$$g(v_1 \otimes_A v_2 \otimes_A \cdots \otimes_A v_n) = g(v_1) \otimes_A g(v_2) \otimes_A \cdots \otimes_A g(v_n).$$

We form the skew group algebra $T_A(V)\#G$.

Proposition 3.10. *Let V be a G-compatible A-A-bimodule with $V\#G$ the induced bimodule over $A\#G$. Then we have an isomorphism $T_A(V)\#G \simeq T_{A\#G}(V\#G)$ of algebras.*

Proof. By applying Lemma 3.9 repeatedly, we infer an isomorphism

$$\phi_n \colon V^{\otimes_A n}\#G \longrightarrow (V\#G)^{\otimes_{A\#G} n}$$

for each $n \geq 1$, which sends $(v_1 \otimes_A v_2 \otimes_A \cdots \otimes_A v_n)\#g$ to

$$(v_1\#1_G) \otimes_{A\#G} (v_2\#1_G) \otimes_{A\#G} \cdots \otimes_{A\#G} (v_n\#g).$$

It is direct to verify that these isomorphisms ϕ_n give rise to the required isomorphism of algebras. □

3.2. Skew group algebras of preprojective algebras

Let M be a G-compatible A-A-bimodule. Consider the *dual A-A-bimodule* $\mathrm{Hom}_{A^e}(M, A^e)$. It is naturally G-compatible by the contragredient G-action: for each $g \in G$ and $\theta\colon M \to A^e$, we define $g(\theta)\colon M \to A^e$ such that $g(\theta)(m) = g(\theta(g^{-1}(m)))$. Then we have the induced bimodule $\mathrm{Hom}_{A^e}(M, A^e)\#G$ over $A\#G$.

The following result is essentially due to [26, Lemma 3.3.1], where the diagonal subalgebra Δ plays a role.

Lemma 3.11. *Let M be a G-compatible A-A-bimodule. Then we have an isomorphism of $(A\#G)$-$(A\#G)$-bimodules*

$$\mathrm{Hom}_{A^e}(M, A^e)\#G \simeq \mathrm{Hom}_{(A\#G)^e}(M\#G, (A\#G)^e).$$

Proof. Recall that $\Delta = A^e\#G$. By Lemma 3.8(1), we identify the induced bimodule $M\#G$ with $(A\#G)^e \otimes_\Delta M$. By the Hom-tensor adjunction, we have the first isomorphism in the following identity.

$$\begin{aligned}
\mathrm{Hom}_{(A\#G)^e}(M\#G, (A\#G)^e) &\simeq \mathrm{Hom}_\Delta(M, (A\#G)^e) \\
&\simeq \mathrm{Hom}_\Delta(M, \Delta) \otimes_\Delta (A\#G)^e \\
&\simeq \mathrm{Hom}_{A^e}(M, A^e) \otimes_\Delta (A\#G)^e \\
&\simeq \mathrm{Hom}_{A^e}(M, A^e)\#G
\end{aligned}$$

Here, the second isomorphism follows since $(A\#G)^e$ is a finitely generated free left Δ-module, the third one follows by applying Lemma 3.1(2) to the left Δ-module M, and the last one follows from Lemma 3.8(2). □

Let M be a G-compatible A-A-bimodule. For each $n \geq 1$, we observe that the A-A-bimodule $\mathrm{Ext}^n_{A^e}(M, A^e)$ is naturally G-compatible. Indeed, we take a projective resolution $P^\bullet$ of M as a left Δ-module. Therefore, each component P^{-n} is a G-compatible A-A-bimodule, whose underlying A-A-bimodule is projective. Each component of the dual complex $\mathrm{Hom}_{A^e}(P^\bullet, A^e)$ is naturally G-compatible. Consequently,

$$\mathrm{Ext}^n_{A^e}(M, A^e) = H^n(\mathrm{Hom}_{A^e}(P^\bullet, A^e))$$

is a naturally G-compatible A-A-bimodule. Then we form the induced bimodule $\mathrm{Ext}^n_{A^e}(M, A^e)\#G$ over $A\#G$.

Lemma 3.12. *Let M be a G-compatible A-A-bimodule. Then for each $n \geq 1$, we have an isomorphism of $(A\#G)$-$(A\#G)$-bimodules*

$$\mathrm{Ext}^n_{A^e}(M, A^e)\#G \simeq \mathrm{Ext}^n_{(A\#G)^e}(M\#G, (A\#G)^e).$$

Proof. By $\mathrm{Ext}^n_{A^e}(M, A^e) = H^n(\mathrm{Hom}_{A^e}(P^\bullet, A^e))$, we infer that

$$\mathrm{Ext}^n_{A^e}(M, A^e)\#G = H^n(\mathrm{Hom}_{A^e}(P^\bullet, A^e)\#G).$$

Lemma 3.11 implies that there is an isomorphism of complexes

$$\mathrm{Hom}_{A^e}(P^\bullet, A^e)\#G \simeq \mathrm{Hom}_{(A\#G)^e}(P^\bullet\#G, (A\#G)^e).$$

We observe that $P^\bullet\#G$ is a projective resolution of the induced bimodule $M\#G$. Then we infer an isomorphism

$$H^n(\mathrm{Hom}_{A^e}(P^\bullet, A^e)\#G) \simeq \mathrm{Ext}^n_{(A\#G)^e}(M\#G, (A\#G)^e).$$

This completes the proof. □

Let G be a finite group and A an algebra with a G-action. For $n \geq 2$, the A-A-bimodule $\mathrm{Ext}^{n-1}_{A^e}(A, A^e)$ is naturally G-compatible. Therefore, the n-preprojective algebra $\Pi_n(A) = T_A(\mathrm{Ext}^{n-1}_{A^e}(A, A^e))$ has an induced G-action. We form the skew group algebra $\Pi_n(A)\#G$.

The derived analogue of the following result is due to [26, Theorem 3.5.4(1)].

Theorem 3.13. *Keep the assumptions above. Then we have an isomorphism $\Pi_n(A)\#G \simeq \Pi_n(A\#G)$ of algebras.*

Proof. By Lemma 3.12, we have an isomorphism of $(A\#G)$-$(A\#G)$-bimodules.

$$\mathrm{Ext}^{n-1}_{(A\#G)^e}(A\#G, (A\#G)^e) \simeq \mathrm{Ext}^{n-1}_{A^e}(A, A^e)\#G$$

It follows that $\Pi_n(A\#G)$ is isomorphic to the following tensor algebra

$$T_{A\#G}(\mathrm{Ext}^{n-1}_{A^e}(A, A^e)\#G).$$

By Proposition 3.10, the algebra above is isomorphic to

$$T_A(\mathrm{Ext}^{n-1}_{A^e}(A, A^e))\#G = \Pi_n(A)\#G.$$

This completes the proof. □

Let G be a finite group which acts on a finite quiver Q by quiver automorphisms. Then G acts on the path algebra $\mathbb{K}Q$ and the preprojective algebra $\Pi(Q)$. Here, we observe that $g(\alpha^*) = g(\alpha)^*$ for $g \in G$ and $\alpha \in Q_1$. We form the corresponding skew group algebras $\mathbb{K}Q\#G$ and $\Pi(Q)\#G$.

Proposition 3.14. *Keep the assumptions above. Then we have an isomorphism $\Pi(Q)\#G \simeq \Pi_2(\mathbb{K}Q\#G)$ of algebras.*

Proof. The G-action on $\mathbb{K}Q$ induces a G-action on the 2-preprojective algebra $\Pi_2(\mathbb{K}Q)$. By Lemma 2.4, we have an isomorphism $\Pi(Q) \simeq \Pi_2(\mathbb{K}Q)$. We claim that this isomorphism is compatible with the two G-actions.

For the claim, it suffices to prove that the following statement is true: the induced G-action on $\mathrm{Ext}^1_{(\mathbb{K}Q)^e}(\mathbb{K}Q, (\mathbb{K}Q)^e)$ coincides with the one given by $g(\alpha^*) = g(\alpha)^*$. Indeed, the projective resolution (2.3) is a complex of G-compatible $\mathbb{K}Q$-$\mathbb{K}Q$-bimodules. Applying $\mathrm{Hom}_{(\mathbb{K}Q)^e}(-, (\mathbb{K}Q)^e)$ to (2.3), we obtain (2.4), whose induced G-action is given as follows: $g(ae_i \otimes e_i b) = g(a)e_{g(i)} \otimes e_{g(i)}g(b)$ and

$$g(ae_{s(\alpha)} \otimes \alpha^* \otimes e_{t(\alpha)}b) = g(a)e_{s(g(\alpha))} \otimes g(\alpha)^* \otimes e_{t(g(\alpha))}g(b)$$

for each $i \in Q_0$ and $\alpha \in Q_1$. Since $\mathrm{Ext}^1_{(\mathbb{K}Q)^e}(\mathbb{K}Q, (\mathbb{K}Q)^e)$ is identified with the cokernel of ∂', the statement above holds.

The claim above implies an isomorphism $\Pi(Q)\#G \simeq \Pi_2(\mathbb{K}Q)\#G$. Then the required isomorphism follows from Theorem 3.13 immediately. $\square$

4. Morita equivalences and bimodules

In this section, we recall known facts on Morita equivalences between two algebras. We prove that any Morita equivalence between two algebras extends to a Morita equivalence between their preprojective algebras; see Proposition 4.7.

Let A and B be two algebras. We fix a $\mathbb{K}$-linear Morita equivalence

$$F\colon A\text{-Mod} \longrightarrow B\text{-Mod}$$

between A and B. There is an invertible B-A-bimodule P such that $F \simeq P \otimes_A -$. Moreover, the bimodule P fits into a set $(P, Q; \phi, \psi)$ of *equivalence data*, where Q is an A-B-bimodule, $\phi\colon P \otimes_A Q \to B$ is an isomorphism of B-B-bimodules, and $\psi\colon Q \otimes_B P \to A$ is an isomorphism of A-A-bimodules. Moreover, these data satisfy the associativity condition, that is,

$$\phi(x \otimes_A y)x' = x\psi(y \otimes_B x') \text{ and } y'\phi(x \otimes_A y) = \psi(y' \otimes_B x)y, \tag{4.1}$$

for any $x, x' \in P$ and $y, y' \in Q$. We refer to [1, Chapter II, § 3] for details.

The equivalence data induce another Morita equivalence

$$F^e = P \otimes_A - \otimes_A Q\colon A^e\text{-Mod} \longrightarrow B^e\text{-Mod},$$

whose quasi-inverse might be chosen as $Q \otimes_B - \otimes_B P$.

Remark 4.1. Denote by $\mathcal{E}^c(A\text{-Mod})$ the category of continuous endofunctors on A-Mod, that is, endofunctors which preserve arbitrary coproducts. By Eilenberg-Watt's theorem, we have an equivalence

$$A^e\text{-Mod} \longrightarrow \mathcal{E}^c(A\text{-Mod}),$$

sending any A-A-bimodule X to the endofunctor $X \otimes_A -$. Then we have the following square, which commutes up to a natural isomorphism.

$$\begin{tikzcd} A^e\text{-Mod} \arrow[r, "F^e"] \arrow[d] & B^e\text{-Mod} \arrow[d] \\ \mathcal{E}^c(A\text{-Mod}) \arrow[r] & \mathcal{E}^c(B\text{-Mod}) \end{tikzcd}$$

Here, the functor at the bottom sends H to FHF^{-1}, where F^{-1} is a quasi-inverse of F. This justifies the notation F^e to some extent.

For each left B-module Y, we denote by add Y the full subcategory of B-Mod formed by direct summands of finite sums of Y.

Lemma 4.2. *Let X be an A-A-bimodule. Then we have* add $F(X) =$ add $F^e(X)$ *in* B-Mod.

Proof. Recall that $F(X) = P \otimes_A X$ and $F^e(X) = (P \otimes_A X) \otimes_A Q$. Then the required equality follows from the fact that the underlying left A-module Q is a finitely generated projective generator. $\square$

Let I be a two-sided ideal of A. The following subspace of B

$$\begin{aligned}\Phi_F(I) &= \{\text{finite sums } \sum_i \phi(x_i \otimes_A y_i) \mid x_i \in PI, y_i \in Q\} \\ &= \{\text{finite sums } \sum_i \phi(x_i \otimes_A y_i) \mid x_i \in P, y_i \in IQ\}\end{aligned}$$

is a two-sided ideal of B.

The following result is implicit in [1, Chapter II, Theorem 3.5(6)].

Proposition 4.3. *The assignment $I \mapsto \Phi_F(I)$ yields an isomorphism*

$$\Phi_F \colon \mathbf{I}(A) \longrightarrow \mathbf{I}(B)$$

between the ideal monoids. Moreover, we have an isomorphism $\Phi_F(I) \simeq F^e(I)$ of B-B-bimodules.

Proof. We identify two-sided ideals of an algebra with its sub bimodules. Consider the following sequence of morphisms between B-B-bimodules.

$$F^e(I) \hookrightarrow F^e(A) = P \otimes_A A \otimes_A Q \simeq P \otimes_A Q \xrightarrow{\phi} B$$

Here, the leftmost morphism is induced by the inclusion $I \hookrightarrow A$. The image of this composite morphism equals $\Phi_F(I)$. This proves the final statement.

Since $F^e(A) \simeq B$, the Morita equivalence F^e induces a bijection between two-sided ideals of A and those of B. This bijection is essentially the same as Φ_F. Using the associativity condition (4.1), one verify that $\Phi_F(II') = \Phi_F(I)\Phi_F(I')$ for any two-sided ideals I and I' of A. □

The following result will be useful to determine the isomorphism Φ_F.

Proposition 4.4. *Assume that I is a two-sided ideal of A and that J is a two-sided ideal of B. Then $\Phi_F(I) = J$ if and only if* add $F(A/I) =$ add (B/J) *in* B-Mod.

Proof. Set $\Phi_F(I) = J'$. By Proposition 4.3, we identify $F^e(A)$ with B, and $F^e(I)$ with J'. Applying F^e to the following canonical exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0,$$

we infer that $F^e(A/I) \simeq B/J'$. By Lemma 4.2, we have

$$\text{add } F(A/I) = \text{add } (B/J')$$

in B-Mod. Then the lemma below implies the required statement. □

The following fact is well known.

Lemma 4.5. *Assume that both J and J' are two-sided ideals of B. Then $J = J'$ if and only if* add $(B/J) =$ add (B/J') *in* B-Mod.

Proof. It suffices to prove the "if" part. The annihilator ideal of the left B-module B/J equals J. Moreover, for any $E \in$ add (B/J), J is contained in the annihilator ideal of E. Then the required equality follows immediately. □

The following result is implicit in [24, Proposition 3.10(e)].

Lemma 4.6. *Let $F\colon A\text{-Mod} \to B\text{-Mod}$ be the given Morita equivalence. Then for each $i \geq 0$, the following diagram*

$$\begin{tikzcd}
A\text{-Mod} \arrow[r, "F"] \arrow[d, "\mathrm{Ext}^i_{A^e}(A,A^e)\otimes_A -"'] & B\text{-Mod} \arrow[d, "\mathrm{Ext}^i_{B^e}(B,B^e)\otimes_B -"] \\
A\text{-Mod} \arrow[r, "F"] & B\text{-Mod}
\end{tikzcd}$$

commutes up to a natural isomorphism.

Proof. Since $F \simeq P \otimes_A -$, it suffices to show an isomorphism of B-A-bimodules

$$P \otimes_A \mathrm{Ext}^i_{A^e}(A, A^e) \simeq \mathrm{Ext}^i_{B^e}(B, B^e) \otimes_B P.$$

For this end, we observe that the Morita equivalence F^e sends A to B, and A^e to $P \otimes Q$. Therefore, we have the following isomorphism of A-A-bimodules.

$$\mathrm{Ext}^i_{A^e}(A, A^e) \simeq \mathrm{Ext}^i_{B^e}(B, P \otimes Q) \tag{4.2}$$

Since $P \otimes Q$ is a finitely generated projective B^e-module, we have

$$\mathrm{Ext}^i_{B^e}(B, P \otimes Q) \simeq \mathrm{Ext}^i_{B^e}(B, B^e) \otimes_{B^e} (P \otimes Q) = Q \otimes_B \mathrm{Ext}^i_{B^e}(B, B^e) \otimes_B P. \tag{4.3}$$

Consequently, we have the following isomorphisms.

$$\begin{aligned}
P \otimes_A \mathrm{Ext}^i_{A^e}(A, A^e) &\simeq P \otimes_A \mathrm{Ext}^i_{B^e}(B, P \otimes Q) \\
&\simeq P \otimes_A (Q \otimes_B \mathrm{Ext}^i_{B^e}(B, B^e) \otimes_B P) \\
&\simeq \mathrm{Ext}^i_{B^e}(B, B^e) \otimes_B P
\end{aligned}$$

Here, the first isomorphism uses (4.2), the second one uses (4.3) and the last one uses the isomorphism $\phi\colon P \otimes_A Q \to B$. This completes the proof. □

The following result shows that any Morita equivalence between A and B extends to a Morita equivalence between their n-preprojective algebras $\Pi_n(A)$ and $\Pi_n(B)$ for $n \geq 2$. We emphasize that the result is essentially due to [24, Proposition 4.2].

Proposition 4.7. *Let $F\colon A\text{-Mod} \to B\text{-Mod}$ be a Morita equivalence and $n \geq 2$. Then there is a Morita equivalence $\widetilde{F}$ making the following diagram commute up to a natural isomorphism.*

$$\begin{tikzcd}
A\text{-Mod} \arrow[r, "F"] \arrow[d] & B\text{-Mod} \arrow[d] \\
\Pi_n(A)\text{-Mod} \arrow[r, "\widetilde{F}"] & \Pi_n(B)\text{-Mod}
\end{tikzcd}$$

Here, we identify any A-module X with the corresponding $\Pi_n(A)$-module X on which $\mathrm{Ext}^{n-1}_{A^e}(A, A^e)$ acts trivially. This yields the unnamed vertical arrow on the left side. Similar remarks work for the right side.

Proof. By combining Lemmas 4.6 and 2.1, the given Morita equivalence F induces an equivalence

$$\widetilde{F}\colon \mathrm{rep}(\mathrm{Ext}^{n-1}_{A^e}(A, A^e)\otimes_A -) \longrightarrow \mathrm{rep}(\mathrm{Ext}^{n-1}_{B^e}(B, B^e)\otimes_B -).$$

By the isomorphism (2.1), we identify $\Pi_n(A)$-Mod with $\mathrm{rep}(\mathrm{Ext}^{n-1}_{A^e}(A, A^e)\otimes_A -)$, and $\Pi_n(B)$-Mod with $\mathrm{rep}(\mathrm{Ext}^{n-1}_{B^e}(B, B^e)\otimes_B -)$. This completes the proof. □

5. Weyl groups and monoids

In this section, we study Weyl groups and monoids associated to quivers and Cartan matrices. In the folding process, we prove that the Weyl monoid of a Cartan matrix is isomorphic to the invariant monoid of a quiver; see Proposition 5.4. We refer to [23, Section 3] for Weyl groups and [37] for Weyl monoids.

Let Q be a finite acyclic quiver. Denote by $W(Q)$ its *Weyl group*. It is generated by simple reflections $\{s_i \mid i \in Q_0\}$, which are subject to the following relations: $s_i^2 = 1$; $(s_is_j)^2 = 1$ if there is no arrow between i and j; $(s_is_j)^3 = 1$ if there is precisely one arrow between i and j.

Denote by $WM(Q)$ the *Weyl monoid*, which is a monoid generated by $\{h_i \mid i \in Q_0\}$ subject to the following relations $h_i^2 = h_i$; $h_ih_j = h_jh_i$ if there is no arrow between i and j; $h_ih_jh_i = h_jh_ih_j$ if there is precisely one arrow between i and j.

By [37, Theorem 1], there is a bijection

$$\rho_Q\colon W(Q) \longrightarrow WM(Q)$$

given as follows: for any reduced expression $\omega = s_{i_1}s_{i_2}\cdots s_{i_n}$ in $W(Q)$ with $i_1, i_2, \cdots, i_n \in Q_0$, we have $\rho_Q(\omega) = h_{i_1}h_{i_2}\cdots h_{i_n}$.

The above consideration works well for Cartan matrices. Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a *symmetrizable generalized Cartan matrix*. Therefore, the following conditions are fulfilled.

(C1) $c_{ii} = 2$ for each i;
(C2) $c_{ij} \leq 0$ for all $i \neq j$, and $c_{ij} < 0$ if and only if $c_{ji} < 0$;
(C3) There exists a diagonal matrix $D = \mathrm{diag}(c_1, c_2, \cdots, c_n)$ with each c_i a positive integer such that DC is symmetric.

Such a matrix D is called a *symmetrizer* of C.

Denote by $W(C)$ the associated Weyl group, which is generated by simple reflections $\{r_1, r_2, \cdots, r_n\}$ subject to the following relations: $r_i^2 = 1$; $(r_ir_j)^2 = 1$ if $c_{ij} = 0$; $(r_ir_j)^3 = 1$ if $c_{ij}c_{ji} = 1$; $(r_ir_j)^4 = 1$ if $c_{ij}c_{ji} = 2$; $(r_ir_j)^6 = 1$ if $c_{ij}c_{ji} = 3$.

Similarly, the Weyl monoid $WM(C)$ is a monoid generated by $\{f_1, f_2, \cdots, f_n\}$ subject to the relations: $f_i^2 = f_i$; $f_if_j = f_jf_i$ if $c_{ij} = 0$; $f_if_jf_i = f_jf_if_j$ if $c_{ij}c_{ji} = 1$; $(f_if_j)^2 = (f_jf_i)^2$ if $c_{ij}c_{ji} = 2$; $(f_if_j)^3 = (f_jf_i)^3$ if $c_{ij}c_{ji} = 3$.

By [37, Theorem 1], there is a bijection

$$\rho_C\colon W(C) \longrightarrow WM(C)$$

given as follows: any reduced expression $\omega = r_{i_1}r_{i_2}\cdots r_{i_m}$ in $W(C)$, we have $\rho_C(\omega) = f_{i_1}f_{i_2}\cdots f_{i_m}$.

Remark 5.1. For any Coxeter group W with the Coxeter matrix M, one defines the corresponding *Coxeter monoid* WM in [37]. There is a similar bijection $\rho\colon W \to WM$; see [37, Theorem 1]. When M arises from a Cartan matrix C, the bijection ρ coincides with ρ_C. We mention that the monoid algebra of WM is isomorphic to the 0-Hecke algebra [30]; compare [4, Theorem 4.4].

Each finite acyclic quiver Q gives rise to a symmetric Cartan matrix C in the following way: the rows and columns of C are indexed by Q_0; for $i \neq j$, we have

$$c_{ij} = -|\{\text{arrows between } i \text{ and } j\}|.$$

The two Weyl groups $W(Q)$ and $W(C)$ coincide. Similarly, the two Weyl monoids $WM(Q)$ and $WM(C)$ coincide. Moreover, $\rho_Q = \rho_C$.

The following example, known as the folding process, is our main concern.

Example 5.2. Let G be a finite group acting on a finite acyclic quiver Q. Denote by Q_0/G the set of G-orbits, whose elements will be denoted by bold letters. Since Q is acyclic, there is no arrow between any two vertices in the same orbit.

We associate a Cartan matrix C to this action as follows. The rows and columns of C are indexed by Q_0/G. The entries are given by

$$c_{\mathbf{i},\mathbf{j}} = \frac{-N_{\mathbf{i},\mathbf{j}}}{|\mathbf{j}|},$$

where $N_{\mathbf{i},\mathbf{j}}$ counts all arrows in Q between the orbits $\mathbf{i}$ and $\mathbf{j}$. The symmetrizer $D = \mathrm{diag}(c_{\mathbf{i}})_{\mathbf{i}\in Q_0/G}$ of C is given such that

$$c_{\mathbf{i}} = \frac{|G|}{|\mathbf{i}|}.$$

The group G acts on $W(Q)$ by group automorphisms. By [34, §11] and [17, Proposition 3.4], there is a well-known isomorphism of groups

$$\psi\colon W(C) \longrightarrow W(Q)^G,$$

which sends $r_{\mathbf{i}}$ to $\prod_{i\in\mathbf{i}} s_i$. Here, we denote by $W(Q)^G$ the fixed subgroup. We refer to [18, Lemma 3(3)] and [11, Theorem 1] for more recent treatments.

The following useful fact can be found in [11, Lemma 2]; compare [17, Proposition 3.4].

Lemma 5.3. *Keep the notation above. Let $\omega = r_{\mathbf{i}_1} r_{\mathbf{i}_2} \cdots r_{\mathbf{i}_m}$ be a reduced expression in $W(C)$. Then the expression $\psi(\omega) = (\prod_{i_1\in\mathbf{i}_1} s_{i_1})(\prod_{i_2\in\mathbf{i}_2} s_{i_2}) \cdots (\prod_{i_m\in\mathbf{i}_m} s_{i_m})$ is also reduced in $W(Q)$.* $\square$

The G-action on Q induces a G-action on $WM(Q)$ by monoid automorphisms. Denote by $MW(Q)^G$ the submonoid formed by G-invariant elements. We observe that the bijection ρ_Q is compatible with the G-actions. In other words, we have

$$\rho_Q(g(\omega)) = g(\rho_Q(\omega))$$

for any $g \in G$ and $\omega \in W(Q)$. Here, we use implicitly the fact that the G-action preserves reduced expressions in $W(Q)$. Consequently, we have the restricted bijection

$$\rho_Q^G\colon W(Q)^G \longrightarrow WM(Q)^G.$$

We obtain an analogue of the isomorphism ψ for Weyl monoids in the following result, which seems to be expected by experts.

Proposition 5.4. *Keep the assumptions in Example 5.2. Then there is a unique isomorphism $\psi' \colon WM(C) \to WM(Q)^G$ of monoids making the following diagram commute.*

$$\begin{tikzcd} W(C) \arrow[r, "\psi"] \arrow[d, "\rho_C"'] & W(Q)^G \arrow[d, "\rho_Q^G"] \\ WM(C) \arrow[r, "\psi'"] & WM(Q)^G \end{tikzcd}$$

Proof. The commutativity implies that $\psi' = \rho_Q^G \circ \psi \circ \rho_C^{-1}$, which shows the uniqueness of such a map. It remains to show that this map ψ' is indeed a morphism of monoids.

We claim that the elements in $\{\psi'(f_{\mathbf{i}}) = \prod_{i\in\mathbf{i}} h_i \mid \mathbf{i} \in Q_0/G\}$ satisfy the defining relations of $WM(C)$. Since there are no arrows between vertices in the G-orbit $\mathbf{i}$, we infer from $h_i^2 = h_i$ that $\psi'(f_{\mathbf{i}})^2 = \psi'(f_{\mathbf{i}})$. Similarly, if $c_{\mathbf{i},\mathbf{j}} = 0$, there are no arrows between $\mathbf{i}$ and $\mathbf{j}$. Then we deduce that $\psi'(f_{\mathbf{i}})\psi'(f_{\mathbf{j}}) = \psi'(f_{\mathbf{j}})\psi'(f_{\mathbf{i}})$.

Assume that $c_{\mathbf{i},\mathbf{j}}c_{\mathbf{j},\mathbf{i}} = 1$. Since $r_{\mathbf{i}}r_{\mathbf{j}}r_{\mathbf{i}}$ is a reduced expression in $W(C)$, Lemma 5.3 implies that $(\prod_{i\in\mathbf{i}} s_i)(\prod_{j\in\mathbf{j}} s_j)(\prod_{i\in\mathbf{i}} s_i)$ is a reduced expression in $W(Q)$. Then by the construction of ρ_Q, we have the second equality in the following identity.

$$\begin{aligned} \psi'(f_{\mathbf{i}})\psi'(f_{\mathbf{j}})\psi'(f_{\mathbf{i}}) &= (\prod_{i\in\mathbf{i}} h_i)(\prod_{j\in\mathbf{j}} h_j)(\prod_{i\in\mathbf{i}} h_i) \\ &= \rho_Q((\prod_{i\in\mathbf{i}} s_i)(\prod_{j\in\mathbf{j}} s_j)(\prod_{i\in\mathbf{i}} s_i)) \\ &= \rho_Q\psi(r_{\mathbf{i}}r_{\mathbf{j}}r_{\mathbf{i}}). \end{aligned}$$

Similarly, we have

$$\psi'(f_{\mathbf{j}})\psi'(f_{\mathbf{i}})\psi'(f_{\mathbf{j}}) = \rho_Q\psi(r_{\mathbf{j}}r_{\mathbf{i}}r_{\mathbf{j}}).$$

Since $r_{\mathbf{i}}r_{\mathbf{j}}r_{\mathbf{i}} = r_{\mathbf{j}}r_{\mathbf{i}}r_{\mathbf{j}}$ holds in $W(C)$, we infer the desired identity

$$\psi'(f_{\mathbf{i}})\psi'(f_{\mathbf{j}})\psi'(f_{\mathbf{i}}) = \psi'(f_{\mathbf{j}})\psi'(f_{\mathbf{i}})\psi'(f_{\mathbf{j}})$$

in $WM(Q)$. A similar proof works for the cases $c_{\mathbf{i},\mathbf{j}}c_{\mathbf{j},\mathbf{i}} = 2$ and 3. This completes the proof of the claim.

By the claim, we have a well-defined morphism

$$\psi'' \colon WM(C) \longrightarrow WM(Q)^G$$

between monoids such that $\psi''(f_{\mathbf{i}}) = \psi'(f_{\mathbf{i}})$ for each $\mathbf{i} \in Q_0/G$. By Lemma 5.3, it is direct to check that $\psi'' \circ \rho_C = \rho_Q^G \circ \psi$. This will force that $\psi'' = \psi'$, which completes the proof. $\square$

Let Q be a finite acyclic quiver. For each $i \in Q_0$, write $I_i = \Pi(Q)(1-e_i)\Pi(Q)$; it is a two-sided ideal of $\Pi(Q)$. Denote by $\langle I_i \mid i \in Q_0\rangle$ the sub monoid of $\mathbf{I}(\Pi(Q))$ generated by these ideals I_i.

The following isomorphism is essentially due to [3, Theorem III.1.9] and [29, Theorem 2.14].

Theorem 5.5. *Let Q be a finite acyclic quiver. There is an isomorphism*

$$\Theta'_Q \colon WM(Q) \longrightarrow \langle I_i \mid i \in Q_0\rangle$$

between monoids, which sends h_i to I_i for each $i \in Q_0$. Consequently, we have a bijection

$$\Theta_Q = \Theta'_Q \circ \rho_Q \colon W(Q) \longrightarrow \langle I_i \mid i \in Q_0 \rangle.$$

Proof. The well-definedness of the morphism Θ'_Q is due to [3, Proposition III.1.8]. It is clearly surjective. For the injectivity, we refer to the proofs of [3, Theorem III.1.9] and [29, Theorem 2.14]. $\square$

Let G be a finite group which acts on Q. For each $\mathbf{i} \in Q_0/G$, we write

$$I_{\mathbf{i}} = \prod_{i \in \mathbf{i}} I_i.$$

This is well defined, since I_i and I_j commute for $i, j \in \mathbf{i}$; see [3, Proposition III.1.8]. We observe that $I_{\mathbf{i}}$ is a G-invariant ideal of $\Pi(Q)$. Then $I_{\mathbf{i}}\#G$ is a two-sided ideal of $\Pi(Q)\#G$; see Proposition 3.2. Denote by $\langle I_{\mathbf{i}}\#G \mid \mathbf{i} \in Q_0/G \rangle$ the sub monoid of $\mathbf{I}(\Pi(Q)\#G)$ generated by $I_{\mathbf{i}}\#G$.

Proposition 5.6. *Let G be a finite group which acts on a finite acyclic quiver Q. Then there is an isomorphism of monoids $WM(Q)^G \to \langle I_{\mathbf{i}}\#G \mid \mathbf{i} \in Q_0/G \rangle$.*

Proof. The bijection ρ_Q restricts to the bijection $\rho_Q^G \colon W(Q)^G \to WM(Q)^G$. Since $W(Q)^G$ is generated by $\{\prod_{i\in \mathbf{i}} s_i \mid \mathbf{i} \in Q_0/G\}$, it follows that the monoid $WM(Q)^G$ is generated by $\{\prod_{i\in\mathbf{i}} h_i \mid \mathbf{i} \in Q_0/G\}$. The isomorphism in Theorem 5.5 implies the following observation: $\langle I_i \mid i \in Q_0\rangle^G$ is generated by $\{I_{\mathbf{i}} \mid \mathbf{i} \in Q_0/G\}$.

By the isomorphism in Proposition 3.2(3), we have an injective morphism of monoids

$$\langle I_i \mid i \in Q_0 \rangle^G \hookrightarrow \mathbf{I}(\Pi(Q)\#G),\ I \mapsto I\#G.$$

The observation above implies that the image of this morphism is precisely $\langle I_{\mathbf{i}}\#G \mid \mathbf{i} \in Q_0/G \rangle$. Combining this with the isomorphism Θ'_Q in Theorem 5.5, we complete the proof. $\square$

Remark 5.7. Recall from Example 5.2 that the Cartan matrix C is associated to the G-action. Recall that $\Theta_Q = \Theta'_Q \circ \rho_Q$. By the commutative square in Proposition 5.4, we have the following commutative square consisting of bijections.

$$\begin{tikzcd}
W(C) \arrow[r, "\psi"] \arrow[d] & W(Q)^G \arrow[d, "\Theta_Q^G"] \\
\langle I_{\mathbf{i}}\#G \mid \mathbf{i} \in Q_0/G \rangle & \langle I_i \mid i \in Q_0 \rangle^G \arrow[l, "-\#G"']
\end{tikzcd}$$

Here, the unnamed vertical arrow is given by $(-\#G) \circ ({\Theta'}_Q^G) \circ \psi' \circ \rho_C$.

6. Algebras associated to Cartan triples

In this section, we recall the generalized preprojective algebras from [14] and prove Proposition 6.5, which is Proposition A in Introduction.

Recall from [8] that a *Cartan triple* (C, D, Ω) consists of a Cartan matrix $C = (c_{ij}) \in M_n(\mathbb{Z})$, its symmetrizer $D = \mathrm{diag}(c_1, c_2, \cdots, c_n)$ and an acyclic orientation Ω. Here, we recall that an *acyclic orientation* Ω on C is a subset of $\{1, 2, \cdots, n\} \times \{1, 2, \cdots, n\}$ subject to the following conditions.

(O1) $\{(i, j), (j, i)\} \cap \Omega \neq \emptyset$ if and only if $c_{ij} < 0$;
(O2) for each sequence $(i_1, i_2, \cdots, i_t, i_{t+1})$ with $t \geq 1$ such that $(i_s, i_{s+1}) \in \Omega$ for all $1 \leq s \leq t$, we necessarily have $i_1 \neq i_{t+1}$.

Let $Q = Q(C, \Omega)$ be the finite quiver with the set of vertices $Q_0 = \{1, 2, \cdots, n\}$ and with the set of arrows

$$Q_1 = \{\alpha_{ij}^{(g)} : j \to i \mid (i,j) \in \Omega, 1 \le g \le \gcd(c_{ij}, c_{ji})\} \sqcup \{\varepsilon_i : i \to i \mid 1 \le i \le n\}.$$

Here, for any nonzero integers c, c', their greatest common divisor $\gcd(c, c')$ is defined to be positive.

Following [14, Subsection 1.4] and [13], we associate a finite dimensional algebras $H(C, D, \Omega)$ to any Cartan triple (C, D, Ω) as follows

$$H(C, D, \Omega) = \mathbb{K}Q/I,$$

where $\mathbb{K}Q$ is the path algebra of $Q = Q(C, \Omega)$, and I is the two-sided ideal of $\mathbb{K}Q$ generated by the following set

$$\{\varepsilon_k^{c_k},\ \varepsilon_i^{\frac{c_i}{\gcd(c_i,c_j)}} \alpha_{ij}^{(g)} - \alpha_{ij}^{(g)} \varepsilon_j^{\frac{c_j}{\gcd(c_i,c_j)}} \mid k \in Q_0,\ (i,j) \in \Omega,\ 1 \le g \le \gcd(c_{ij}, c_{ji})\}.$$

Here, $\varepsilon_k^{c_k}$ is called the *nilpotency relation*, and $\varepsilon_i^{\frac{c_i}{\gcd(c_i,c_j)}} \alpha_{ij}^{(g)} - \alpha_{ij}^{(g)} \varepsilon_j^{\frac{c_j}{\gcd(c_i,c_j)}}$ the *commutativity relation*.

Let (C, D, Ω) be a Cartan triple. The opposite orientation of Ω is $\Omega^{\mathrm{op}} = \{(j,i) \mid (i,j) \in \Omega\}$. Set $\overline{\Omega} = \Omega \cup \Omega^{\mathrm{op}}$. For each $(i,j) \in \overline{\Omega}$, we define

$$\operatorname{sgn}(i,j) = \begin{cases} 1, & \text{if } (i,j) \in \Omega; \\ -1, & \text{if } (i,j) \in \Omega^{\mathrm{op}}. \end{cases}$$

Denote by $\widetilde{Q} = \widetilde{Q}(C, \Omega)$ the quiver obtained from $Q = Q(C, \Omega)$ by adding a new arrow $\alpha_{ji}^{(g)} : i \to j$ for each arrow $\alpha_{ij}^{(g)} : j \to i$. We mention that $\widetilde{Q}$ is not the double quiver of Q, since we do not double the loops.

The following definition is taken from [14, Subsection 1.4]; compare [5, Subsection 5.1] and [38, Section 3]. We refer to [14, Subsection 1.7], where the relationships among these constructions presented in the aforementioned literature are explicitly discussed.

Definition 6.1. Let (C, D, Ω) be a Cartan triple and set $\widetilde{Q} = \widetilde{Q}(C, \Omega)$. The *generalized preprojective algebra* is defined to be

$$\Pi(C, D, \Omega) = \mathbb{K}\widetilde{Q}/\widetilde{I},$$

where the two-sided ideal $\widetilde{I}$ of $\mathbb{K}\widetilde{Q}$ is determined by the following relations.

(P1) For each vertex $i \in Q_0$, we have the nilpotency relation $\varepsilon_i^{c_i} = 0$.

(P2) For each $(i,j) \in \overline{\Omega}$ and $1 \le g \le \gcd(c_{ij}, c_{ji})$, we have the commutativity relation $\varepsilon_i^{\frac{c_i}{\gcd(c_i,c_j)}} \alpha_{ij}^{(g)} - \alpha_{ij}^{(g)} \varepsilon_j^{\frac{c_j}{\gcd(c_i,c_j)}}$.

(P3) For each vertex $i \in Q_0$, we have the *mesh relation*

$$\sum_{\{j \in Q_0 \mid (i,j) \in \overline{\Omega}\}} \sum_{g=1}^{\gcd(c_{ij},c_{ji})} \sum_{l=0}^{\frac{c_i}{\gcd(c_i,c_j)}-1} \operatorname{sgn}(i,j)\ \varepsilon_i^l \alpha_{ij}^{(g)} \alpha_{ji}^{(g)} \varepsilon_i^{\frac{c_i}{\gcd(c_i,c_j)}-1-l} = 0.$$

The following result is essentially due to [14, Theorem 1.6].

Proposition 6.2. *Let (C, D, Ω) be a Cartan triple with $H = H(C, D, \Omega)$. Then we have an isomorphism of algebras*

$$\Pi(C, D, \Omega) \simeq \Pi_2(H).$$

Proof. Since the algebra H is finite dimensional, by Lemma 2.2 we have an isomorphism $\mathrm{Ext}^1_{H^e}(H, H^e) \simeq \mathrm{Ext}^1_H(DH, H)$ of H-H-bimodules. Then the required isomorphism follows from [14, Theorem 1.6]. □

For each vertex $i \in Q_0$, we denote by e_i the corresponding idempotent of $\Pi(C, D, \Omega)$. Denote by L_i the two-sided ideal generated by $1 - e_i$. Denote by $\langle L_i \mid i \in Q_0\rangle$ the sub monoid of $\mathbf{I}(\Pi(C, D, \Omega))$ generated by these L_i's. Consider the Weyl group $W(C)$, whose simple reflections are denoted by r_i for each $i \in Q_0$.

The following result is essentially due to [10, Theorem 4.7].

Theorem 6.3. *Let (C, D, Ω) be a Cartan triple. There is an isomorphism*

$$\Theta'_C : WM(C) \longrightarrow \langle L_i \mid i \in Q_0\rangle$$

of monoids, which sends f_i to L_i. Consequently, we have a bijection

$$\Theta_C = \Theta'_C \circ \rho_C : W(C) \longrightarrow \langle L_i \mid i \in Q_0\rangle.$$

Proof. The well-definedness of the morphism Θ'_C is due to [10, Proposition 4.6], which is clearly surjective. For the injectivity, we refer to the proof of [10, Theorem 4.7]. □

In the example below, we see that the isomorphism in Theorem 6.3 extends the one in Theorem 5.5.

Example 6.4. Let Q be a finite acyclic quiver. Then it corresponds to a symmetric Cartan matrix C. Denote by I_{Q_0} the identity matrix with rows and columns indexed by Q_0. The set Q_1 of arrows yields an acyclic orientation Ω on Q_0 in the obvious manner: there is an arrow from i to j in Q if and only if (j, i) belongs to Ω.

We have algebra isomorphisms $\mathbb{K}Q \simeq H(C, I_{Q_0}, \Omega)$ and $\Pi(Q) \simeq \Pi(C, I_{Q_0}, \Omega)$. Since $WM(Q) = WM(C)$, therefore in this situation, the isomorphisms in Theorem 6.3 coincide with the one in Theorem 5.5.

Let G be a finite group, which acts on a finite acyclic quiver Q. Following the folding process in Example 5.2, we denote by (C, D, Ω) the associated Cartan triple. Here, the rows and columns of C and D are indexed by the orbit set Q_0/G. The acyclic orientation Ω is defined to such that $(\mathbf{j}, \mathbf{i})$ belongs to Ω if and only if there is some arrow from the orbit $\mathbf{i}$ to the orbit $\mathbf{j}$ in Q. We mention that each Cartan triple arises in this way; compare [8, Remark 6.9] and [28, Section 14.1]

Write $\Pi = \Pi(C, D, \Omega)$. Each vertex $\mathbf{i}$ in $Q(C, \Omega)_0 = Q_0/G$ gives rise to an idempotent $e_{\mathbf{i}}$ of Π. Recall that $L_{\mathbf{i}} = \Pi(1 - e_{\mathbf{i}})\Pi$.

The following square compares the bijections in Theorems 5.5 and 6.3.

Proposition 6.5. *Keep the assumptions above. Then there is a unique isomorphism Ψ between monoids making the following diagram commute.*

$$\begin{tikzcd}
W(C) \arrow[r, "\psi"] \arrow[d, "\Theta_C"'] & W(Q)^G \arrow[d, "\Theta_Q^G"] \\
\langle L_{\mathbf{i}} \mid \mathbf{i} \in Q_0/G\rangle \arrow[r, "\Psi"] & \langle I_i \mid i \in Q_0\rangle^G
\end{tikzcd}$$

We mention that $\Psi(L_{\mathbf{i}}) = I_{\mathbf{i}}$ for each $\mathbf{i} \in Q_0/G$.

Proof. The uniqueness of Ψ is clear, since the other three maps in the square are all bijections. Recall that $\Theta_C = \Theta'_C \circ \rho_C$ and $\Theta_Q = \Theta'_Q \circ \rho_Q$. In view of the commutative square in Proposition 5.4, it suffices to take

$$\Psi = \Theta'^{G}_Q \circ \psi' \circ (\Theta'_C)^{-1}.$$

As a composition of three isomorphisms between monoids, Ψ is an isomorphism of monoids. □

7. The main result

In this section, we establish a Morita equivalence involving preprojective algebras in the folding process. We assume that $\mathrm{char}(\mathbb{K}) = p > 0$ and that $G = \langle \sigma \mid \sigma^{p^a} = 1\rangle$ is a cyclic p-group for some $a \geq 1$.

Let Q be a finite acyclic quiver. For each vertex $i \in Q_0$, we denote by S_i the 1-dimensional simple $\mathbb{K}Q$-module concentrated in the vertex i. It is naturally a simple $\Pi(Q)$-module. Recall that $I_i = \Pi(Q)(1 - e_i)\Pi(Q)$. We observe that $I_i = \sum\limits_{j \neq i} \Pi(Q)e_j\Pi(Q)$. Then we have an exact sequence

$$0 \longrightarrow I_i \longrightarrow \Pi(Q) \longrightarrow S_i \longrightarrow 0$$

of left $\Pi(Q)$-modules.

Fix a G-action on Q. We assume that the action satisfies the following condition.

$(*)$ For each arrow α in Q, we have $G_\alpha = G_{s(\alpha)} \cap G_{t(\alpha)}$. Here, G_α, $G_{s(\alpha)}$ and $G_{t(\alpha)}$ denote their stabilizers.

We denote by (C, D, Ω) the associated Cartan triple. Write $H = H(C, D, \Omega)$ and $\Pi = \Pi(C, D, \Omega)$. Each vertex $\mathbf{i}$ in $Q(C, \Omega)_0 = Q_0/G$ gives rise to an idempotent $e_{\mathbf{i}}$ of H, which is also an idempotent of Π. Following [14, Subsection 3.2], we denote by $E_{\mathbf{i}} = H/H(1 - e_{\mathbf{i}})H$ the *generalized simple module* over H concentrated in $\mathbf{i}$. We mention that $E_{\mathbf{i}}$ is naturally isomorphic to $\mathbb{K}[\varepsilon]/(\varepsilon^{c_{\mathbf{i}}})$. We view $E_{\mathbf{i}}$ as a module over Π. Recall that $L_{\mathbf{i}} = \Pi(1 - e_{\mathbf{i}})\Pi$. Then we have an exact sequence

$$0 \longrightarrow L_{\mathbf{i}} \longrightarrow \Pi \longrightarrow E_{\mathbf{i}} \longrightarrow 0 \tag{7.1}$$

of left Π-modules.

The following Morita equivalence is due to [8], which is based on [7].

Proposition 7.1. *Keep the assumptions above. Then there is a Morita equivalence*

$$U\colon \mathbb{K}Q\#G\text{-Mod} \longrightarrow H(C, D, \Omega)\text{-Mod}$$

such that $U((\mathbb{K}Q\#G) \otimes_{\mathbb{K}Q} S_i) \simeq E_{\mathbf{i}}$ *for each* $\mathbf{i} \in Q_0/G$ *and* $i \in \mathbf{i}$.

Proof. This is due to [8, Theorem 7.8]. We observe that the folding projection $\mathbf{f}$ therein sends simple roots to simple roots. It follows that the Morita equivalence U sends $(\mathbb{K}Q\#G) \otimes_{\mathbb{K}Q} S_i$ to $E_{\mathbf{i}}$. □

Recall the bijection $\Theta_Q\colon W(Q) \to \langle I_i \mid i \in Q_0\rangle$ in Section 5 and the bijection $\Theta_C\colon W(C) \to \langle L_{\mathbf{i}} \mid \mathbf{i} \in Q_0/G\rangle$ in Section 6. The bijection Θ_Q restricts to a bijection

$$\Theta_Q^G\colon W(Q)^G \longrightarrow \langle I_i \mid i \in Q_0\rangle^G$$

between the subsets formed by G-invariant elements. Recall from Example 5.2 the isomorphism $\psi\colon W(C) \to W(Q)^G$, which sends the simple reflections $r_{\mathbf{i}}$ to $\prod_{j\in\mathbf{i}} s_j$.

The main result of this work is as follows.

Theorem 7.2. *Assume that* $\mathrm{char}(\mathbb{K}) = p > 0$ *and that* G *is a cyclic* p*-group which acts on a finite acyclic quiver* Q *satisfying* $(*)$. *Keep the notation above. Then there is a Morita equivalence*

$$F \colon \Pi(Q)\#G\text{-Mod} \longrightarrow \Pi(C, D, \Omega)\text{-Mod}$$

such that the following diagram commutes.

$$\begin{tikzcd}
W(C) \arrow[r, "\psi"] \arrow[d, "\Theta_C"'] & W(Q)^G \arrow[d, "\Theta_Q^G"] \\
\langle L_{\mathbf{i}} \mid \mathbf{i} \in Q_0/G \rangle & \langle I_i \mid i \in Q_0 \rangle^G \arrow[l, "\Phi_F \circ (-\#G)"']
\end{tikzcd}$$

For the bottom map $\Phi_F \circ (-\#G)$, we observe that each element J in $\langle I_i \mid i \in Q_0 \rangle^G$ is G-invariant, and that $J\#G$ is a two-sided ideal of $\Pi(Q)\#G$. The isomorphism $\Phi_F \colon \mathbf{I}(\Pi(Q)\#G) \to \mathbf{I}(\Pi(C, D, \Omega))$ is given in Proposition 4.3.

Proof. Write $H = H(C, D, \Omega)$. We divide the proof into four steps.

Step 1. By Lemma 2.4, we identify $\Pi(Q)$ with the 2-preprojective algebra $\Pi_2(\mathbb{K}Q)$. Therefore, by Theorem 3.13 the skew group algebra $\Pi(Q)\#G$ is isomorphic to $\Pi_2(\mathbb{K}Q\#G)$. Applying Proposition 4.7 to the Morita equivalence in Proposition 7.1, we obtain a Morita equivalence

$$\widetilde{U} \colon \Pi_2(\mathbb{K}Q\#G)\text{-Mod} \longrightarrow \Pi_2(H)\text{-Mod}.$$

By Proposition 6.2, we identify $\Pi_2(H)$ with $\Pi(C, D, \Omega)$. Combining these isomorphisms and $\widetilde{U}$, we obtain the required Morita equivalence F.

Step 2. We claim that

$$F((\Pi(Q)\#G) \otimes_{\Pi(Q)} S_i) \simeq E_{\mathbf{i}}$$

for each $\mathbf{i} \in Q_0/G$ and $i \in \mathbf{i}$.

For the claim, we recall the isomorphism

$$\theta \colon U((\mathbb{K}Q\#G) \otimes_{\mathbb{K}Q} S_i) \to E_{\mathbf{i}}$$

in H-Mod. We view $\mathbb{K}Q\#G$ as a quotient algebra of $\Pi(Q)\#G$, and H as a quotient algebra of $\Pi(C, D, \Omega)$. Then θ might be viewed as an isomorphism in $\Pi(C, D, \Omega)$-Mod. The natural map

$$(\Pi(Q)\#G) \otimes_{\Pi(Q)} S_i \longrightarrow (\mathbb{K}Q\#G) \otimes_{\mathbb{K}Q} S_i$$

is an isomorphism, since both vector spaces are naturally identified with $\mathbb{K}G \otimes S_i$; compare [8, (7.1)]. Since $\widetilde{U}$ extends U, we deduce the claim.

Step 3. For each $\mathbf{i} \in Q_0/G$, we recall that $I_{\mathbf{i}} = \prod_{j \in \mathbf{i}} I_j$, which is a G-invariant ideal of $\Pi(Q)$. We claim that $\Phi_F(I_{\mathbf{i}}\#G) = L_{\mathbf{i}}$.

Since the ideal $I_{\mathbf{i}}$ is generated by $\{e_l \mid l \notin \mathbf{i}\}$, the quotient algebra $\Pi(Q)/I_{\mathbf{i}}$ is isomorphic to a product of $\mathbb{K}$ indexed by $\mathbf{i}$. Consequently, as a left $\Pi(Q)$-module, we have

$$\Pi(Q)/I_{\mathbf{i}} \simeq \bigoplus_{j \in \mathbf{i}} S_j. \tag{7.2}$$

Here, we mention that $\Pi(Q)/I_{\mathbf{i}}$ is naturally a G-compatible bimodule over $\Pi(Q)$. Therefore, we have the induced bimodule $(\Pi(Q)/I_{\mathbf{i}})\#G$ over $\Pi(Q)\#G$.

By combining Remark 3.6 and (7.2), we have the second isomorphism in the following identity consisting of isomorphisms in $\Pi(C,D,\Omega)$-Mod.

$$\begin{aligned} F((\Pi(Q)\#G)/(I_{\mathbf{i}}\#G)) &\simeq F((\Pi(Q)/I_{\mathbf{i}})\#G) \\ &\simeq \bigoplus_{j\in\mathbf{i}} F((\Pi(Q)\#G)\otimes_{\Pi(Q)} S_j) \\ &\simeq \bigoplus_{j\in\mathbf{i}} E_{\mathbf{i}} \simeq \bigoplus_{j\in\mathbf{i}} \Pi(C,D,\Omega)/L_{\mathbf{i}} \end{aligned}$$

Here, the third isomorphism uses the claim in Step 2, and the last one follows from (7.1). Apply Proposition 4.4, we infer that $\Phi_F(I_{\mathbf{i}}\#G) = L_{\mathbf{i}}$, as required.

Step 4. Recall from the proof of Proposition 5.6 that

$$\langle I_i \mid i \in Q_0\rangle^G = \langle I_{\mathbf{i}} \mid \mathbf{i} \in Q_0/G\rangle.$$

Then by Step 3, the following composition of morphisms between monoids is well-defined.

$$\langle I_i \mid i \in Q_0\rangle^G \xrightarrow{-\#G} \langle I_{\mathbf{i}}\#G \mid \mathbf{i} \in Q_0/G\rangle \xrightarrow{\Phi_F} \langle L_{\mathbf{i}} \mid \mathbf{i} \in Q_0/G\rangle$$

By the isomorphisms in Propositions 3.2(3) and 4.3, this composite morphism is an isomorphism. Since it sends $I_{\mathbf{i}}$ to $L_{\mathbf{i}}$, we infer that it coincides with the inverse of Ψ in Proposition 6.5. Now, the required commutativity follows from Proposition 6.5 immediately. □

Remark 7.3.

(1) It might be of interest to compare the two commutative squares in Remark 5.7 and Theorem 7.2.
(2) When Q is of type A and G is of order 2, such a Morita equivalence F is established in [25, Lemma 5.4] independently by a different method.

We illustrate the main result with an explicit example.

Example 7.4. Let $\mathbb{K}$ be a field of characteristic two, and let Q be the following quiver of type A_3.

$$2 \xrightarrow{\alpha} 1 \xleftarrow{\alpha'} 2'$$

The preprojective algebra $\Pi(Q)$ is given by the following quiver

$$\begin{tikzcd} 2 \arrow[r, shift left, "\alpha"] & 1 \arrow[l, shift left, "\alpha^*"] \arrow[r, shift right, "\alpha'^*"'] & 2' \arrow[l, shift right, "\alpha'"'] \end{tikzcd}$$

subject to the relations $\alpha^*\alpha = 0 = \alpha'^*\alpha'$ and $\alpha\alpha^* + \alpha'\alpha'^* = 0$. Let $G = \{1_G, \sigma\}$ be a cyclic group of order two, and let σ act on Q by interchanging α and α'. This action extends a G-action on $\Pi(Q)$ by $\sigma(\alpha^*) = \alpha'^*$ and $\sigma(\alpha'^*) = \alpha^*$.

The associated Cartan triple (C,D,Ω) is of type B_2 and given as follows:

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix},\ D = \mathrm{diag}(2,1),\ \text{and } \Omega = \{(\mathbf{1},\mathbf{2})\}.$$

The generalized preprojective algebra $\Pi(C, D, \Omega)$ is given by the following quiver

$$\begin{tikzcd} \mathbf{1} \arrow[loop left, "\varepsilon_{\mathbf{1}}"] \arrow[r, bend right, "\alpha_{\mathbf{21}}"'] & \mathbf{2} \arrow[l, bend right, "\alpha_{\mathbf{12}}"'] \arrow[loop right, "\varepsilon_{\mathbf{2}}"] \end{tikzcd}$$

subject to relations $\varepsilon_{\mathbf{1}}^2 = 0 = \varepsilon_{\mathbf{2}}$, $\varepsilon_{\mathbf{1}}\alpha_{\mathbf{12}}\alpha_{\mathbf{21}} + \alpha_{\mathbf{12}}\alpha_{\mathbf{21}}\varepsilon_{\mathbf{1}} = 0$, and $\alpha_{\mathbf{21}}\alpha_{\mathbf{12}} = 0$. In practice, we omit the loop $\varepsilon_{\mathbf{2}}$. Theorem 7.2 yields a Morita equivalence F between $\Pi(Q)\#G$ and $\Pi(C, D, \Omega)$.

Consider the isomorphism $\psi\colon W(C) \to W(Q)^G$, which sends $r_{\mathbf{1}}$ to s_1, and $r_{\mathbf{2}}$ to $s_2s_{2'} = s_{2'}s_2$. We have

$$\Phi_F(I_1\#G) = L_{\mathbf{1}} \text{ and } \Phi_F((I_2I_{2'})\#G) = L_{\mathbf{2}}.$$

The bijection Θ_C sends the longest element $r_{\mathbf{1}}r_{\mathbf{2}}r_{\mathbf{1}}r_{\mathbf{2}}$ to the zero ideal, that is,

$$L_{\mathbf{1}}L_{\mathbf{2}}L_{\mathbf{1}}L_{\mathbf{2}} = 0$$

in $\Pi(C, D, \Omega)$; see [10, Proposition 4.2]. Accordingly, the bijection Θ_Q sends the longest element $\psi(r_{\mathbf{1}}r_{\mathbf{2}}r_{\mathbf{1}}r_{\mathbf{2}}) = s_1s_2s_{2'}s_1s_2s_{2'}$ to the zero ideal, that is,

$$I_1I_2I_{2'}I_1I_2I_{2'} = 0$$

holds in $\Pi(Q)$; compare [29, Theorem 2.30].

CRediT authorship contribution statement

Xiao-Wu Chen: Writing – original draft. **Ren Wang:** Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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