Gorenstein Homological Algebra of Artin Algebras

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Acknowledgements

Abstract

Gorenstein homological algebra is a kind of relative homological algebra which has been developed to a high level since more than four decades.

In this report we review the basic theory of Gorenstein homological algebra of artin algebras. It is hoped that such a theory will help to understand the famous Gorenstein symmetric conjecture of artin algebras.

With only few exceptions all the results in this report are contained in the existing literature. We have tried to keep the exposition as self-contained as possible. This report can be viewed as a preparation for learning the newly developed theory of virtually Gorenstein algebras.

In Chapter 2 we recall the basic notions in Gorenstein homological algebra with particular emphasis on finitely generated Gorenstein-projective modules, Gorenstein algebras and CM-finite algebras.

In Chapter 3 based on a theorem by Beligiannis we study the Gorenstein-projective resolutions and various Gorenstein dimensions; we also discuss briefly Gorenstein derived categories in the sense of Gao and Zhang.

We include three appendixes: Appendix A treats cotorsion pairs; Appendix B sketches a proof of the theorem by Beligiannis; Appendix C provides a list of open problems in Gorenstein homological algebra of artin algebras.

Keywords: Gorenstein-projective modules, Gorenstein dimensions, Gorenstein algebras, CM-finite algebras, virtually Gorenstein algebras.

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Chapter 1

Introduction

The main idea of homological algebra in module categories is to replace a module by its projective (or injective) resolution. In this way one defines derived functors of a given functor, which provide more information on the given functor. Roughly speaking, the information obtained measures how far the given functor is from being exact. Here the notions of projective module and projective resolution play a central role.

Relative homological algebra is initiated by Eilenberg and Moore ([29]). The idea of relative homological algebra is that one might replace projective modules by certain classes of modules and then "pretends" that these modules are projective. Let us call these modules temporarily relatively projective. Suppose that one has a resolution of any module with respect to these relatively projective modules. Then for a given functor one defines certain derived functors via the resolution. These derived functors will be the hero in the theory of relative homological algebra. Evidently the choice of relatively projective modules will be vital in relative homological algebra. As a matter of fact, a different choice of such modules will lead to a different theory.

There is another point of view on relative homological algebra. For the chosen class of relatively projective modules, one can associate a class of short exact sequences on which theses chosen modules behave like projective modules. Such a class of short exact sequences provides a new exact structure on the module category and then one gets a new exact category in the sense of Quillen ([57]). Then it follows that relative homological algebra is just homological algebra of certain exact categories. Again these exact categories depend on the choice of these relatively projective modules.

Gorenstein homological algebra is a kind of relative homological algebra, where the relatively projective modules are chosen to be Gorenstein-projective modules. Finitely generated Gorenstein-projective modules over a noetherian ring are introduced by Auslander and Bridger under the name "modules of G-dimension zero" ([3]). Over a commutative Gorenstein ring these modules are equal to the maximal Cohen-Macaulay

modules. Auslander and Bridger introduce the notion of G-dimension for a finitely generated module and then they generalize the famous Auslander-Buchbaum formula with projective dimension replaced by the G-dimension. The notion of arbitrary Gorenstein-projective modules over an arbitrary ring is invented by Enochs and Jenda ([26]). Later the theory of Gorenstein-projective modules is studied intensively by Enochs's school and others. Gorenstein derived functors are then defined using a Gorenstein-projective resolution of a module ([28, 36]). However it is not a priori that such a resolution exists for an arbitrary module. A recent and remarkable result due to Jørgensen states that for a large class of rings such a resolution always exists ([44]). Inspired by these results Gao and Zhang introduce the notion of Gorenstein derived category ([31]; also see [22]), which is a category with a higher structure in the theory of Gorenstein homological algebra.

Dual to Gorenstein-projective modules one has the notion of Gorenstein-injective module. These modules play the role of injective modules in the classical homological algebra. Using Gorenstein-injective modules one can define the Gorenstein-injective coresolutions of modules and then define the corresponding derived functors for a given functor. However, in general it is not clear how these derived functors are related to the ones given by Gorenstein-projective resolutions.

In this report, we study the Gorenstein homological algebra of artin algebras. The restriction to artin algebras is mainly because of a matter of taste. Due to a work by Auslander and Reiten Gorenstein-projective modules are closely related to the famous Gorenstein symmetric conjecture in the theory of artin algebras ([5, 6]; also see [34]). We hope that an intensive study of Gorenstein homological algebra of artin algebras will help to understand this conjecture.

This report is organized as follows.

In Chapter 2, we provide some preliminaries on Gorenstein homological algebra: we treat the category of finitely generated Gorenstein-projective modules in detail; we also discuss other classes of modules which are important in Gorenstein homological algebra; we briefly discuss Gorenstein algebras, CM-finite algebras, CM-free algebras and virtually Gorenstein algebras.

Chapter 3 treats the main topic in Gorenstein homological algebra: we study the Gorenstein-projective extension groups in detail; we study various Gorenstein dimensions of modules and algebras and study the class of modules having finite Gorenstein dimension; we briefly discuss Gorenstein derived categories.

We include three appendixes: Appendix A treats cotorsion pairs and related notions; Appendix B sketches a proof of an important theorem due to Beligiannis; Appendix C

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collects some open problems, most of which are related to CM-finite algebras.

Let us finally point out that with only few exceptions the results in this report are contained in the existing literature. This report may be viewed as a preparation for reading the beautiful theory of virtually Gorenstein algebras developed in [15, 12, 13, 14].

Chapter 2

Preliminaries

2.1 Gorenstein-Projective Modules

In this section we study for an artin algebra the category of finitely generated Gorenstein-projective modules. Such modules are also known as maximal Cohen-Macaulay modules. These modules play a central role in the theory of Gorenstein homological algebra.

Throughout A will be an artin R-algebra where R is a commutative artinian ring. Denote by A-mod the category of finitely generated left A-modules. In this section all modules are considered to be finitely generated. A left A-module X is often written as ${}_A X$ and a right A-module Y is written as Y_A. Right A-modules are viewed as left A^{op}-modules. Here A^{op} denotes the opposite algebra of A. In what follows, A-modules always mean left A-modules.

For an A-module X, write $DX = \mathrm{Hom}_R(X, E)$ its Matlis dual where E is the minimal injective cogenerator for R. Note that DX has a natural right A-module structure and then it is viewed as an A^{op}-module.

A complex $C^{\bullet} = (C^n, d^n)_{n \in \mathbb{Z}}$ of A-modules consists of a family $\{C^n\}_{n \in \mathbb{Z}}$ of A-modules and differentials $d^n \colon C^n \to C^{n+1}$ satisfying $d^n \circ d^{n-1} = 0$. Sometimes a complex is written as a sequence of A-modules $\cdots \to C^{n-1} \overset{d^{n-1}}{\to} C^n \overset{d^n}{\to} C^{n+1} \to \cdots$. For each $n \in \mathbb{Z}$ denote by $B^n(C^{\bullet}) = \mathrm{Im}\ d^{n-1}$ and $Z^n(C^{\bullet}) = \mathrm{Ker}\ d^n$ the n-th coboundary and cocycle of $C^{\bullet}$, respectively. Note that $B^n(C^{\bullet}) \subseteq Z^n(C^{\bullet})$. Denote by $H^n(C^{\bullet}) = Z^n(C^{\bullet})/B^n(C^{\bullet})$ the n-th cohomology of the complex $C^{\bullet}$.

A complex $C^{\bullet}$ of A-modules is *acyclic* provided that it is exact as a sequence, or equivalently, $H^n(C^{\bullet}) = 0$ for all n. Following [10, p.400] a complex $P^{\bullet}$ of projective A-modules is said to be *totally acyclic* provided it is acyclic and the Hom complex $\mathrm{Hom}_A(P^{\bullet}, A)$ is also acyclic.

Following Enochs and Jenda [26, 28] we have the following definition.

Definition 2.1.1. *An A-module M is said to be (finitely generated) Gorenstein-projective provided that there is a totally acyclic complex $P^\bullet$ of projective modules such that its 0-th cocycle $Z^0(P^\bullet)$ is isomorphic to M.*

We will denote by A-Gproj the full subcategory of A-mod consisting of Gorenstein-projective modules.

In Definition 2.1.1 the complex $P^\bullet$ is said to be a *complete resolution* of M. Observe that each cocycle in a totally acyclic complex is Gorenstein-projective. Note that any projective module P is Gorenstein-projective, since its complete resolution can be taken as $\cdots \to 0 \to P \overset{\mathrm{Id}_P}{\to} P \to 0 \to \cdots$. Therefore, we have A-proj $\subseteq A$-Gproj.

For an A-module M write $M^* = \operatorname{Hom}_A(M, A)$ which has a canonical right A-module structure. This gives rise to a contravariant functor $(-)^* \colon A$-mod $\to A^{\mathrm{op}}$-mod. For a complex $M^\bullet$ we denote by $(M^\bullet)^*$ the Hom complex $\operatorname{Hom}_A(M^\bullet, A)$.

Recall that for an A-module M, one has the following *evaluation morphism*

$$\mathrm{ev}_M \colon M \longrightarrow M^{**} = \operatorname{Hom}_{A^{\mathrm{op}}}(\operatorname{Hom}_A(M, A), A)$$

such that $\mathrm{ev}_M(m)(f) = f(m)$; M is called *reflexive* if ev_M is an isomorphism. For example, (finitely generated) projective modules are reflexive.

Denote by $^\perp A$ the full subcategory of A-mod consisting of modules M with the property $\operatorname{Ext}^i_A(M, A) = 0$ for $i \geq 1$. By a dimension-shift argument, one obtains that $\operatorname{Ext}^i_A(M, L) = 0$ for all $i \geq 1$, $M \in {}^\perp A$ and L having finite projective dimension.

Lemma 2.1.2. *Let $P^\bullet$ be a complex of projective A-modules. Then the following statements are equivalent:*

(1) the complex $P^\bullet$ is totally acyclic;

(2) the complex $P^\bullet$ is acyclic and each cocycle $Z^i(P^\bullet)$ lies in $^\perp A$;

(3) the complex $(P^\bullet)^*$ is totally acyclic.

Proof. Note that for a complex $P^\bullet$ of projective modules the evaluation morphisms induce an isomorphism $P^\bullet \simeq (P^\bullet)^{**}$ of complexes. Then the equivalence $(1) \Leftrightarrow (3)$ follows from the definition. The equivalence $(1) \Leftrightarrow (2)$ follows from the following observation: for an acyclic complex $P^\bullet$ of projective modules and for each $i \in \mathbb{Z}$, the truncated complex $\cdots \to P^{i-2} \to P^{i-1} \to 0$ is a projective resolution of the cocycle $Z^i(P^\bullet)$ and then we have $H^{-i+k+1}((P^\bullet)^*) \simeq \operatorname{Ext}^k_A(Z^i(P^\bullet), A)$ for all $k \geq 1$. $\square$

Corollary 2.1.3. *We have that A-Gproj $\subseteq {}^\perp A$. Then for a Gorenstein-projective module M we have:*

(1) $\mathrm{Ext}_A^i(M, L) = 0 = \mathrm{Tor}_i^A(L', M)$ *for all* $i \geq 1$, $_A L$ *of finite projective dimension and* L'_A *of finite injective dimension;*

(2) M *is either projective or has infinite projective dimension.*

Proof. Note that $D\mathrm{Tor}_i^A(L', M) \simeq \mathrm{Ext}_A^i(M, DL')$. Then the first statement follows from Lemma 2.1.2(2). For the second, we apply the first statement. Then it follows from the fact that a module X of finite projective dimension d satisfies that $\mathrm{Ext}_A^d(X, A) \neq 0$. $\quad\square$

Lemma 2.1.4. *Let* $M \in A\text{-mod}$. *Then the following statements are equivalent:*

(1) M *is Gorenstein-projective;*

(2) *there exists a long exact sequence* $0 \to M \to P^0 \to P^1 \to P^2 \to \cdots$ *with each* P^i *projective and each cocycle in* $^\perp A$;

(3) $M \in {}^\perp A$, $M^* \in {}^\perp(A_A)$ *and* M *is reflexive.*

Proof. The implication "(1) $\Rightarrow$ (2)" follows from Lemma 2.1.2(2). For the other direction, assume (2) and take a projective resolution $\cdots \to P^{-2} \to P^{-1} \to M \to 0$. By splicing we get an acyclic complex $P^\bullet$ such that $Z^0(P^\bullet) \simeq M$. Note that since $^\perp A \subseteq A\text{-mod}$ is closed under taking kernels of epimorphisms. It follows that all the cocycles in $P^\bullet$ lie in $^\perp A$. By Lemma 2.1.2(2) the complex $P^\bullet$ is totally acyclic. We are done.

To see "(1) $\Rightarrow$ (3)", first note that $M \in {}^\perp A$; see Corollary 2.1.3. For others, take a complete resolution $P^\bullet$ of M. Note that $(P^\bullet)^*$ is totally acyclic and that $Z^i((P^\bullet)^*) = (Z^{-i+1}(P^\bullet))^*$. It follows that M^* is Gorenstein-projective and then $M^* \in {}^\perp A_A$. For the same reason we have $Z^i((P^\bullet)^{**}) = (Z^{-i+1}((P^\bullet)^*))^* = Z^i(P^\bullet)^{**}$. Note that evaluation morphisms induce an isomorphism $P^\bullet \simeq (P^\bullet)^{**}$ of complexes. Then it follows that M is reflexive.

For "(3) $\Rightarrow$ (1)", take projective resolutions $\cdots \to P^{-3} \to P^{-2} \to P^{-1} \to M \to 0$ and $\cdots \to Q^{-2} \to Q^{-1} \to Q^0 \to M^* \to 0$. Apply the functor $(-)^*$ to the second resolution. By (3) the resulting complex $0 \to (M^*)^* \to (Q^0)^* \to (Q^{-1})^* \to (Q^{-2})^* \to \cdots$ is acyclic. Note that M is reflexive. Then by splicing the first and the third complexes together we obtain a complete resolution of M. $\quad\square$

Remark 2.1.5. *In view of Lemma 2.1.4(3) Gorenstein-projective modules are the same as* totally reflexive modules *in [10, section 2]. Sometimes they are also called* modules of G-dimension zero *([2, 3]) or* maximal Cohen-Macaulay modules *([12, Definition 3.2]). In view of Lemma 2.1.4(2) we note that the subcategory* $A\text{-Gproj}$ *is a special case of the categories studied in [5, Proposition 5.1].*

The following is an immediate consequence of Lemma 2.1.4(3).

Corollary 2.1.6. *There is a duality* $(-)^*\colon A\text{-Gproj} \xrightarrow{\sim} A^{\mathrm{op}}\text{-Gproj}$ *with its quasi-inverse given by* $(-)^* = \mathrm{Hom}_{A^{\mathrm{op}}}(-, A)$. $\qquad\square$

Recall that a full additive subcategory $\mathcal{X}$ of A-mod is *resolving* provided that it contains all projective modules and is closed under extensions, taking kernels of epimorphisms and direct summands ([3]). For example, A-proj $\subseteq$ A-mod is resolving. We will see shortly that A-Gproj $\subseteq$ A-mod is resolving.

The following result collects some important properties of Gorenstein-projective modules. (1)-(3) are due to [5, Proposition 5.1] (compare [10, Lemma 2.3] and [35, Theorem 2.5]) and (4) is [35, Corollary 2.11].

Proposition 2.1.7. *Let* $\xi\colon 0 \to N \xrightarrow{f} M \xrightarrow{g} L \to 0$ *be a short exact sequence of A-modules. Then we have the following statements:*

(1) if N, L are Gorenstein-projective, then so is M;

(2) if ξ splits and M is Gorenstein-projective, then so are N, L;

(3) if M, L are Gorenstein-projective, then so is N;

(4) if $\mathrm{Ext}_A^1(L, A) = 0$ and N, M are Gorenstein-projective, then so is L.

Proof. (1). Since N and L are Gorenstein-projective, we may take monomorphisms $N \xrightarrow{i_N} P$ and $L \xrightarrow{i_L} Q$ such that P and Q are projective and the cokernels N^1 and L^1 of i_N and i_L, respectively, are Gorenstein-projective. Since $\mathrm{Ext}_A^1(L, P) = 0$, from the long exact sequence obtained by applying the functor $\mathrm{Hom}_A(-, P)$ to ξ we infer that the induced map $\mathrm{Hom}_A(M, P) \to \mathrm{Hom}_A(N, P)$ is surjective. In particular, there is a morphism $a\colon M \to P$ such that $a \circ f = i_N$. Therefore we have the following exact diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_N} & & \downarrow{\scriptstyle \binom{a}{i_L \circ g}} & & \downarrow{\scriptstyle i_L} & & \\
0 & \longrightarrow & P & \xrightarrow{\binom{1}{0}} & P \oplus Q & \xrightarrow{(0\ 1)} & Q & \longrightarrow & 0.
\end{array}
$$

By Snake Lemma the middle column map is monic and there is an induced short exact sequence $0 \to N^1 \to M^1 \to L^1 \to 0$ where M^1 is the cokernel of the middle column map. Note that N^1, L^1 are Gorenstein projective. In particular, $N^1, L^1 \in {}^{\perp}A$ and then we have $M^1 \in {}^{\perp}A$. Iterating this argument, using Lemma 2.1.4(2) we show that M is Gorenstein-projective.

The statement (2) amounts to the fact that a direct summand of a Gorenstein-projective module is again Gorenstein-projective. For this end, let $N \oplus L$ be Gorenstein-projective. Note that $N \oplus L \in {}^{\perp}A$ and then $N \in {}^{\perp}A$. Take a short exact sequence $0 \to N \oplus L \to P \to G \to 0$ such that P is projective and G is Gorenstein-projective. Then the cokernel N^1 of the monomorphism $N \to P$ fits into a short exact sequence $\eta \colon 0 \to L \to N^1 \to G \to 0$. We add the trivial exact sequence $0 \to N \overset{\mathrm{Id}_N}{\to} N \to 0 \to 0$ to η. Note that both $L \oplus N$ and G are Gorenstein-projective. By (1) we infer that $N^1 \oplus N$ is Gorenstein-projective. Note that we have the short exact sequence $0 \to N \to P \to N^1 \to 0$ and that $N^1 \in {}^{\perp}A$. We repeat the argument by replacing N with N^1 to get N^2 and a short exact sequence $0 \to N^1 \to P^1 \to N^2 \to 0$. Continue this argument. Then we get a required long exact sequence in Lemma 2.1.4(2), proving that N is Gorenstein-projective. Similarly L is Gorenstein-projective.

For the statement (3), take a short exact sequence $0 \to L' \to P \to L \to 0$ such that P is projective and L' is Gorenstein-projective. Consider the following pullback diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L' & = & L' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \dashrightarrow & E & \dashrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Consider the short exact sequence in the middle column. By (1) E is Gorenstein-projective. Note that the short exact sequence in the middle row splits since P is projective. Hence $E \simeq N \oplus P$. By (2) N is Gorenstein-projective.
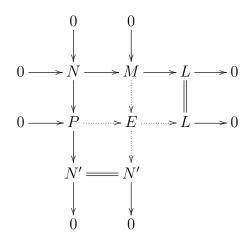
For the statement (4), take a short exact sequence $0 \to N \to P \to N' \to 0$ such that P is projective and N' is Gorenstein-projective. Consider the following pushout

diagram.

$$
\begin{array}{ccc}
& 0 & 0 \\
& \downarrow & \downarrow \\
0 \longrightarrow N \longrightarrow & M \longrightarrow L \longrightarrow 0 \\
\downarrow & \downarrow & \| \\
0 \longrightarrow P \dashrightarrow & E \dashrightarrow L \longrightarrow 0 \\
\downarrow & \downarrow & \\
N' =\!\!= & N' & \\
\downarrow & \downarrow & \\
0 & 0 &
\end{array}
$$

Consider the short exact sequence in the middle column. By (1) we infer that E is Gorenstein-projective. By the assumption the short exact sequence in the middle row splits and then $E \simeq P \oplus L$. Now applying (2) we are done. $\hfill\square$

We denote by A-$\underline{\mathrm{mod}}$ the *stable category* of A-mod modulo projective modules: the objects are the same as A-mod while the morphism space between two objects M and N, denote by $\underline{\mathrm{Hom}}_A(M, N)$, is by definition the quotient R-module $\mathrm{Hom}_A(M, N)/P(M, N)$ where $P(M, N)$ is the R-submodule of $\mathrm{Hom}_A(M, N)$ consisting of morphisms factoring through projective modules. The stable category A-$\underline{\mathrm{mod}}$ is additive and projective modules are zero objects; for details, see [8, p.104]. Moreover, two modules M and N become isomorphic in A-$\underline{\mathrm{mod}}$ if and only if there exist projective modules P and Q such that $M \oplus P \simeq N \oplus Q$; compare [3, Proposition 1.41].

For a module M take a short exact sequence $0 \to \Omega M \to P \to M \to 0$ with P projective. The module ΩM is called a *syzygy module* of M. Note that syzygy modules of M are not uniquely determined, while they are naturally isomorphic to each other in A-$\underline{\mathrm{mod}}$. In this sense we say that ΩM is "the" syzygy module of M. Moreover, we get the *syzygy functor* $\Omega \colon A$-$\underline{\mathrm{mod}} \to A$-$\underline{\mathrm{mod}}$. For each $i \geq 1$, denote by Ω^i the i-th power of Ω and then for a module M, $\Omega^i M$ is the i-th *syzygy module* of M; for details, see [8, p.124].

Recall the following basic property of these syzygy modules[1].

Lemma 2.1.8. *Let M, N be A-modules and let $k \geq 1$. Then there exists a natural epimorphism*

$$\underline{\mathrm{Hom}}_A(\Omega^k M, N) \twoheadrightarrow \mathrm{Ext}^k_A(M, N).$$

If in addition $\mathrm{Ext}^i_A(M, A) = 0$ for $1 \leq i \leq k$, then the above map is an isomorphism. $\hfill\square$

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[1]The author thanks Rene Marczinzik for pointing out an error in the previous version.

The second part of the following observation seems of interest.

Corollary 2.1.9. *Let M be an A-module and let $d \geq 1$. Then we have the following statements:*

(1) if M is a Gorenstein-projective module, then so are $\Omega^i M$ for $i \geq 1$;

(2) if $\operatorname{Ext}_A^i(M, A) = 0$ for $1 \leq i \leq d$ and $\Omega^d M$ is Gorenstein-projective, then so is M.

Proof. The first statement follows by applying Proposition 2.1.7(3) repeatedly. Just consider the long exact sequence $\eta: 0 \to \Omega^d M \to P^{d-1} \to \cdots \to P^0 \to M \to 0$. For the second one, note that from the assumption and a dimension-shift argument we have that $\operatorname{Ext}_A^1(-, A)$ vanishes on all the cocycles of η. Then the second statement follows by applying Proposition 2.1.7(4) repeatedly. $\square$

Recall the construction of the *transpose* $\operatorname{Tr}M$ of a module M: take a projective presentation $P^{-1} \to P^0 \to M \to 0$ and then define the right A-module $\operatorname{Tr}M$ to be the cokernel of the morphism $(P^0)^* \to (P^{-1})^*$. Again the module $\operatorname{Tr}M$ is not uniquely determined, while it is unique when viewed as an object in A-mod. This defines the *transpose functor* $\operatorname{Tr}: A\text{-}\underline{\mathrm{mod}} \to A^{\mathrm{op}}\text{-}\underline{\mathrm{mod}}$ which is contravariant; it is even a duality of categories. Observe that there is a natural isomorphism $\Omega^2 M \simeq (\operatorname{Tr}M)^*$ (certainly in the stable category A-mod). For details, see [3] and [8, p.105].

The following result is interesting.

Proposition 2.1.10. *Let M be an A-module. Then M is Gorenstein-projective if and only if $\operatorname{Tr}M$ is Gorenstein-projective. Moreover, there is an isomorphism $\operatorname{Tr}M \simeq (\Omega^2 M)^*$ in A-$\underline{\mathrm{mod}}$ which is functorial in $M \in A$-Gproj.*

Proof. For the "only if" part of the first statement, assume that M is Gorenstein-projective. Take a complete resolution $P^\bullet$ of M. By definition $\operatorname{Tr}M$ is isomorphic to $Z^3((P^\bullet)^*)$ (in the stable category). Then $\operatorname{Tr}M$ is Gorenstein-projective, since the complex $(P^\bullet)^*$ is totally acyclic; see Lemma 2.1.2(3).

To see the "if" part, first note the following exact sequence ([8, Chapter IV, Proposition 3.2])

$$0 \longrightarrow \operatorname{Ext}_{A^{\mathrm{op}}}^1(\operatorname{Tr}M, A) \longrightarrow M \xrightarrow{\operatorname{ev}_M} M^{**} \longrightarrow \operatorname{Ext}_{A^{\mathrm{op}}}^2(\operatorname{Tr}M, A) \longrightarrow 0.$$

Since $\operatorname{Tr}M$ is Gorenstein-projective the two end terms vanish; see Corollary 2.1.3(1). Then M is reflexive. Take a projective presentation $P^{-1} \to P^0 \to M \to 0$. Then we have an exact sequence $0 \to M^* \to (P^{-1})^* \to (P^0)^* \to \operatorname{Tr}M \to 0$. Applying

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Proposition 2.1.7(3) twice we obtain that M^* is Gorenstein-projective. By Corollary 2.1.6 $M \simeq (M^*)^*$ is Gorenstein-projective.

Note that $\Omega^2 M \simeq (\mathrm{Tr}M)^*$ and that $\mathrm{Tr}M$ is reflexive. Then the second statement follows. $\qquad\square$

Recall that an *exact category* in the sense of Quillen is an additive category endowed with an *exact structure*, that is, a distinguished class of ker-coker sequences which are called *conflations*, subject to certain axioms ([45, Appendix A]). For example, an extension-closed subcategory of an abelian category is an exact category such that conflations are short exact sequences with terms in the subcategory.

Recall that an exact category $\mathcal{A}$ is *Frobenius* provided that it has enough projective and enough injective objects and the class of projective objects coincides with the class of injective objects. For a Frobenius exact category, denote by $\underline{\mathcal{A}}$ its stable category modulo projective objects; it is a *triangulated category* such that its shift functor is the quasi-inverse of the syzygy functor and triangles are induced by conflations. For details, see [33, Chapter I, section 2].

In what follows we denote by A-$\underline{\mathrm{Gproj}}$ the full subcategory of A-$\underline{\mathrm{mod}}$ consisting of Gorenstein-projective modules.

Proposition 2.1.11. *Let A be an artin algebra. Then we have*

(1) *the category A-Gproj is a Frobenius exact category, whose projective objects are equal to projective modules;*

(2) *the stable category A-$\underline{\mathrm{Gproj}}$ modulo projective modules is triangulated.*

Proof. By Proposition 2.1.7(1) A-Gproj $\subseteq$ A-mod is closed under extensions, and then A-Gproj is an exact category. Note that A-Gproj $\subseteq {}^\perp A$. Then projective modules are projective and injective in A-Gproj. Then (1) follows, while (2) follows from (1). $\qquad\square$

Remark 2.1.12. *Note that $\Omega\colon A$-$\underline{\mathrm{Gproj}} \to A$-$\underline{\mathrm{Gproj}}$ is invertible and its quasi-inverse Σ is the shift functor for the triangulated category A-$\underline{\mathrm{Gproj}}$; see [33, p.13]. Moreover, we have a natural isomorphism $\Sigma M \simeq \Omega(M^*)^*$ for $M \in A$-Gproj.*

In fact, applying $(-)^*$ to the short exact sequence $0 \to \Omega(M^*) \to P \to M^* \to 0$ with P_A projective, we get an exact sequence $0 \to M \to P^* \to \Omega(M^*)^* \to 0$ (here we use that M is reflexive and $\mathrm{Ext}^1_{A^{\mathrm{op}}}(M^*, A) = 0$). Then we conclude that $\Sigma M \simeq \Omega(M^*)^*$. Let us remark that one can also infer this natural isomorphism from Corollary 2.1.14 below.

The following observation is of interest.

Lemma 2.1.13. *Let M be a non-projective indecomposable Gorenstein-projective A-module. Consider its projective cover $\pi\colon P(M) \to M$. Then $\operatorname{Ker} \pi$ is non-projective and indecomposable.*

Proof. Note that $\operatorname{Ker} \pi$ is isomorphic to ΩM in $A\text{-}\underline{\mod}$. By Remark 2.1.12 we infer that $\operatorname{Ker} \pi$ is indecomposable in $A\text{-}\underline{\mod}$. By Krull-Schmidt Theorem we have $\operatorname{Ker} \pi \simeq N \oplus P$ such that N is non-projective and indecomposable and P is projective. Since $\operatorname{Ext}^1_A(M, P) = 0$, then the composite inclusion $P \hookrightarrow \operatorname{Ker} \pi \hookrightarrow P(M)$ splits. On this other hand, the morphism π is a projective cover. This forces that P is zero. $\square$

Note that it follows from Corollary 2.1.3 that the duality functors $(-)^*\colon A\text{-Gproj} \xrightarrow{\sim} A^{\mathrm{op}}\text{-Gproj}$ and $(-)^*\colon A^{\mathrm{op}}\text{-Gproj} \xrightarrow{\sim} A\text{-Gproj}$ are exact; see Corollary 2.1.6. Moreover, they restrict to the well-known duality $A\text{-proj} \simeq A^{\mathrm{op}}\text{-proj}$. Then the following result follows immediately (consult [33, p.23, Lemma]).

Corollary 2.1.14. *There is a duality $(-)^*\colon A\text{-}\underline{\mathrm{Gproj}} \xrightarrow{\sim} A^{\mathrm{op}}\text{-}\underline{\mathrm{Gproj}}$ of triangulated categories such that its quasi-inverse is given by $(-)^* = \operatorname{Hom}_{A^{\mathrm{op}}}(-, A)$.* $\square$

The following observation is of independent interest. For the notion of cohomological functor, we refer to [33, p.4].

Proposition 2.1.15. *Let M be an A-module. Then the functors $\underline{\operatorname{Hom}}_A(M, -)\colon A\text{-}\underline{\mathrm{Gproj}} \to R\text{-mod}$ and $\underline{\operatorname{Hom}}_A(-, M)\colon A\text{-}\underline{\mathrm{Gproj}} \to R\text{-mod}$ are cohomological.*

Proof. We only show that the first functor is cohomological and the second can be proved dually. Since triangles in $A\text{-}\underline{\mathrm{Gproj}}$ are induced by short exact sequences in $A\text{-Gproj}$, it suffices to show that for any short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ of Gorenstein-projective modules, the induced sequence $\underline{\operatorname{Hom}}_A(M, X) \to \underline{\operatorname{Hom}}_A(M, Y) \to \underline{\operatorname{Hom}}_A(M, Z)$ is exact in the middle.

This amounts to proving the following statement: given a morphism $\alpha\colon M \to Y$ such that $g \circ \alpha$ factors though a projective module, then there is a morphism $\beta\colon M \to X$ such that $\alpha - f \circ \beta$ factors through a projective module. Assume that there is a morphism $\pi\colon P \to Z$ with P projective such that there is a morphism $t\colon M \to P$ satisfying $g \circ \alpha = \pi \circ t$. Since P is projective, we may lift π along g to a morphism $\pi'\colon P \to Y$, that is, $g \circ \pi' = \pi$. Note that $g \circ (\alpha - \pi' \circ t) = 0$ and then we infer that there exists a morphism $\beta\colon M \to X$ such that $\alpha - \pi' \circ t = f \circ \beta$. In particular, $\alpha - f \circ \beta$ factors through the projective module P. We are done. $\square$

2.2 Other Relevant Modules

In this section we discuss other classes of modules in Gorenstein homological algebra: finitely generated Gorenstein-injective modules, large Gorenstein-projective modules and large Gorenstein-injective modules; here "large" means "not necessarily finitely generated". Strongly Gorenstein-projective modules are also briefly discussed.

Let A be an artin algebra. Denote by A-inj the full subcategory of A-mod consisting of finitely generated injective A-modules. Consider the *Nakayama functors* $\nu = DA \otimes_A -\colon A\text{-mod} \to A\text{-mod}$ and $\nu^- = \operatorname{Hom}_A(DA, -)\colon A\text{-mod} \to A\text{-mod}$. Note that (ν, ν^{-1}) is an adjoint pair. Then for any module $_AM$ we have natural morphisms $\nu\nu^- M \to M$ and $M \to \nu^-\nu M$. Moreover, for projective modules P we have $P \xrightarrow{\sim} \nu^-\nu P$; for injective modules I we have $\nu\nu^- I \xrightarrow{\sim} I$. In this way we get mutually inverse equivalences $\nu\colon A\text{-proj} \xrightarrow{\sim} A\text{-inj}$ and $\nu^-\colon A\text{-inj} \xrightarrow{\sim} A\text{-proj}$.

Note that we have natural isomorphisms $\nu \simeq D \circ (-)^*$ and $\nu^- \simeq (-)^* \circ D$ of functors. Hence a module M is reflexive if and only if the natural morphism $M \to \nu^-\nu M$ is an isomorphism; while the natural morphism $\nu\nu^- M \to M$ is an isomorphism if and only if the right A-module DM is reflexive.

An acyclic complex $I^\bullet$ of injective A-modules is said to be *cototally acyclic*[2] provided that the Hom complex $\nu^- I^\bullet = \operatorname{Hom}_A(DA, I^\bullet)$ is acyclic. An A-module N is said to be (finitely generated) *Gorenstein-injective* provided that there is a cototally acyclic complex $I^\bullet$ such that its zeroth coboundary $B^0(I^\bullet)$ is isomorphic to N. Finitely generated Gorenstein-injective modules are also known as *maximal co-Cohen-Macaulay modules* ([15, 12]).

We denote by A-Ginj the full subcategory of A-mod consisting of Gorenstein-injective modules. Observe that A-inj $\subseteq A$-Ginj.

Denote by $(DA)^\perp$ the full subcategory of A-mod consisting of modules N with the property $\operatorname{Ext}_A^i(DA, N) = 0$ for all $i \geq 1$.

We note the following analogue of Lemma 2.1.2.

Lemma 2.2.1. *Let $I^\bullet$ be a complex of injective modules. Then the following statements are equivalent:*

(1) the complex $I^\bullet$ is cototally acyclic;

(2) the complex $I^\bullet$ is acyclic and each coboundary $B^i(P^\bullet)$ lies in $(DA)^\perp$;

(3) the complex $\nu^- I^\bullet$ is totally acyclic.

---

[2]The terminology, which certainly is not standard, is introduced to avoid possible confusion.

Proof. The proof is analogous to the one of Lemma 2.1.2. Just note that for $(1) \Leftrightarrow (3)$, one uses that the natural chain map $\nu\nu^- I^\bullet \xrightarrow{\sim} I^\bullet$ is an isomorphism. $\qquad\square$

Remark 2.2.2. *Note that $\nu^- I^\bullet = (DI^\bullet)^*$. By Lemma 2.1.2 we infer that $I^\bullet$ is cototally acyclic if and only if $DI^\bullet$ is totally acyclic. Consequently, a module M is Gorenstein-injective if and only if $(DM)_A$ is Gorenstein-projective. Moreover, one has a duality $D \colon A\text{-Ginj} \xrightarrow{\sim} A^{\mathrm{op}}\text{-Gproj}$.*

Let us remark that from Lemma 2.2.1(2) it follows that $A\text{-Ginj} \subseteq (DA)^\perp$. Note that in the case of the lemma above, we have that $Z^0(\nu^- I^\bullet) \simeq \nu^- B^0(I^\bullet)$. Dually a complex $P^\bullet$ of projective modules is totally acyclic if and only if the complex $\nu P^\bullet$ is cototally acyclic; in this case, we have $B^0(\nu P^\bullet) \simeq \nu Z^0(P^\bullet)$.

We conclude from the above discussion the following result.

Proposition 2.2.3. *Let M be an A-module. Then we have:*

(1) *M is Gorenstein-injective if and only if $\nu^- M$ is Gorenstein-projective and the natural morphism $\nu\nu^- M \to M$ is an isomorphism;*

(2) *M is Gorenstein-projective if and only if νM is Gorenstein-injective and the natural morphism $M \to \nu^- \nu M$ is an isomorphism.*

Consequently, we have an equivalence $\nu^- \colon A\text{-Ginj} \xrightarrow{\sim} A\text{-Gproj}$ of categories with its quasi-inverse given by ν. $\qquad\square$

The following result is analogous to Lemma 2.1.4.

Lemma 2.2.4. *Let N be an A-module. Then the following statements are equivalent:*

(1) *N is Gorenstein-injective;*

(2) *there exists a long exact sequence $\cdots \to I^{-3} \to I^{-2} \to I^{-1} \to N \to 0$ with each I^{-i} injective and each coboundary in $(DA)^\perp$;*

(3) *$N \in (DA)^\perp$, $\mathrm{Tor}_i^A(DA, \nu^- N) = 0$ for $i \geq 1$ and the natural morphism $\nu\nu^- N \to N$ is an isomorphism.*

Proof. The proof is analogous to the one of Lemma 2.1.4. We apply Lemma 2.2.1 and note that we have $Z^0(\nu^- I^\bullet) \simeq \nu^- B^0(I^\bullet)$ for a cototally acyclic complex $I^\bullet$. $\qquad\square$

Recall that $A\text{-inj} \subseteq A\text{-Ginj}$. Dual to Proposition 2.1.7 one can show that $A\text{-Ginj} \subseteq A\text{-mod}$ is a *coresolving subcategory*, that is, it contains all the injective modules and is closed under extensions, taking cokernels of monomorphisms and direct summands.

Since A-Ginj is closed under extensions, it becomes an exact category; moreover, it is Frobenius such that its projective objects are equal to injective A-modules.

Denote by A-$\overline{\mathrm{mod}}$ the *stable category* of A-mod modulo injective modules. Recall that for a module M the *cosyzygy module* $\Omega^- M$ is defined to be the cokernel of a monomorphism $M \to I$ with I injective; this gives rise to the *cosyzygy functor* $\Omega^-\colon A$-$\overline{\mathrm{mod}} \to A$-$\overline{\mathrm{mod}}$. For each $i \geq 1$ denote by Ω^{-i} the i-th power of Ω^-.

We denote by A-$\overline{\mathrm{Ginj}}$ the full subcategory of A-$\overline{\mathrm{mod}}$ consisting of Gorenstein-injective modules. Dual to Corollary 2.1.9 one observes that for a Gorenstein-injective module M its cozysygy modules $\Omega^{-i}M$ are Gorenstein-injective. In particular, one has an induced functor $\Omega^-\colon A$-$\overline{\mathrm{Ginj}} \to A$-$\overline{\mathrm{Ginj}}$.

Lemma 2.2.5. *The stable category A-$\overline{\mathrm{Ginj}}$ is triangulated with the shift functor given by Ω^-. Moreover, we have an equivalence $\nu^-\colon A$-$\overline{\mathrm{Ginj}} \xrightarrow{\sim} A$-$\underline{\mathrm{Gproj}}$ of triangulated categories with its quasi-inverse given by ν.*

Proof. The first statement is dual to Proposition 2.1.11(2), while the second follows from Proposition 2.2.3. Note that the second statement also follows from Corollary 2.1.14 and Remark 2.2.2. □

Remark 2.2.6. *The cosyzygy functor $\Sigma = \Omega^-$ is invertible on A-$\overline{\mathrm{Ginj}}$; its quasi-inverse is given by $\Sigma^{-1} = \nu\Omega\nu^-$.*

Recall that $\tau = D \circ \mathrm{Tr}\colon A$-$\underline{\mathrm{mod}} \xrightarrow{\sim} A$-$\overline{\mathrm{mod}}$ is the *Auslander-Reiten translation*; it is an equivalence with its quasi-inverse given by $\tau^{-1} = \mathrm{Tr} \circ D$; see [8, p.106].

The following result is of independent interest.

Proposition 2.2.7. *Let M be an A-module. Then M is Gorenstein-projective if and only if τM is Gorenstein-injective; dually M is Gorenstein-injective if and only if $\tau^{-1}M$ is Gorenstein-projective.*

Proof. We apply Proposition 2.1.10 and Remark 2.2.2. □

From now on we will study for an artin algebra A the category A-Mod of "large" A-modules, that is, modules which are not necessarily finitely generated. We will consider modules and complexes in A-Mod.

Denote by A-Proj (resp. A-Inj) the full subcategory of A-Mod consisting of projective (resp. injective) modules. Note that the Nakayama functor induces an equivalence $\nu\colon A$-Proj $\xrightarrow{\sim} A$-Inj with its quasi-inverse given by ν^-; see [40, Lemma 5.4].

We note the following well-known fact.

Lemma 2.2.8. *Let A be an artin algebra. Then a module is projective if and only if it is a direct summand of a product of $_AA$; a module is injective if and only if it is a direct summand of a coproduct of $D(A_A)$.* $\square$

An acyclic complex $P^\bullet$ of projective A-modules is *totally acyclic* if for each projective A-module Q the Hom complex $\mathrm{Hom}_A(P^\bullet, Q)$ is acyclic; dually an acyclic complex $I^\bullet$ of injective A-modules is *cototally acyclic* if for each injective A-module J the Hom complex $\mathrm{Hom}_A(J, I^\bullet)$ is acyclic.

Denote by $^\perp(A\text{-Proj})$ the full subcategory of A-Mod consisting of modules M such that $\mathrm{Ext}_A^i(M, Q) = 0$ for all $i \geq 1$ and Q projective.

We note the following equalities

$$^\perp(A\text{-Proj}) = \{M \in A\text{-Mod} \mid \mathrm{Ext}_A^i(M, A) = 0 \text{ for all } i \geq 1\}$$
$$= \{M \in A\text{-Mod} \mid \mathrm{Tor}_i^A(D(_AA), M) = 0 \text{ for all } i \geq 1\},$$

where the first equality follows from Lemma 2.2.8 and the second from the fact that $D\mathrm{Tor}_i^A(N, M) \simeq \mathrm{Ext}_A^i(M, DN)$. It then follows that $^\perp(A\text{-Proj}) \subseteq A\text{-Mod}$ is closed under products and $^\perp(A\text{-Proj}) \cap A\text{-mod} = {}^\perp A$.

The following is analogous to Lemma 2.2.1.

Lemma 2.2.9. *Let $P^\bullet$ be a complex of projective A-modules. Then the following are equivalent:*

(1) the complex $P^\bullet$ is totally acyclic;

(2) the complex $P^\bullet$ is acyclic and each cocycle $Z^i(P^\bullet)$ lies in $^\perp(A\text{-Proj})$;

(3) the complexes $P^\bullet$ and $\nu P^\bullet$ are both acyclic;

(4) the complex $\nu P^\bullet$ is cototally acyclic. $\square$

The following notion was first introduced by Enochs and Jenda ([26]).

Definition 2.2.10. *A module M is said to be (large) Gorenstein-projective provided that it is the zeroth cocycle of a totally acyclic complex; a module N is said to be (large) Gorenstein-injective provided that it is the zero coboundary of a cototally acyclic complex.*

Denote by A-GProj (resp. A-GInj) the full subcategory of A-Mod consisting of Gorenstein-projective (resp. Gorenstein-injective) modules. Note that A-Proj $\subseteq A$-GProj and A-Inj $\subseteq A$-GInj.

The following is analogous to Lemma 2.1.4.

Lemma 2.2.11. *Let $M \in A$-Mod. Then the following statements are equivalent:*

(1) *M is Gorenstein-projective;*

(2) *there is a long exact sequence $0 \to M \to P^0 \to P^1 \to P^2 \to \cdots$ with each P^i projective and each cocycle in $^\perp(A$-Proj$)$;*

(3) *$\mathrm{Tor}_i^A(DA, M) = 0 = \mathrm{Ext}_A^i(DA, \nu M)$ for $i \geq 1$ and the natural morphism $M \to \nu^- \nu M$ is an isomorphism.* $\qquad\square$

Let us remark that we have an analogue of Proposition 2.1.7 for A-GProj. In particular, A-GProj is closed under taking direct summands.

The second part of the following result is contained in [66, Proposition 1.4] (also see [21, Lemma 3.4]).

Proposition 2.2.12. *Let A be an artin algebra. Then we have*

(1) *the subcategory A-GProj $\subseteq A$-Mod is closed under coproducts, products and filtered colimits;*

(2) *A-GProj $\cap$ A-mod $= A$-Gproj.*

Proof. Note that the functors ν, ν^-, $\mathrm{Tor}_i^A(DA, -)$ and $\mathrm{Ext}_A^i(DA, -)$ commute with coproducts, products and filtered colimits. Then (1) follows from Lemma 2.2.11(3).

For (2), note first that $^\perp A \subseteq {}^\perp(A$-Proj$)$. It follows from Lemmas 2.1.4(2) and 2.2.11(2) that A-Gproj $\subseteq A$-GProj $\cap$ A-mod. On the other hand, let $M \in A$-GProj $\cap$ A-mod. Take a short exact sequence $0 \to M \to P \to M' \to 0$ with P projective and M' Gorenstein-projective. We may assume that P is free. Since M is finitely generated, there is a decomposition $P \simeq P^0 \oplus P'$ such that P^0 is finitely generated containing M. Then we have a short exact sequence $0 \to M \to P^0 \to M^1 \to 0$. Note that $M^1 \oplus P' \simeq M'$ and that M' is Gorenstein-projective. Then M^1 is also Gorenstein-projective. Therefore $M^1 \in A$-GProj$\cap A$-mod. Observe that $M^1 \in {}^\perp A$. Repeat the argument with M replaced by M^1. We get a long exact sequence $0 \to M \to P^0 \to P^1 \to \cdots$. Now we apply Lemma 2.1.4(2). $\qquad\square$

Note that we have a version of Proposition 2.2.3 for large modules. In particular, there is an equivalence $\nu \colon A$-GProj $\xrightarrow{\sim} A$-GInj with its quasi-inverse given by ν^-.

Note that Auslander-Reiten translations allow a natural extension on A-Mod as follows: for an A-module M take a projective presentation $P^{-1} \to P^0 \to M \to 0$ and define τM to be the kernel of the induced morphism $\nu P^{-1} \to \nu P^0$. Similarly one extends

τ^{-1}. Then we have a duality $\tau \colon A\text{-}\underline{\mathrm{Mod}} \xrightarrow{\sim} A\text{-}\overline{\mathrm{Mod}}$ with its quasi-inverse given by τ^{-1}; for details, see [40, section 5] and [12, Remark 2.3(ii)].

The following result extends Proposition 2.2.7; see [12, Proposition 3.4].

Proposition 2.2.13. *Let M be an A-module. Then M is Gorenstein-projective if and only if τM is Gorenstein-injective; M is Gorenstein-injective if and only if $\tau^{-1}M$ is Gorenstein-projective.*

Proof. The proof is similar as the one of Proposition 2.1.10. The following is analogous to [8, Chapter IV, Proposition 3.2]: for an A-module M, we have the following exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_A(DA, \tau M) \longrightarrow M \longrightarrow \nu^-\nu M \longrightarrow \mathrm{Ext}^2_A(DA, \tau M) \longrightarrow 0,$$

where the middle morphism is the natural map associated to the adjoint pair (ν, ν^-). Then we apply a version of Propositions 2.1.7(3) and 2.2.3 for large modules. $\square$

We make the following observation; see [12, Lemma 8.6].

Proposition 2.2.14. *Let M (resp. N) be a Gorenstein-projective (resp. Gorenstein-injective) A-module. Then we have*

(1) DM (resp. M^*) is a right Gorenstein-injective (resp. Gorenstein-projective) A-module;

(2) DN is a right Gorenstein-projective A-module.

Proof. Observe that for a complex $P^\bullet$ of projective modules, we have $\nu^- DP^\bullet \simeq D\nu P^\bullet$. It follows that for a totally acyclic complex $P^\bullet$ the complex $DP^\bullet$ is cototally acyclic; see Lemma 2.2.1. Then for a Gorenstein-projective A-module M, DM is Gorenstein-injective. Dually observe that for a cototally acyclic complex $I^\bullet$ there is a totally acyclic complex $P^\bullet$ such that $\nu P^\bullet = I^\bullet$, and note that $DI^\bullet = \nu^- DP^\bullet$ is totally acyclic; see Lemma 2.2.1. This proves that for a Gorenstein-injective module N, DN is Gorenstein-projective. Note that $M^* = D\nu M$. Since νM is Gorenstein-injective, then by (2) M^* is Gorenstein-projective. $\square$

The category $A\text{-GProj}$ is a Frobenius exact category such that its projective objects are equal to (large) projective A-modules. We denote by $A\text{-}\underline{\mathrm{GProj}}$ its stable category modulo projective modules which is triangulated; compare Proposition 2.1.11.

Note that the shift functor on $A\text{-}\underline{\mathrm{GProj}}$ is given by $\Sigma = \nu^-\Omega^-\nu$ whose quasi-inverse is given by Ω (compare Remark 2.1.12). Note that the inclusion $A\text{-Gproj} \subseteq A\text{-GProj}$ induces a fully faithful triangle functor $A\text{-}\underline{\mathrm{Gproj}} \hookrightarrow A\text{-}\underline{\mathrm{GProj}}$.

Dually we have that the stable category $A\text{-}\overline{\text{GInj}}$ of A-GInj modulo injective A-modules is triangulated and that the inclusion $A\text{-inj} \hookrightarrow A\text{-GInj}$ induces a full embedding $A\text{-}\overline{\text{Ginj}} \hookrightarrow A\text{-}\overline{\text{GInj}}$ of triangulated categories.

Observe that the equivalence $\nu \colon A\text{-GProj} \xrightarrow{\sim} A\text{-GInj}$ and its quasi-inverse ν^- are both exact. Hence we have

Lemma 2.2.15. *Let A be an artin algebra. Then the Nakayama functor induces a triangle equivalence $\nu \colon A\text{-}\underline{\text{GProj}} \xrightarrow{\sim} A\text{-}\overline{\text{GInj}}$ with its quasi-inverse given by ν^-.* $\qquad\square$

Recall that in a triangulated category $\mathcal{T}$ with arbitrary coproducts an object C is *compact* if the functor $\text{Hom}_{\mathcal{T}}(C, -) \colon \mathcal{T} \to \text{Ab}$ commutes with coproducts. Here "Ab" denotes the category of abelian groups. Denote by $\mathcal{T}^c$ the full subcategory consisting of compact objects; it is a thick triangulated subcategory.

The following observation is easy; see Proposition 2.2.12(1).

Lemma 2.2.16. *Let A be artin algebra. Then the triangulated category $A\text{-}\underline{\text{GProj}}$ has arbitrary coproducts and products, and the natural full embedding $A\text{-}\underline{\text{Gproj}} \hookrightarrow A\text{-}\underline{\text{GProj}}$ induces $A\text{-}\underline{\text{Gproj}} \hookrightarrow (A\text{-}\underline{\text{GProj}})^c$.* $\qquad\square$

In what follows we will discuss very briefly strongly Gorenstein-projective modules. They play the role as "free objects" in Gorenstein homological algebra. Let us first study finitely generated strongly Gorenstein-projective modules.

Let A be an artin algebra and let $n \geq 1$. Following Bennis and Mahdou ([17]) a totally acyclic complex is said to be *n-strong* if it is of the following form

$$\cdots \to P^{-1} \xrightarrow{d^{n-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \to \cdots \to P^{n-1} \xrightarrow{d^{n-1}} P^n \xrightarrow{d^0} P^{n+1} \to \cdots$$

A finitely generated A-module M is said to be *n-strongly Gorenstein-projective* provided that it is the zeroth cocycle of a totally acyclic complex which is n-strong. Denote by $n\text{-}A\text{-SGproj}$ the full subcategory of A-mod consisting of such modules. 1-strongly Gorenstein projective modules are called *strongly Gorenstein-projective* and $1\text{-}A\text{-SGproj}$ is also denoted by $A\text{-SGproj}$ ([16]). Observe that if n divides m we have $n\text{-}A\text{-SGproj} \subseteq m\text{-}A\text{-SGproj}$. Observe that a projective module P is strongly Gorenstein-projective, since we may take its complete resolution

$$\cdots \to P \oplus P \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} P \oplus P \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} P \oplus P \to \cdots$$

which is 1-strong. Then we have $A\text{-proj} \subseteq A\text{-SGproj} \subseteq n\text{-}A\text{-SGproj} \subseteq A\text{-Gproj}$.

The following characterizes n-strongly Gorenstein-projective modules.

Proposition 2.2.17. *An A-module M is n-strongly Gorenstein-projective if and only if $\Omega^n M \simeq M$ (in the stable category) and $\operatorname{Ext}_A^i(M, A) = 0$ for $1 \le i \le n$.*

Proof. The "only if" part is easy. For the "if" part, by dimension-shift we infer that $M \in {}^{\perp}A$. Take an exact sequence $0 \to K \to P^{n-1} \to \cdots \to P^0 \to M \to 0$ such that each P^i is projective. By assumption K and M are isomorphic in A-$\underline{\mathrm{mod}}$. Then there exist projective modules P and Q such that $K \oplus P \simeq M \oplus Q$. By adding P to P^{n-1} we may assume that $K \simeq M \oplus Q$. Denote by M' the image of $P^{n-1} \to P^{n-2}$. By dimension-shift we have $\operatorname{Ext}_A^1(M', Q) \simeq \operatorname{Ext}_A^n(M, Q) = 0$. Consider the short exact sequence $0 \to M \oplus Q \to P^{n-1} \to M' \to 0$. We conclude from $\operatorname{Ext}_A^1(M', Q) = 0$ that there is a decomposition $P^{n-1} = P' \oplus Q$ such that there is a short exact sequence $0 \to M \to P' \to M' \to 0$. Then we get a long exact sequence $0 \to M \to P' \to P^{n-2} \to \cdots \to P^0 \to M \to 0$ from which we construct an n-strong complete resolution for M immediately. $\square$

We note the following immediate consequence.

Corollary 2.2.18. *An A-module M is a direct summand of an n-strongly Gorenstein-projective module if and only if M is Gorenstein-projective and $\Omega^{nd}M \simeq M$ for some $d \ge 1$.*

Proof. For the "if" part, take $N = \oplus_{i=0}^{d-1} \Omega^{ni} M$; it is n-strongly Gorenstein-projective by Proposition 2.2.17. Conversely, assume that M is a direct summand of an n-strongly Gorenstein-projective N. It suffices to show the result in the case that M is indecomposable. Note that by Proposition 2.2.17 we have $\Omega^{ni}N \simeq N$ and then for all $i \ge 1$, $\Omega^{ni}M$ are direct summands of N. By Krull-Schmidt Theorem we infer that $M \simeq \Omega^{nd}M$ for some $d \ge 1$. $\square$

For an additive subcategory $\mathcal{X}$ of A-mod, denote by add $\mathcal{X}$ its *additive closure*, that is, add $\mathcal{X}$ consists of direct summands of the modules in $\mathcal{X}$. Then it follows from the above results that

$$\text{add } A\text{-SGproj} = \bigcup_{n \ge 1} n\text{-}A\text{-SGproj}.$$

There is a large version of n-strongly Gorenstein-projective A-modules. The subcategory of A-Mod consisting of (large) n-strongly Gorenstein-projective modules is denoted by n-A-SGProj; 1-A-SGProj is also denoted by A-SGProj. As above we have inclusions A-Proj $\subseteq A$-SGProj $\subseteq n$-A-SGProj $\subseteq A$-GProj.

Note that Proposition 2.2.17 works for any module $M \in A$-Mod. Then it follows that n-A-SGProj $\cap A$-mod $= n$-A-SGproj for all $n \ge 1$.

The following result is of interest; see [16, Theorem 2.7]. Note that Corollary 2.2.18 does not apply to large modules.

Proposition 2.2.19. *Let A be an artin algebra. Then we have* add A-SGProj $=$ A-GProj.

Proof. The inclusion add A-SGProj $\subseteq A$-GProj is clear. On the other hand, we need to show that each Gorenstein-projective module M is a direct summand of a strong Gorenstein-projective module. Take a complete resolution $P^\bullet$ for M. For each $i \in \mathbb{Z}$ denote by $P^\bullet(i)$ the shifted complex of $P^\bullet$, which is defined by $(P^\bullet(i))^n = P^{n+i}$ and $d_{P(i)}^n = d_P^{n+i}$. Consider the complex $\oplus_{i\in\mathbb{Z}}P^\bullet(i)$; it is a strong totally acyclic complex. Note that its zeroth cocycle is $N = \oplus_{i\in\mathbb{Z}}Z^i(P^\bullet)$ and by definition N is strong Gorenstein-projective. Observe that M is a direct summand of N. $\qquad\qquad\square$

2.3   Gorenstein Algebras

In this section we will study Gorenstein-projective modules over Gorenstein algebras. This is the case where Gorenstein-projective modules behave the best. Other related notions such as virtually Gorenstein algebras, CM-finite algebras and CM-free algebras will be discussed very briefly.

Recall that an artin algebra A is *self-injective* provided that its regular module $_AA$ is injective; this is equivalent to that projective modules are injective and vice verse; see [8, Chapter IV, section 3].

The following result is easy.

Proposition 2.3.1. *Let A be an artin algebra. Then the following statements are equivalent:*

  (1) the algebra A is self-injective;

  (2) A-mod $= A$-Gproj;

  (2') A-mod $= A$-Ginj;

  (3) A-inj $\subseteq A$-Gproj;

  (3') A-proj $\subseteq A$-Ginj.

Proof. Note that for a self-injective algebra A and an A-module M, we may splice the projective resolution and the injective resolution of M to get a complete resolution for M. This shows " (1) $\Rightarrow$ (2)". The implication "(2) $\Rightarrow$ (3)" is trivial. For "(3) $\Rightarrow$ (1)", note

that then $D(A_A)$ is Gorenstein-projective, in particular, it is a submodule of a projective module. Since $D(A_A)$ is injective, the submodule is necessarily a direct summand. Hence $D(A_A)$ is projective and then the algebra A is self-injective. $\qquad\square$

Recall that an artin algebra A is *Gorenstein* provided that the regular module A has finite injective dimension on both sides ([34]). We have that an algebra A is Gorenstein is equivalent to that any A-module has finite projective dimension if and only if it has finite injective dimension.

Observe that for a Gorenstein algebra A we have inj.dim $_AA$ = inj.dim A_A ([5, Lemma 6.9]); the common value is denoted by G.dim A. If G.dim $A \leq d$, we say that A is *d-Gorenstein*.[3] Note that 0-Gorenstein algebras are the same as self-injective algebras. An algebra of finite global dimension d is d-Gorenstein.

Let us begin with the following observation.

Lemma 2.3.2. *Let A be a d-Gorenstein algebra and let $M \in A$-Mod. If M has finite projective dimension, then* proj.dim $M \leq d$ *and* inj.dim $M \leq d$.

Proof. Note that inj.dim $_AA$ = proj.dim $D(A_A) \leq d$. We use the following fact: an A-module M of finite projective dimension n satisfies that $\operatorname{Ext}_A^n(M, Q) \neq 0$ for some projective A-module Q; by Lemma 2.2.8 this is equivalent to $\operatorname{Ext}_A^n(M, A) \neq 0$. This shows that proj.dim $M \leq d$. Similarly one shows that inj.dim $M \leq d$. $\qquad\square$

For each $d \geq 1$ denote by $\Omega^d(A\text{-mod})$ the class of modules of the form $\Omega^d M$ for a module M. In addition we identify $\Omega^0(A\text{-mod})$ with A-mod. Note that A-Gproj $\subseteq \Omega^d(A\text{-mod})$ for all $d \geq 0$. Dually we have the notations $\Omega^{-d}(A\text{-mod})$ for $d \geq 0$.

The following result, which is implicitly contained in [10, Theorem 3.2], characterizes d-Gorenstein algebras; it generalizes part of of Proposition 2.3.1.

Theorem 2.3.3. *Let A be an artin algebra and let $d \geq 0$. Then the following statements are equivalent:*

(1) the algebra A is d-Gorenstein;

(2) A-Gproj $= \Omega^d(A\text{-mod})$;

(2') A-Ginj $= \Omega^{-d}(A\text{-mod})$.

In this case, we have A-Gproj $= {}^\perp A$ and A-Ginj $= (DA)^\perp$.

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[3]In the literature there is a different notion of d-Gorenstein algebra; see [7].

Proof. We only show the result concerning Gorenstein-projective modules.

For "(1) $\Rightarrow$ (2)", assume that A is d-Gorenstein. Note that by dimension-shift we have $\Omega^d(A\text{-mod}) \subseteq {}^\perp A$. We have already observed that A-Gproj $\subseteq \Omega^d(A\text{-mod})$. Hence it suffices to show that any module M in ${}^\perp A$ is Gorenstein-projective. For this, take a projective resolution $\cdots \to P^{-1} \to P^0 \to M \to 0$. By assumption we apply $(-)^*$ to get an exact sequence $\xi\colon 0 \to M^* \to (P^0)^* \to (P^{-1})^* \to \cdots$. Since A_A has finite injective dimension, using dimension-shift on ξ we infer that $M^* \in {}^\perp(A_A)$; moreover, all the cocylces in ξ lie in ${}^\perp(A_A)$. Hence applying $(-)^*$ to ξ we still get an exact sequence. Note that each P^{-i} is reflexive. From this we conclude that M is reflexive. Then by Lemma 2.1.4(3) M is Gorenstein-projective.

To show "(2) $\Rightarrow$ (1)", assume that A-Gproj $= \Omega^d(A\text{-mod})$. For each module M and $k \geq 1$, we have $\mathrm{Ext}_A^{d+k}(M, A) \simeq \mathrm{Ext}_A^k(\Omega^d M, A) = 0$, since $\Omega^d M$ is Gorenstein-projective. Therefore inj.dim ${}_A A \leq d$. On the other hand, consider the long exact sequence $0 \to \Omega^d D(A_A) \to P^{1-d} \to \cdots \to P^{-1} \to P^0 \to D(A_A) \to 0$. For any Gorenstein-projective module G, by dimension-shift we infer that $\mathrm{Ext}_A^{d+1}(G, \Omega^d D(A_A)) \simeq \mathrm{Ext}_A^1(G, D(A_A)) = 0$ (we can apply dimension-shift because of $G \in {}^\perp A$). Note that by assumption $\Omega^d D(A_A)$ is Gorenstein-projective, and then we may take a long exact sequence $0 \to \Omega^d D(A_A) \xrightarrow{\varepsilon} Q^0 \to Q^1 \to \cdots \to Q^d \to G \to 0$ with each Q^i projective and G Gorenstein-projective. Take G' to be the cokernel of ε. Then by dimension-shift again we have $\mathrm{Ext}_A^1(G', \Omega^d D(A_A)) \simeq \mathrm{Ext}_A^{d+1}(G, \Omega^d D(A_A)) = 0$. It follows that ε splits and then $\Omega^d D(A_A)$ is projective. Hence proj.dim $D(A_A) \leq d$ and then inj.dim $A_A \leq d$, completing the proof.

The equality A-Gproj $= {}^\perp A$ is shown in the proof of "(1) $\Rightarrow$ (2)". $\qquad\square$

Remark 2.3.4. *For a Gorenstein algebra A, the modules in ${}^\perp A$ are often called* maximal Cohen-Macaulay modules *([19]).*

For an artin algebra A and $n \geq 0$, denote by $\mathcal{P}^{\leq n}(A\text{-mod})$ the full subcategory of A-mod consisting of modules having projective dimension at most n. Denote by $\mathcal{P}^{<\infty}(A\text{-mod})$ the union of these categories. Dually we have the notations $\mathcal{I}^{\leq n}(A\text{-mod})$ and $\mathcal{I}^{<\infty}(A\text{-mod})$.

For a d-Gorenstein algebra A, by Lemma 2.3.2 we have

$$\mathcal{P}^{<\infty}(A\text{-mod}) = \mathcal{P}^{\leq d}(A\text{-mod}) = \mathcal{I}^{\leq d}(A\text{-mod}) = \mathcal{I}^{<\infty}(A\text{-mod}).$$

By Theorem 2.3.3 we may apply Auslander-Buchweitz's result (Theorem A.0.12) to obtain the following important result; compare [12, Proposition 3.10].

Proposition 2.3.5. *Let A be a d-Gorenstein algebra. Then $(A\text{-Gproj}, \mathcal{P}^{\leq d}(A\text{-mod}))$ and $(\mathcal{I}^{\leq d}(A\text{-mod}), A\text{-Ginj})$ are cotorsion pairs in A-mod.* $\qquad\square$

Recall that a full additive subcategory $\mathcal{X} \subseteq A$-mod is said to be *contravariantly finite* provided that each module M admits a morphism $f_M \colon X_M \to M$ with $X_M \in \mathcal{X}$ such that each morphism from a module in $\mathcal{X}$ to M factors through f_M. Such a morphism f_M is called a *right $\mathcal{X}$-approximation* of M. Dually one has the notion of *covariantly finite subcategory*. A subcategory is said to be *functorially finite* provided that it is both contravariantly finite and covariantly finite.

The following is well known; compare [5, Corollary 5.10(1)]. It is contained in [14, Theorem 5].

Corollary 2.3.6. *Let A be a d-Gorenstein artin algebra. Then all the three subcategories A-Gproj, $\mathcal{P}^{\leq d}(A\text{-mod}) = \mathcal{I}^{\leq d}(A\text{-mod})$, A-Ginj are functorially finite in A-mod.*

Proof. It follows from the two cotorsion pairs above that A-Gproj is contravriantly finite, $\mathcal{P}^{\leq d}(A\text{-mod}) = \mathcal{I}^{\leq d}(A\text{-mod})$ is functorially finite and A-Ginj is covariantly finite. The rest follows from the following fact (and its dual): a resolving contravariantly finite subcategory in A-mod is functorially finite; see [42, Corollary 0.3]. $\qquad\square$

There are analogues of the results above for large modules. For each $d \geq 0$ denote by $\Omega^d(A\text{-Mod})$ the class of modules of the form $\Omega^d M$ for an A-module M. Similarly we have the notation $\Omega^{-d}(A\text{-Mod})$.

Then we have the following result.

Theorem 2.3.7. *Let A be an artin algebra and let $d \geq 0$. Then the following statements are equivalent:*

(1) the algebra A is d-Gorenstein;

(2) A-GProj $= \Omega^d(A\text{-Mod})$;

(2') A-GInj $= \Omega^{-d}(A\text{-Mod})$.

In this case, we have A-GProj $= \{M \in A\text{-Mod} \mid \operatorname{Ext}_A^i(M, A) = 0, i \geq 1\}$ and A-GInj $= \{M \in A\text{-Mod} \mid \operatorname{Ext}_A^i(DA, M) = 0, i \geq 1\}$. Moreover, we have two cotorsion pairs $(A\text{-GProj}, \mathcal{P}^{\leq d}(A\text{-Mod}))$ and $(\mathcal{I}^{\leq d}(A\text{-Mod}), A\text{-GInj})$ in A-Mod.

Proof. We just comment on the proof of the results concerning Gorenstein-projective modules. Note that the condition (2) implies that A-Gproj $= \Omega^d(A\text{-mod})$ and then "(2) $\Rightarrow$ (1)" follows from Theorem 2.3.3. To see "(1) $\Rightarrow$ (2)", first observe that A-GProj $\subseteq \Omega^d(A\text{-Mod}) \subseteq \{M \in A\text{-Mod} \mid \operatorname{Ext}_A^i(M, A) = 0, i \geq 1\}$. Take an A-module M with the property $\operatorname{Ext}_A^i(M, A) = 0$ for all $i \geq 1$. We have to show that M is Gorenstein-projective. We apply Lemma 2.2.11(3). Note that M satisfies that

$\mathrm{Tor}_i^A(DA, M) = 0$ for all $i \geq 1$. We replace $(-)^*$ by ν in the proof of "(1) $\Rightarrow$ (2)" in Theorem 2.3.3. $\qquad\square$

Following Beligiannis and Reiten an artin algebra A is said to be *virtually Gorenstein* provided that $(A\text{-GProj})^\perp = {}^\perp(A\text{-GInj})$; see [15, Chapter X, Definition 3.3]. Then it follows from the two cotorsion pairs above that a Gorenstein artin algebra is virtually Gorenstein. For more on virtually Gorenstein algebras, see [12, 14].

For an additive subcategory $\mathcal{X}$ of A-Mod we denote by $\varinjlim \mathcal{X}$ the full subcategory of A-Mod consisting of filtered colimits of modules in $\mathcal{X}$, or equivalently, consisting of direct limits of modules in $\mathcal{X}$ ([5, Chapter 1, Theorem 1.5]). By Proposition 2.2.12 the full subcategory A-GProj is closed under filtered colimits. In particular we have $\varinjlim A\text{-Gproj} \subseteq A\text{-GProj}$. Similarly we have $\varinjlim A\text{-Ginj} \subseteq A\text{-GInj}$.

The following result is of interest. It is contained in [14, Theorem 5], while its first part is contained in [27].

Proposition 2.3.8. *Let A be a Gorenstein artin algebra. Then we have $\varinjlim A\text{-Gproj} = A\text{-GProj}$ and $\varinjlim A\text{-Ginj} = A\text{-GInj}$.*

Proof. Assume that the algebra A is d-Gorenstein. Recall from Proposition 2.3.5 the cotorsion pair $(A\text{-Gproj}, \mathcal{P}^{\leq d}(A\text{-mod}))$ in A-mod. Then by [42, Theorem 2.4(2)] we have $\varinjlim A\text{-Gproj} = \{M \in A\text{-Mod} \mid \mathrm{Ext}_A^i(M, L) = 0, i \geq 1, L \in \mathcal{P}^{\leq d}(A\text{-mod})\}$. By dimension-shift one infer that an A-module M has the property that $\mathrm{Ext}_A^i(M, L) = 0$ for all $i \geq 1$ and all L in $\mathcal{P}^{\leq d}(A\text{-mod})$ if and only if M satisfies $\mathrm{Ext}_A^i(M, A) = 0$ for all $i \geq 1$. Then it follows from Theorem 2.3.7 that this is equivalent to that M lies in A-GProj. $\qquad\square$

Recall that finitely generated Gorenstein-projective modules over an artin algebra A are also called maximal Cohen-Macaulay modules. Following [12, Example 8.4(2)] an artin algebra A is said to be *CM-finite* provided that up to isomorphism there are only finitely many indecomposable modules in A-Gproj. Observe that an algebra of finite representation type is CM-finite. By Corollary 2.1.6 A is CM-finite if and only if so is A^{op}.

Recently CM-finite Gorenstein artin algebras attract considerable attentions; see [20, 13, 48]. It is believed that the theory of Gorenstein-projective modules over a CM-finite Gorenstein artin algebra is the simplest one among all the interesting cases.

As an extreme case of CM-finite algebras, we call that an artin algebra A is *CM-free* provided that A-Gproj $= A$-proj (compare [50]). Roughly speaking, for these kinds of algebras the theory of Gorenstein-projective modules is boring. Note that by Corollary 2.1.3(2) an algebra of finite global dimension is CM-finite. On the other hand, by

Theorem 2.3.3(2) a Gorenstein algebra is CM-finite if and only if it has finite global dimension.

Take k to be a field and consider the three dimensional truncated polynomial algebra $A = k[x, y]/(x^2, xy, y^2)$. It is well-known that the algebra A is CM-free. In what follows we will give a general result, which generalizes [62, Proposition 2.4].

Recall that for an artin algebra A its *Ext-quiver* $Q(A)$ is defined such that the vertices are given by a complete set $\{S_1, S_2, \cdots, S_n\}$ of pairwise non-isomorphic simple A-modules and there is an arrow from S_i to S_j if and only if $\mathrm{Ext}_A^1(S_i, S_j) \neq 0$. Recall that the algebra A is connected if and only if the underlying graph of $Q(A)$ is connected.

Theorem 2.3.9. *Let A be a connected artin algebra such that $\mathbf{r}^2 = 0$. Here $\mathbf{r}$ is the Jacobson radical of A. Then either A is self-injective or CM-free.*

Proof. Assume that A is not CM-finite. Take $M \in A$-Gproj to be non-projective and indecomposable. Note that there is a short exact sequence $0 \to M \to P \xrightarrow{\pi} M' \to 0$ with P projective and $M' \in A$-Gproj. It follows that π is a projective cover and then $M \subseteq \mathbf{r}P$. Note that $\mathbf{r}^2 = 0$. Hence $\mathbf{r}M = 0$ and then M is semisimple. Note that M is indecomposable. Then we conclude that M is a simple module.

Let $S_1 = M$ be the above simple module. Take a short exact sequence $\eta\colon 0 \to S_2 \xrightarrow{i_2} P_1 \xrightarrow{\pi_1} S_1 \to 0$ such that π_1 is a projective cover. Observe that $S_2 \neq 0$. By Lemma 2.1.13 S_2 is indecomposable. Then by above we infer that S_2 is simple. Moreover we claim that a simple A-module S with $\mathrm{Ext}_A^1(S, S_2) \neq 0$ is isomorphic to S_1.

To prove the claim, let us assume on the contrary that S is not isomorphic to S_1. Take a short exact sequence $0 \to K \to P \xrightarrow{\pi} S \to 0$ such that π is a projective cover. As above we infer that K is semisimple. Observe that $\mathrm{Ext}_A^1(S, S_2) \neq 0$ implies that $\mathrm{Hom}_A(K, S_2) \neq 0$. Then S_2 is a direct summand of K. Thus we get a nonzero morphism $S_2 \hookrightarrow K \hookrightarrow P$ which is denoted by l. Note that $\mathrm{Ext}_A^1(S_1, P) = 0$ since S_1 is Gorenstein-projective. By the long exact sequence obtained by applying $\mathrm{Hom}_A(-, P)$ to η we have an epimorphism $\mathrm{Hom}_A(P_1, P) \to \mathrm{Hom}_A(S_2, P)$ induced by i_2. It follows then there exists a morphism $a\colon P_1 \to P$ such that $a \circ i_2 = l$. Note that S_2 is the socle of P_1 on which a is nonzero. It follows that the morphism a is monic. On the other hand, since S is not isomorphic to S_1, the composite $P_1 \xrightarrow{a} P \xrightarrow{\pi} S$ is zero. This implies that the monomorphism a factors through K. Note that K is semisimple while the module P_1 is not semisimple. This is absurd. We are done with the claim.

Similarly we define S_3 by the short exact sequence $0 \to S_3 \xrightarrow{i_3} P_2 \xrightarrow{\pi_2} S_2 \to 0$ such that π_2 is a projective cover. As above S_3 is simple and satisfies that any simple A-module S with $\mathrm{Ext}_A^1(S, S_3) \neq 0$ is isomorphic to S_2. In this way we define S_n for $n \geq 1$.

Choose $n \geq 1$ minimal with the property that $S_n \simeq S_m$ for some $m < n$. Then such an m is unique. Note that $m = 1$. Otherwise $\mathrm{Ext}^1_A(S_{m-1}, S_n) \simeq \mathrm{Ext}^1_A(S_{m-1}, S_m) \neq 0$ while S_{m-1} is not isomorphic to S_{n-1}. This contradicts the claim above for S_n. Then we get a set $\{S_1, S_2, \cdots, S_{n-1}\}$ of pairwise non-isomorphic simple A-modules; moreover each S_i satisfies that any simple A-module S with $\mathrm{Ext}^1_A(S, S_i)$ is isomorphic to S_{i-1}, and clearly from the construction of S_i's any simple A-module S with $\mathrm{Ext}^1_A(S_i, S)$ is isomorphic to S_{i+1} (here we identify S_0 with S_{n-1}, S_n with S_1). It follows then that the full sub quiver of $Q(A)$ with vertices $\{S_1, S_2, \cdots, S_{n-1}\}$ is a connected component. Since A is connected, these are all the simple A-modules. Then all the indecomposable projective A-modules are given by $\{P_1, P_2, \cdots, P_{n-1}\}$. Observe that each of them is of length 2 and has a different simple socle. It follows immediately that the algebra A is self-injective either by [18, 1.6, Ex.2] or by [8, Chapter IV, Ex.12]. □

Chapter 3

Gorenstein Homological Algebra

In this chapter we will study the central topic of Gorenstein homological algebra: we study proper Gorenstein-projective resolutions and Gorenstein-projective dimensions; we study the class of modules having finite Gorenstein-projective dimension; we also briefly discuss Gorenstein derived categories.

3.1   Gorenstein Resolutions and Dimensions

In this section we will study proper Gorenstein-projective resolutions of modules and then various Gorenstein dimensions of modules and algebras.

Let A be an artin R-algebra where R is a commutative artinian ring. Denote by A-GProj the category of Gorenstein-projective A-modules. Recall that the stable category A-$\underline{\text{GProj}}$ modulo projective modules is triangulated with arbitrary coproducts.

Following Neeman ([52] and [53, Definition 1.7]) a triangulated category $\mathcal{T}$ with arbitrary coproducts is *compactly generated* provided that the full subcategory $\mathcal{T}^c$ consisting of compact objects is essentially small and for any nonzero object $X \in \mathcal{T}$ there exists a compact object C with $\text{Hom}_{\mathcal{T}}(C, X) \neq 0$. In this case the smallest triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{T}^c$ and is closed under coproducts is $\mathcal{T}$ itself; see [53, Lemma 3.2]. One of the main features of compactly generated triangulated categories is that the Brown representability theorem and its dual hold form them; see [53, Theorem 3.1] and [54]; also see [55, Theorems 8.3.3 and 8.6.1].

The following result, due to Beligiannis ([12, Theorem 6.6]), is one of the basic results in Gorenstein homological algebra. Observe that it is contained in [38, Theorem 5.4].

Theorem 3.1.1. (Beligiannis) *Let A be an artin algebra. Then the triangulated category A-$\underline{\text{GProj}}$ is compactly generated.* □

We will sketch a proof of this theorem in Appendix B. Here we will give an application of the theorem.

The following application of Beligiannis's Theorem is contained in [44, Theorem 2.11]. We will present a stronger result due to Beligiannis-Reiten in Appendix B.

Corollary 3.1.2. *Let A be an artin algebra. Then the subcategory A-$\underline{\text{GProj}} \subseteq A$-Mod is contravariantly finite.*

Proof. Let M be an A-module. Consider the contravariant functor

$$\underline{\text{Hom}}_A(-, M) \colon A\text{-}\underline{\text{GProj}} \to R\text{-Mod},$$

where R-Mod denotes the category of (left) R-modules. This functor sends coproducts to products and by Proposition 2.1.15 it is cohomological. We apply Theorem 3.1.1 and Brown representability theorem ([53, Theorem 3.1]). There exists a Gorenstein-projective module G with an isomorphism $\eta \colon \underline{\text{Hom}}_A(-, G) \simeq \underline{\text{Hom}}_A(-, M)$. This yields a morphism $f \colon G \to M$. Take an epimorphism $\pi \colon P \to M$ with P projective.

We claim that $(f, \pi) \colon G \oplus P \to M$ is a right A-GProj-approximation of M. In fact, given a morphism $g \colon G' \to M$ with G' Gorenstein-projective, by the isomorphism η there exists a morphism $t \colon G' \to G$ such that $g - f \circ t$ becomes zero in the stable category, that is, it factors through a projective module. Since π is epic, $g - f \circ t$ necessarily factors through π. Hence g factors though (f, π). $\qquad \square$

Remark 3.1.3. *Recall from Proposition 2.2.12(1) that the subcategory A-GProj is closed under filtered colimits. Combining the corollary above with [65, Theorem 2.2.8] we infer that the subcategory A-GProj is covering in A-Mod. Using Wakamatsu's Lemma and this remark one deduces (part of) Theorem B.0.24 directly.*

Remark 3.1.4. *By a similar argument as above we prove that the subcategory A-GProj is covariantly finite in A-Mod, and then A-GProj is functorially finite in A-Mod. For each A-module consider the covariant functor $\underline{\text{Hom}}_A(M, -) \colon A\text{-}\underline{\text{GProj}} \to R$-Mod. Then we apply the dual Brown representability theorem to this functor to get a left A-GProj-approximation of M. Here one needs to use the fact that the category A-Proj of projective modules is covariantly finite in A-Mod.*

Note that for an artin algebra A the category A-Gproj of finitely generated Gorenstein-projective modules is not necessarily contravariantly finite in A-mod; see [62, 14]. While Beligiannis's Theorem enables us to define Gorenstein extension groups via resolutions by large Gorenstein-projective modules. This is the main reason why we study Gorenstein homological algebra in the category A-Mod of large modules.

A complex $X^\bullet = (X^n, d_X^n)_{n \in \mathbb{Z}}$ of A-modules is said to be *right* GP-*acyclic* provided that for each Gorenstein-projective module G the Hom complex $\mathrm{Hom}_A(G, X^\bullet)$ is acyclic. A right GP-acyclic complex is necessarily acyclic; moreover, an acyclic complex $X^\bullet$ is right GP-acyclic if and only if each induced morphism $X^n \to Z^{n+11}(X^\bullet)$ induces for each Gorenstein-projective module G a surjective map $\mathrm{Hom}_A(G, X^n) \to \mathrm{Hom}_A(G, Z^{n+1}(X^\bullet))$.

Let M be an A-module. By a *Gorenstein-projective resolution*, or a GP-*resolution* in short, of M we mean an acyclic complex $\cdots \to G^{-2} \to G^{-1} \to G^0 \to M \to 0$ with each $G^{-i} \in A$-GProj; sometimes we write this resolution as $G^\bullet \to M$. A GP-resolution is *proper* provided that in addition it is right GP-acyclic ([10, section 4]). It follows from Corollary 3.1.2 that each A-module admits a proper GP-resolution. Such a proper GP-resolution is necessarily unique.

The following two lemmas are well known.

Lemma 3.1.5. (Comparison Theorem) *Let M and N be A-modules. Consider two proper GP-resolutions $G_M^\bullet \to M$ and $G_N^\bullet \to N$. Let $f \colon M \to N$ be a morphism. Then there is a chain map $f^\bullet \colon G_M^\bullet \to G_N^\bullet$ filling into the following commutative diagram*

$$
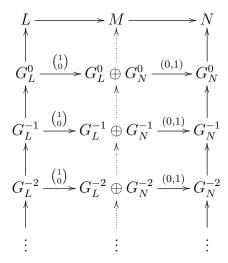\begin{array}{ccccccccc}
\cdots & \longrightarrow & G_M^{-2} & \longrightarrow & G_M^{-1} & \longrightarrow & G_M^0 & \longrightarrow & M & \longrightarrow 0 \\
& & \downarrow{\scriptstyle f^{-2}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f} \\
\cdots & \longrightarrow & G_N^{-2} & \longrightarrow & G_N^{-1} & \longrightarrow & G_N^0 & \longrightarrow & N & \longrightarrow 0
\end{array}
$$

Such a chain map is unique up to homotopy. $\qquad\square$

Lemma 3.1.6. (Horseshoe Lemma) *Let $0 \to L \to M \to N \to 0$ be a right GP-acyclic sequence. Take two proper GP-resolutions $G_L^\bullet \to L$ and $G_N^\bullet \to N$. Then there exists a commutative diagram*

$$
\begin{array}{ccccc}
L & \longrightarrow & M & \longrightarrow & N \\
\uparrow & & \uparrow & & \uparrow \\
G_L^0 & \xrightarrow{\binom{1}{0}} & G_L^0 \oplus G_N^0 & \xrightarrow{(0,1)} & G_N^0 \\
\uparrow & & \uparrow & & \uparrow \\
G_L^{-1} & \xrightarrow{\binom{1}{0}} & G_L^{-1} \oplus G_N^{-1} & \xrightarrow{(0,1)} & G_N^{-1} \\
\uparrow & & \uparrow & & \uparrow \\
G_L^{-2} & \xrightarrow{\binom{1}{0}} & G_L^{-2} \oplus G_N^{-2} & \xrightarrow{(0,1)} & G_N^{-2} \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots
\end{array}
$$

such that the middle column is a proper GP-resolution. $\qquad\square$

One of the central notions in Gorenstein homological algebra is Gorenstein extension group defined below.

Definition 3.1.7. *Let M and N be A-modules. Let $n \geq 0$. Take a proper GP-resolution $G_M^\bullet \to M$. Define the n-th GP-extension group of N by M to be $\operatorname{Ext}_{\mathrm{GP}}^n(M, N) = H^n(\operatorname{Hom}_A(G^\bullet, N))$. We set $\operatorname{Ext}_{\mathrm{GP}}^{-n}(M, N) = 0$ for $n \geq 1$.*

Remark 3.1.8. *By Comparison Theorem the GP-extension groups do not depend on the choice of the proper GP-resolution. As an immediate consequence we have $\operatorname{Ext}_{\mathrm{GP}}^0(M, N) = \operatorname{Hom}_A(M, N)$, and $\operatorname{Ext}_{\mathrm{GP}}^n(M, N) = 0$ if $n \geq 1$ and M is Gorenstein-projective.*

Note that the n-th GP-extension groups $\operatorname{Ext}_{\mathrm{GP}}^n(M, N)$ are functorial both in M and N. Moreover we have the following well-known results.

Lemma 3.1.9. (Long Exact Sequence Theorem I) *Let $0 \to M' \to M \to M'' \to 0$ be a right GP-acyclic sequence and let N be an A-module. Then there is a long exact sequence*

$$0 \to \operatorname{Hom}_A(M'', N) \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, N) \xrightarrow{c^0} \operatorname{Ext}_{\mathrm{GP}}^1(M'', N)$$

$$\to \operatorname{Ext}_{\mathrm{GP}}^1(M, N) \to \operatorname{Ext}_{\mathrm{GP}}^1(M', N) \xrightarrow{c^1} \operatorname{Ext}_{\mathrm{GP}}^2(M'', N) \to \operatorname{Ext}_{\mathrm{GP}}^2(M, N) \to \cdots$$

where the morphisms c^i are the connecting morphisms *and the other morphisms are induced by the corresponding functors.* $\square$

Lemma 3.1.10. (Long Exact Sequence Theorem II) *Let $0 \to N' \to N \to N'' \to 0$ be a right GP-acyclic sequence and let M be an A-module. Then there is a long exact sequence*

$$0 \to \operatorname{Hom}_A(M, N') \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, N'') \xrightarrow{c^0} \operatorname{Ext}_{\mathrm{GP}}^1(M, N')$$

$$\to \operatorname{Ext}_{\mathrm{GP}}^1(M, N) \to \operatorname{Ext}_{\mathrm{GP}}^1(M, N'') \xrightarrow{c^1} \operatorname{Ext}_{\mathrm{GP}}^2(M, N') \to \operatorname{Ext}_{\mathrm{GP}}^2(M, N) \to \cdots$$

where the morphisms c^i are the connecting morphisms *and the other morphisms are induced by the corresponding functors.* $\square$

One of the main reasons to study the GP-extension groups is that they provide certain numerical invariants for modules and algebras.

Definition 3.1.11. *For an A-module M we define its* Gorenstein-projective dimension *by GP.dim $M = \sup\{n \geq 0 \mid \operatorname{Ext}_{\mathrm{GP}}^n(M, -) \neq 0\}$. The* global Gorenstein-projective dimension *of the algebra A, denoted by gl.GP.dim A, is defined to be the supreme*

of the Gorenstein-projective dimensions of all modules. The large (resp. small) finistic Gorenstein-projective dimension of the algebra A, denoted by Fin.GP.dim A *(resp.* fin.Gp.dim A), *is defined to be the supreme of the Gorenstein-projective dimensions of all (resp. finitely generated) A-modules of finite Gorenstein-projective dimension.*

Remark 3.1.12. *It follows from the definitions that* fin.Gp.dim $A \leq$ Fin.GP.dim $A \leq$ gl.GP.dim A; *if* gl.GP.dim $A < \infty$ *or* Fin.GP.dim $A = \infty$, *then* Fin.GP.dim $A =$ gl.GP.dim A $\qquad\square$

The following result is basic.

Proposition 3.1.13. *Let M be an A-module and let $n \geq 0$. Then the following statements are equivalent:*

(1) GP.dim $M \leq n$;

(2) $\mathrm{Ext}_{\mathrm{GP}}^{n+1}(M, -) = 0$;

(3) for each right GP-acyclic complex $0 \to K \to G^{1-n} \to \cdots \to G^0 \to M \to 0$ with each G^i Gorenstein-projective we have that K is Gorenstein-projective.

Proof. The implications "(1) $\Rightarrow$ (2)" and "(3) $\Rightarrow$ (1)" are trivial. To see "(2) $\Rightarrow$ (3)", assume that $\mathrm{Ext}_{\mathrm{GP}}^{n+1}(M, -) = 0$ and that we are given a right GP-acyclic complex $0 \to K \to G^{1-n} \to \cdots \to G^0 \to M \to 0$ with each G^i Gorenstein-projective. Take a right GP-acyclic sequence $0 \to K' \xrightarrow{j} G^{-n} \to K \to 0$ with G^{-n} Gorenstein-projective; see Corollary 3.1.2. Denote by K^{-i} the image of $G^{-i} \to G^{1-i}$; we identify K^{-n} with K, K^0 with M. Note that each sequence $0 \to K^{-i} \to G^{1-i} \to K^{1-i} \to 0$ is right GP-acyclic. Here we identify K^{-n-1} with K'. By Lemma 3.1.9 we can apply dimension-shift to the these sequences. Then we get $\mathrm{Ext}_{\mathrm{GP}}^1(K^{-n}, K') \simeq \mathrm{Ext}_{\mathrm{GP}}^{n+1}(M, K') = 0$. By Lemma 3.1.9 again this implies that the induced morphism $\mathrm{Hom}_A(G^{-n}, K') \to \mathrm{Hom}_A(K', K')$ by j is epic and then the monomorphism j splits. Hence K is a direct summand of G^{-n} and then it is Gorenstein-projective. $\qquad\square$

Denote by $(A\text{-GProj})^\perp$ the full subcategory of A-Mod consisting of modules M with the property that $\mathrm{Ext}_A^n(G, M) = 0$ for all $n \geq 1$ and G Gorenstein-projective.

The following observation is of interest.

Lemma 3.1.14. *Let M be an A-module. The following statements are equivalent:*

(1) $M \in (A\text{-GProj})^\perp$;

(2) $\underline{\mathrm{Hom}}_A(G, M) = 0$ *for all Gorenstein-projective modules G;*

(3) any epimorphism $P \to M$ with P projective is a right A-GProj-approximation.

Proof. Note that the full subcategory A-GProj of A-Mod is closed under syzygies and that every Gorenstein-projective module is a syzygy module. Then "(1) $\Leftrightarrow$ (2)" follows from Lemma 2.1.8. Note that (2) just means that any morphism from a Gorenstein-projective module to M factors through a projective module, and then factors though any fixed epimorphism $P \to M$ with P projective. Then "(2) $\Leftrightarrow$ (3)" follows immediately. $\square$

We observe the following result.

Corollary 3.1.15. *Let M be an A-module. Then $M \in (A\text{-GProj})^{\perp}$ if and only if $\Omega M \in (A\text{-GProj})^{\perp}$.*

Proof. Note that projective modules lie in $(A\text{-GProj})^{\perp}$ and by dimension-shift the full subcategory $(A\text{-GProj})^{\perp}$ is closed under taking cokernels of monomorphisms. Then the "if" part follows.

For the "only if" part, assume that $M \in (A\text{-GProj})^{\perp}$. Take a Gorenstein-projective module G and consider the short exact sequence $\xi \colon 0 \to \Omega M \to P \xrightarrow{f} M \to 0$ with P projective. By Lemma 3.1.14(3) the morphism f is a right A-GProj-approximation. Then from the long exact sequence obtained by applying $\operatorname{Hom}_A(G, -)$ to ξ one deduces that for all $n \geq 1$, $\operatorname{Ext}_A^n(G, \Omega M) = 0$. $\square$

We have the following comparison between GP-extension groups and the usual extension groups.

Proposition 3.1.16. *Let $M \in (A\text{-GProj})^{\perp}$. Then any projective resolution of M is a proper GP-resolution. Consequently, we have natural isomorphisms*

$$\operatorname{Ext}_{\operatorname{GP}}^n(M, N) \simeq \operatorname{Ext}_A^n(M, N)$$

for all $n \geq 0$ and all modules N.

Proof. Consider a projective resolution $P^{\bullet} \to M$. By an iterated application of Corollary 3.1.15 we infer that all the cocycles of $P^{\bullet}$ lie in $(A\text{-GProj})^{\perp}$. Then applying Lemma 3.1.14(3) repeatedly we infer that the projective resolution is a proper GP-resolution. $\square$

For an artin algebra A denote by Fin.dim A (resp. fin.dim A) the large (resp. small) *finistic dimension* of A. Observe that modules of finite projective dimension lie in $(A\text{-GProj})^{\perp}$; see Corollary 2.1.3. Then we have the following immediate consequence of the proposition above.

Corollary 3.1.17. *Let A be an artin algebra. Then we have* Fin.dim $A \leq$ Fin.GP.dim A *and* fin.dim $A \leq$ fin.Gp.dim A. $\qquad \square$

Let us remark that the results and the arguments in this section carry over to Gorenstein-injective modules without any difficulty. In particular we define the GI-extension groups $\mathrm{Ext}^n_{\mathrm{GI}}(M, N)$ by using the *proper Gorenstein-injective coresolution* of the module N.

3.2 Modules of Finite Gorenstein Dimension

In this section we study the class of modules having finite Gorenstein-projective dimension.

Let A be an artin algebra. Recall that for an A-module M its Gorenstein-projective dimension is denote by GP.dim M. For each $n \geq 0$ denote by $\mathrm{GP}^{\leq n}(A)$ the full subcategory of A-Mod consisting of modules M with GP.dim $M \leq n$, and denote by $\mathrm{GP}^{<\infty}(A)$ the full subcategory of A-Mod consisting of modules of finite Gorenstein-projective dimension. Observe that $\mathrm{GP}^{\leq 0}(A) = A$-GProj.

The following result is basic; also see [35, Theorem 2.10].

Lemma 3.2.1. (Auslander-Buchweitz) *Let M be an A-module which fits into an exact sequence $0 \to G^{-n} \to \cdots \to G^{-1} \to G^0 \to M \to 0$ with each G^{-i} Gorenstein-projective. Then there is a proper GP-resolution $0 \to P^{-n} \to \cdots \to P^{-2} \to P^{-1} \to G \to M \to 0$ such that G is Gorenstein-projective and each P^{-i} is projective. In particular, we have GP.dim $M \leq n$.*

Proof. The existence of the second exact sequence follows from [4, Theorem 1.1]. We will show that the sequence $0 \to P^{-n} \overset{d^{-n}}{\to} \cdots \to P^{-2} \overset{d^{-2}}{\to} P^{-1} \overset{d^{-1}}{\to} G \overset{\varepsilon}{\to} M \to 0$ is a proper GP-resolution. Note that for each $1 \leq i \leq n$ the module Im d^{-i} has finite projective dimension, and hence it lies in $(A\text{-GProj})^\perp$. By Lemma 3.1.14(3) each morphism $P^{-i} \to$ Im d^{-i} is a right A-GProj-approximation. Since Ker $\varepsilon =$ Im d^{-1} lies in $(A\text{-GProj})^\perp$, it follows immediately that ε is a right A-GProj-approximation. From these we conclude that the sequence is a proper GP-resolution. $\qquad \square$

The following result is contained in [4, Proposition 2.1]; also see [35, Theorem 2.20].

Proposition 3.2.2. *Let M be an A-module of finite Gorenstein-projective dimension and let $n \geq 0$. The following statements are equivalent:*

(1) GP.dim $M \leq n$;

(2) $\mathrm{Ext}_A^i(M, L) = 0$ *for $i \geq n + 1$ and L of finite projective dimension;*

(3) $\mathrm{Ext}_A^i(M, A) = 0$ *for $i \geq n + 1$;*

(4) $\mathrm{Ext}_A^{n+1}(M, L) = 0$ *for L of finite projective dimension.*

Proof. For "(1) $\Rightarrow$ (2)", take an exact sequence $0 \to G^{-n} \to \cdots \to G^{-1} \to G^0 \to M \to 0$ with each G^{-i} Gorenstein-projective. Note that $\mathrm{Ext}_A^i(G^{-j}, L) = 0$ for all $i \geq 1$ and $0 \leq j \leq n$. By dimension-shift we have that for $i \geq n + 1$, $\mathrm{Ext}_A^i(M, L) \simeq \mathrm{Ext}_A^{i-1}(M^{-1}, L) \simeq \mathrm{Ext}_A^{i-2}(M^{-2}, L) \simeq \cdots \simeq \mathrm{Ext}_A^{i-n}(G^{-n}, L) = 0$, where each M^{-i} is the image of $G^{-i} \to G^{1-i}$.

The implication "(2) $\Rightarrow$ (3)" is trivial. For "(3) $\Rightarrow$ (4)", first note that $\mathrm{Ext}_A^i(M, P) = 0$ for $i \geq n + 1$ and all projective modules P. Then (4) follows by applying dimension-shift to a projective resolution of L.

To see "(4) $\Rightarrow$ (1)", we apply the lemma above to get a long exact sequence $0 \to K \to P^{-n} \overset{d^{-n}}{\to} \cdots \to P^{-1} \to G \to M \to 0$ such that G is Gorenstein-projective, each P^{-j} is projective and K has finite projective dimension. Note that $\mathrm{Ext}_A^i(G, K) = 0$ for all $i \geq 1$. We can apply dimension-shift to get that $\mathrm{Ext}_A^1(\mathrm{Im}\, d^{-n}, K) \simeq \mathrm{Ext}_A^{n+1}(M, K) = 0$. Consequently, the monomorphism $K \to P^{-n}$ splits. Then we get a GP-resolution of M of length n; see Lemma 3.2.1. $\qquad\square$

The following important result is contained in [4, section 3]; compare [35, Theorem 2.24].

Proposition 3.2.3. (Auslander-Buchweitz) *Let A be an artin algebra. If any two of the three terms in a short exact sequence of A-modules have finite Gorenstein-projective dimension, then so does the remaining term. Moreover, a direct summand of a module of finite Gorenstein-projective dimension is of finite Gorenstein-projective dimension.* $\square$

We get the following result by applying Propositions 3.2.3 and 3.2.2; compare [35, Proposition 2.18].

Corollary 3.2.4. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of A-modules. Then we have* $\mathrm{GP.dim}\, M \leq \max\{\mathrm{GP.dim}\, L, \mathrm{GP.dim}\, N\}$, $\mathrm{GP.dim}\, N \leq \max\{\mathrm{GP.dim}\, L + 1, \mathrm{GP.dim}\, M\}$ *and* $\mathrm{GP.dim}\, L \leq \max\{\mathrm{GP.dim}\, M, \mathrm{GP.dim}\, N - 1\}$. $\square$

The following result is of interest; compare [35, Theorem 2.20].

Theorem 3.2.5. *Let M be an A-module and let $n \geq 0$. Then the following statements are equivalent:*

(1) GP.dim $M \leq n$;

(2) there exists an exact sequence $0 \to G^{-n} \to \cdots \to G^{-1} \to G^0 \to M \to 0$ with each G^{-i} Gorenstein-projective;

(3) for each exact sequence $0 \to K \to G^{1-n} \to \cdots \to G^{-1} \to G^0 \to M \to 0$ with each G^{-i} Gorenstein-projective we have that K is Gorenstein-projective.

Proof. We apply Lemma 3.2.1. Then the equivalence "(1) $\Leftrightarrow$ (2)" follows directly. The implication "(3) $\Rightarrow$ (2)" is trivial. To see "(1) $\Rightarrow$ (3)", first note by applying Proposition 3.2.3 repeatedly we get that the module K has finite Gorenstein-projective dimension. We apply dimension-shift to the given exact sequence. We get that $\mathrm{Ext}^i_A(K, A) \simeq \mathrm{Ext}^{i+n}_A(M, A) = 0$ for $i \geq 1$. By Proposition 3.2.2 we get GP.dim $K = 0$, that is, K is Gorenstein-projective. $\square$

The following immediate consequence of Theorem 3.2.5 is contained implicitly in [6, Proposition 4.2]; also see [12, Proposition 3.10].

Corollary 3.2.6. *Let A be an artin algebra. Then A is Gorenstein if and only if* gl.GP.dim $A < \infty$. *In this case we have* G.dim $A =$ gl.GP.dim A.

Proof. We apply Theorem 3.2.5 and Theorem 2.3.7. $\square$

The following result is due to Holm; compare [35, Theorem 2.28].

Theorem 3.2.7. (Holm) *Let A be an artin algebra. Then we have* Fin.dim $A =$ Fin.GP.dim A *and* fin.dim $A =$ fin.Gp.dim A.

Proof. We only show the first equality and the second is proved similarly. We have observed that Fin.dim $A \leq$ Fin.GP.dim A in Corollary 3.1.17. By Lemma 3.2.1 we infer that Fin.GP.dim $A \leq$ Fin.dim $A + 1$. Hence if Fin.GP.dim A is infinite we are done.

Now assume that Fin.GP.dim $A = m$ such that $0 < m < \infty$. Take a module M with GP.dim $M = m$. By Lemma 3.2.1 there is a short exact sequence $0 \to K \to G \to M \to 0$ such that G is Gorenstein-projective and proj.dim $K = m - 1$. Take a short exact sequence $0 \to G \to P \to G' \to 0$ with P projective and G' Gorenstein-projective. Hence we get two short exact sequences

$$0 \longrightarrow K \longrightarrow P \longrightarrow L \longrightarrow 0 \text{ and } 0 \longrightarrow M \longrightarrow L \longrightarrow G' \longrightarrow 0.$$

Since M is not Gorenstein-projective, by the second exact sequence we have that L is not Gorenstein-projective; see Proposition 2.1.7. In particular, it is not projective. By the first exact sequence we have proj.dim $L = m$. We are done. $\square$

The following observation is rather easy.

Proposition 3.2.8. *Let A be an artin algebra. Then we have*

$$\text{gl.GP.dim } A = \sup\{\text{GP.dim } M|\ M \in A\text{-mod}\}.$$

Proof. Choose a complete set of representatives of pairwise non-isomorphic simple A-modules $\{S_1, \cdots, S_n\}$. Note that each A-module has a finite filtration with semisimple factors. Using the fact that GP.dim $\oplus_i M_i = \sup\{\text{GP.dim } M_i\}$. We apply Corollary 3.2.4 repeatedly to infer that GP.dim $M \leq \max\{\text{GP.dim } S_1, \cdots, \text{GP.dim } S_n\}$ for all A-modules M. $\qquad\square$

We end this section with a discussion on a certain balanced property of Gorenstein extension groups.

The following observation is contained in the proof of [36, Lemma 3.4].

Lemma 3.2.9. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence with $L \in {}^{\perp}(A\text{-GInj})$ and let X be a Gorenstein-projective module. Then the following induced sequence*

$$0 \longrightarrow \text{Hom}_A(N, X) \longrightarrow \text{Hom}_A(M, X) \longrightarrow \text{Hom}_A(L, X) \longrightarrow 0$$

is exact.

Proof. Denote the morphism $L \to M$ by f. It suffices to show that for each morphism $a\colon L \to X$ there exists a morphism $b\colon M \to X$ such that $b \circ f = a$. Since $L \in {}^{\perp}(A\text{-GInj})$, the morphism a factors through an injective module I, say there are morphisms $a'\colon N \to I$ and $i\colon I \to X$ such that $i \circ a' = a$; compare Lemma 3.1.14. By the injectivity of I there is a morphism $b'\colon M \to I$ with $b' \circ f = a'$. Set $b = i \circ b'$. $\qquad\square$

The following balanced property of Gorenstein extension groups is due to Holm; see [36, Theorem 3.6].

Theorem 3.2.10. (Holm) *Let A be an artin algebra. Let M and N be A-modules with finite Gorenstein-projective (resp. Gorenstein-injective) dimension. Then for each $n \geq 0$ there is an isomorphism*

$$\text{Ext}^n_{\text{GP}}(M, N) \simeq \text{Ext}^n_{\text{GI}}(M, N)$$

which is functorial in both M and N.

Proof. Take a proper GP-resolution $0 \to P^{-n} \to \cdots \to P^{-1} \to G^0 \to M \to 0$ with G Gorenstein-projective and each P^{-i} projective. Write it as $G^\bullet \to M$. Note that all the cocycles of $G^\bullet$ (but M) have finite projective dimension and then lie in $^\perp(A\text{-GInj})$. Let X be a Gorenstein-injective A-module. Applying Lemma 3.2.9 repeatedly we get that the induced sequence $\mathrm{Hom}_A(M, X) \to \mathrm{Hom}_A(G^\bullet, X)$ is acyclic.

Take a proper GI-coresolution $0 \to N \to R \to I^1 \to \cdots \to I^m \to 0$ with R Gorenstein-injective and each I^i injective. Write it as $N \to R^\bullet$. Similarly as above we have that for each Gorenstein-projective module G the induced sequence $\mathrm{Hom}_A(G, N) \to \mathrm{Hom}_A(G, R^\bullet)$ is acyclic. Now consider the Hom bicomplex $\mathrm{Hom}_A(G^\bullet, R^\bullet)$ and the associated two spectral sequences. The two spectral sequences collapse to $\mathrm{Ext}^*_{\mathrm{GP}}(M, N)$ and $\mathrm{Ext}^*_{\mathrm{GI}}(M, N)$, respectively; for details, consult [25, Proposition 2.3]. Then we are done. $\square$

Recall that an artin algebra A is said to be *virtually Gorenstein* provided that $(A\text{-GProj})^\perp = {}^\perp(A\text{-GInj})$; see [15, 12].

We observe the following characterization of virtually Gorenstein algebras; see [15, Chapter X, Theorem 3.4(v)].

Proposition 3.2.11. *Let A be an artin algebra. Then A is virtually Gorenstein if and only if for all modules M and N and $n \geq 0$, there are isomorphisms*

$$\mathrm{Ext}^n_{\mathrm{GP}}(M, N) \simeq \mathrm{Ext}^n_{\mathrm{GI}}(M, N),$$

which are functorial in both M and N.

Proof. For the "if" part, we apply Proposition 3.1.16. Observe that for $M \in (A\text{-GProj})^\perp$ and $R \in A\text{-GInj}$ we have $\mathrm{Ext}^n_A(M, R) \simeq \mathrm{Ext}^n_{\mathrm{GP}}(M, R) \simeq \mathrm{Ext}^n_{\mathrm{GI}}(M, R) = 0$ for $n \geq 1$. This shows that $(A\text{-GProj})^\perp \subseteq {}^\perp(A\text{-GInj})$. Dually one shows that $^\perp(A\text{-GInj}) \subseteq (A\text{-GProj})^\perp$, and then A is virtually Gorenstein.

For the "only if" part, we apply Corollary B.0.25 (and its dual). Then the same proof as in the theorem above works. $\square$

3.3 Gorenstein Derived Categories

In this section we will briefly study Gorenstein derived categories of an artin algebra A. The GP-extension and GI-extension groups of two A-modules are encoded as the Hom spaces between certain objects in the Gorenstein derived categories.

Let A be an artin algebra. Denote by $\mathbf{K}(A\text{-Mod})$ the *homotopy category* of complexes in A-Mod. For a complex $X^\bullet = (X^n, d_X^n)_{n \in \mathbb{Z}}$ its *shifted complex* $X^\bullet[1]$ is defined

such that $(X^\bullet[1])^n = X^{n+1}$ and $d^n_{X^\bullet[1]} = -d^{n+1}_X$. This gives rise to an automorphism $[1]\colon \mathbf{K}(A\text{-Mod}) \to \mathbf{K}(A\text{-Mod})$. We denote by $[n]$ the n-th power of $[1]$ for $n \in \mathbb{Z}$. A module M is usually identified with the *stalk complex* $\cdots \to 0 \to M \to 0 \to \cdots$ concentrated at degree zero. Then for each n the stalk complex $M[n]$ has M at degree $-n$ and zero elsewhere.

For a chain map $f^\bullet\colon X^\bullet \to Y^\bullet$ its *mapping cone* $\mathrm{Cone}(f^\bullet)$ is complex defined such that for each $n \in \mathbb{Z}$

$$\mathrm{Cone}(f^\bullet)^n = Y^n \oplus X^{n+1} \text{ and } d^n_{\mathrm{Cone}(f^\bullet)} = \begin{pmatrix} d^n_Y & f^{n+1} \\ 0 & -d^n_X \end{pmatrix},$$

where d^n_X and d^n_Y are differentials of $X^\bullet$ and $Y^\bullet$, respectively. The homotopy category $\mathbf{K}(A\text{-Mod})$ has a canonical triangulated structure such that all exact triangles are isomorphic to the *standard triangles* $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{\binom{1}{0}} \mathrm{Cone}(f^\bullet) \xrightarrow{(0\ 1)} X^\bullet[1]$ associated to chain maps $f^\bullet$; for details, see [63, 33, 39].

For an additive subcategory $\mathcal{X}$ of A-Mod denote by $\mathbf{K}^-(\mathcal{X})$ (resp. $\mathbf{K}^+(\mathcal{X})$, $\mathbf{K}^b(\mathcal{X})$) the full subcategory of $\mathbf{K}(A\text{-Mod})$ consisting of bounded above (resp. bounded below, bounded) complexes in $\mathcal{X}$; they are triangulated subcategories.

We call a chain map $f^\bullet\colon X^\bullet \to Y^\bullet$ a *right GP-quasi-isomorphism* provided that for each Gorenstein projective module G the induced chain map $\mathrm{Hom}_A(G, f^\bullet)$ is a quasi-isomorphism. Observe that a complex $X^\bullet$ is right GP-acyclic if and only if the trivial map $X^\bullet \to 0$ is a right GP-quasi-isomorphism. Moreover, a chain map $f^\bullet$ is a right GP-quasi-isomorphism if and only if its mapping cone $\mathrm{Cone}(f^\bullet)$ is right GP-acyclic. Denote by GP-ac the full triangulated subcategory of $\mathbf{K}(A\text{-Mod})$ consisting of right GP-acyclic complexes; it is *thick*, that is, the subcategory GP-ac is closed under taking direct summands. Denote by Σ_{GP} the class of all the right GP-quasi-isomorphisms in $\mathbf{K}(A\text{-Mod})$; it is a saturated multiplicative system.

The following is initiated by Gao and Zhang [31]; also see [22].

Definition 3.3.1. *The* Gorenstein-projective derived category $\mathbf{D}_{\mathrm{GP}}(A)$ *of an artin algebra A is defined to be the Verdier quotient category*

$$\mathbf{D}_{\mathrm{GP}}(A)\colon = \mathbf{K}(A\text{-Mod})/\text{GP-ac} = \mathbf{K}(A\text{-Mod})[\Sigma^{-1}_{\mathrm{GP}}].$$

We denote by $Q\colon \mathbf{K}(A\text{-Mod}) \to \mathbf{D}_{\mathrm{GP}}(A)$ *the quotient functor.*

Observe that a chain map $f^\bullet\colon X^\bullet \to Y^\bullet$ is a right GP-quasi-isomorphism if and only if $Q(f^\bullet)$ is an isomorphism in $\mathbf{D}_{\mathrm{GP}}(A)$.

Dually one defines the *Gorenstein-injective derived category* of A to be $\mathbf{D}_{\mathrm{GI}}(A)\colon = \mathbf{K}(A\text{-Mod})/\text{GI-ac}$, where GI-ac is the full triangulated subcategory of $\mathbf{K}$-Mod consisting

of left GI-acyclic complexes. Both $\mathbf{D}_{\mathrm{GP}}(A)$ and $\mathbf{D}_{\mathrm{GI}}(A)$ are called *Gorenstein derived categories* of A. In what follows we will mainly consider the Gorenstein-projective derived category.

The following result is basic.

Lemma 3.3.2. *Let* $X^{\bullet} \in \mathbf{K}^{-}(A\text{-GProj})$ *and* $Y^{\bullet} \in \mathrm{GP}\text{-ac}$. *Then we have*

$$\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(X^{\bullet}, Y^{\bullet}) = 0.$$

Proof. Take a chain map $f^{\bullet} \colon X^{\bullet} \to Y^{\bullet}$. Without loss of generality we assume that $X^n = 0$ for $n > 0$. Then we have $d_Y^0 \circ f^0 = 0$. Note that by assumption the Hom complex $\mathrm{Hom}_A(X^0, Y^{\bullet})$ is acyclic. This implies that there exists $h^0 \colon X^0 \to Y^{-1}$ such that $f^0 = d_Y^{-1} \circ h^0$. Set $h^n = 0$ for $n \geq 1$.

We make induction on $i \geq 0$. Assume that the morphisms $h^n \colon X^n \to Y^{n-1}$ are defined for $n \geq -i$ such that $f^n = d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n$ for all $n \geq -i$. We will construct h^{-i-1}. Note that

$$d_Y^{-i-1} \circ (f^{-i-1} - h^{-i} \circ d_X^{-i-1}) = f^{-i} \circ d_X^{-i-1} - d_Y^{-i-1} \circ h^{-i} \circ d_X^{-i-1}$$
$$= h^{-i+1} \circ d_X^{-i} \circ d_X^{-i-1} = 0.$$

By assumption the Hom complex $\mathrm{Hom}_A(X^{-i-1}, Y^{\bullet})$ is acyclic. It follows that there exists $h^{-i-1} \colon X^{-i-1} \to Y^{-i-2}$ such that $f^{-i-1} - h^{-i} \circ d_X^{-i-1} = d_Y^{-i-2} \circ h^{-i-1}$. Continuing this argument we find a homotopy $\{h^n\}_{n \in \mathbb{Z}}$ of the chain map $f^{\bullet}$. $\square$

We have the following direct consequence.

Corollary 3.3.3. *Let* $X^{\bullet} \in \mathbf{K}(A\text{-Mod})$ *and* $Y^{\bullet} \in \mathbf{K}^{-}(A\text{-Mod}) \cap \mathrm{GP}\text{-ac}$. *Then we have*

$$\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(X^{\bullet}, Y^{\bullet}) = 0.$$

$\square$

The following consequence will be crucial to us.

Corollary 3.3.4. *Let* $X^{\bullet} \in \mathbf{K}^{-}(A\text{-GProj})$ *and* $Y^{\bullet} \in \mathbf{K}(A\text{-Mod})$. *Then the natural map*

$$\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(X^{\bullet}, Y^{\bullet}) \to \mathrm{Hom}_{\mathbf{D}_{\mathrm{GP}}(A)}(X^{\bullet}, Y^{\bullet})$$

sending $f^{\bullet}$ *to* $Q(f^{\bullet})$ *is an isomorphism. In particular, the composite* $\mathbf{K}^{-}(A\text{-GProj}) \hookrightarrow \mathbf{K}(A\text{-Mod}) \xrightarrow{Q} \mathbf{D}_{\mathrm{GP}}(A)$ *is fully faithful.*

Proof. We apply Lemma 3.3.2. Then this result is an immediate consequence of [63, §2, 5-3 Proposition]. $\square$

The following observation highlights Gorenstein derived categories; see [31].

Theorem 3.3.5. *Let M, N be A-modules and let $n \in \mathbb{Z}$. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{GP}}(A)}(M, N[n]) \simeq \mathrm{Ext}^n_{\mathrm{GP}}(M, N).$$

Proof. Take a proper GP-resolution $\varepsilon \colon G^\bullet \to M$. View M as a stalk complex concentrated in degree zero and $G^\bullet$ as a complex belonging to $\mathbf{K}^-(A\text{-GProj})$. Note that ε is a right GP-quasi-isomorphism. Then $G^\bullet$ is isomorphic to M in $\mathbf{D}_{\mathrm{GP}}(A)$. We apply Corollary 3.3.4. Then we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}_{\mathrm{GP}}(A)}(M, N[n]) &\simeq \mathrm{Hom}_{\mathbf{D}_{\mathrm{GP}}(A)}(G^\bullet, N[n]) \\
&\simeq \mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(G^\bullet, N[n]) \\
&\simeq H^n(\mathrm{Hom}_A(G^\bullet, N)).
\end{aligned}$$

By definition we have $\mathrm{Ext}^n_{\mathrm{GP}}(M, N) = H^n(\mathrm{Hom}_A(G^\bullet, N))$. We are done. $\qquad\square$

We will finish this section with a remark on Gorenstein derived categories. For more, we refer to [31] and [22].

Consider $\mathcal{E}_{\mathrm{GP}}$ the class of short exact sequence of A-modules on which each functor $\mathrm{Hom}_A(G, -)$ is exact for $G \in A$-GProj. Then the pair $(A\text{-Mod}, \mathcal{E}_{\mathrm{GP}})$ is an exact category in the sense of Quillen. We will denote this exact category by $A\text{-Mod}_{\mathrm{GP}}$. Following Neeman ([51, section 1]) a complex $X^\bullet = (X^n, d_X^n)_{n \in \mathbb{Z}}$ is *acyclic* in $\mathbf{K}(A\text{-Mod}_{\mathrm{GP}})$ if and only if there are factorizations $d_X^n \colon X^n \xrightarrow{p^n} Z^{n+1} \xrightarrow{i^{n+1}} X^{n+1}$ such that for each n, $0 \to Z^n \xrightarrow{i^n} X^n \xrightarrow{p^n} Z^{n+1} \to 0$ is a short exact sequence belonging to $\mathcal{E}_{\mathrm{GP}}$. Observe that a complex is acyclic in $\mathbf{K}(A\text{-Mod}_{\mathrm{GP}})$ if and only if it is right GP-acyclic.

Following Neeman again ([51, Remark 1.6]; also see [46, sections 11,12]) the *derived category* $\mathbf{D}(A\text{-Mod}_{\mathrm{GP}})$ of the exact category $A\text{-Mod}_{\mathrm{GP}}$ is defined by

$$\mathbf{D}(A\text{-Mod}_{\mathrm{GP}}) \colon = \mathbf{K}(A\text{-Mod}_{\mathrm{GP}})/\text{GP-ac}.$$

Dually one may also consider the exact category $A\text{-Mod}_{\mathrm{GI}}$ with the exact structure given by short exact sequences of A-modules on which $\mathrm{Hom}_A(-, I)$ is exact for each $I \in A$-GInj. Then one has the derived category $\mathbf{D}(A\text{-Mod}_{\mathrm{GI}})$.

The last result can be viewed as a remark: roughly speaking, Gorenstein derived categories are not "new". This remark makes possible to apply the general results on the derived categories of exact categories to Gorenstein homological algebra; see [46].

Proposition 3.3.6. *Let A be an artin algebra. Then we have $\mathbf{D}_{\mathrm{GP}}(A) = \mathbf{D}(A\text{-Mod}_{\mathrm{GP}})$ and $\mathbf{D}_{\mathrm{GI}}(A) = \mathbf{D}(A\text{-Mod}_{\mathrm{GI}})$.* $\qquad\square$

Appendix A

Cotorsion Pairs

In this section we review the theory of cotorsion pairs and other relevant notions. The main references are [59], [28, Chapter 7] and [32, Chapter 2].

Throughout $\mathcal{A}$ is an abelian category. Let $\mathcal{X}$ be a full additive subcategory of $\mathcal{A}$. Let $M \in \mathcal{A}$ be an object. A *right $\mathcal{X}$-approximation* of M is a morphism $f \colon X \to M$ such that $X \in \mathcal{X}$ and any morphism $X' \to M$ from an object $X' \in \mathcal{X}$ factors through f. Dually one has the notion of *left $\mathcal{X}$-approximation* ([9]). A right (resp. left) $\mathcal{X}$-approximation is also known as an *$\mathcal{X}$-precover* (resp. *$\mathcal{X}$-preenvelop*) ([24]). The subcategory $\mathcal{X} \subseteq \mathcal{A}$ is said to be *contravariantly finite* (resp. *covariantly finite*) provided that each object in $\mathcal{A}$ has a right (resp. left) $\mathcal{X}$-approximation. The subcategory $\mathcal{X} \subseteq \mathcal{A}$ is said to be *functorially finite* provided that it is both contravariantly finite and covariantly finite.

Let $\mathcal{X} \subseteq \mathcal{A}$ be a full additive subcategory. Denote by $\mathcal{X}^{\perp_1} = \{Y \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0 \text{ for all } X \in \mathcal{X}\}$. A *special right $\mathcal{X}$-approximation* of an object M is an epimorphism $\phi \colon X \to M$ such that $X \in \mathcal{X}$ and the kernel $\mathrm{Ker}\,\phi$ lies in $\mathcal{X}^{\perp_1}$. Observe that a special right $\mathcal{X}$-approximation is a right $\mathcal{X}$-approximation. Dually one has the notation $^{\perp_1}\mathcal{X}$ and the notion of *special left $\mathcal{X}$-approximation*.

A right $\mathcal{X}$-approximation $f \colon X \to M$ is said to be *minimal* provided that any endomorphism $\theta \colon X \to X$ with $f \circ \theta = f$ is necessarily an isomorphism ([9]). Such a minimal right $\mathcal{X}$-approximation is also known as an *$\mathcal{X}$-cover*. Dually one has the notion of *$\mathcal{X}$-envelop* ([24]).

We have the following useful lemma.[1]

Lemma A.0.7. (Wakamatsu's Lemma) *Let $\mathcal{X} \subseteq \mathcal{A}$ be a full additive subcategory which is closed under extensions. Let $f \colon X \to M$ be an $\mathcal{X}$-cover. Then $\mathrm{Ker}\,f$ lies in $\mathcal{X}^{\perp_1}$. In particular, an epic $\mathcal{X}$-cover is a special right $\mathcal{X}$-approximation.* $\qquad\qquad\square$

---

[1]This lemma is supposed to be found in [64], while I do not find it there. For a proof, see [65, Lemma 2.1.1]; also see [28] and [32, Lemma 2.1.13].

The notion of cotorsion pair presented below is different from the original one[2]; see [59]. In our opinion this one is more useful.

Definition A.0.8. (Salce) *A pair* $(\mathcal{F}, \mathcal{C})$ *of full additive subcategories in* $\mathcal{A}$ *is called a cotorsion pair if the following conditions are satisfied:*

(C0) *the subcategories* $\mathcal{F}$ *and* $\mathcal{C}$ *are closed under taking direct summands;*

(C1) $\mathrm{Ext}^1_{\mathcal{A}}(F, C) = 0$ *for all* $F \in \mathcal{F}$ *and* $C \in \mathcal{C}$;

(C2) *each object* $M \in \mathcal{A}$ *fits into a short exact sequence* $0 \to C \to F \to M \to 0$ *with* $F \in \mathcal{F}$ *and* $C \in \mathcal{C}$;

(C3) *each object* $M \in \mathcal{A}$ *fits into a short exact sequence* $0 \to M \to C' \to F' \to 0$ *with* $C' \in \mathcal{C}$ *and* $F' \in \mathcal{F}$. $\qquad\qquad\qquad \square$

Remark A.0.9. *(1). We have assumed that both* $\mathcal{F}$ *and* $\mathcal{C}$ *are closed under taking direct summands. It follows immediately from the conditions above that* $\mathcal{F} = {}^{\perp_1}\mathcal{C}$ *and* $\mathcal{C} = \mathcal{F}^{\perp_1}$. *In particular, one infers that both* $\mathcal{F}$ *and* $\mathcal{C}$ *are closed under extensions.*

(2). The condition (C2) claims that each object M *has a special right* $\mathcal{F}$*-approximation. Hence the subcategory* $\mathcal{F} \subseteq \mathcal{A}$ *is contravariantly finite. Dually* $\mathcal{C}$ *is covariantly finite.*

Recall that for a full additive subcategory $\mathcal{X}$ in $\mathcal{A}$ we denote by $\mathrm{fac}\,\mathcal{X}$ (resp. $\mathrm{sub}\,\mathcal{X}$) the full subcategory of $\mathcal{A}$ consisting of factor objects (resp. sub objects) of objects in $\mathcal{X}$.

We have the following result; compare [59, Corollary 2.4]. Let us remark that using Wakamatsu's Lemma one deduces [5, Proposition 1.9] from the result below quite easily.
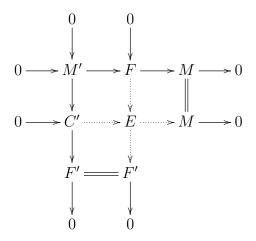
Lemma A.0.10. (Salce's Lemma) *Let* $(\mathcal{F}, \mathcal{C})$ *be a pair of full additive subcategories in* $\mathcal{A}$ *such that* $\mathcal{F} = {}^{\perp_1}\mathcal{C}$ *and* $\mathcal{C} = \mathcal{F}^{\perp_1}$. *Then the following statements are equivalent:*

(1) the pair $(\mathcal{F}, \mathcal{C})$ *is a cotorsion pair;*

(2) $\mathrm{fac}\,\mathcal{F} = \mathcal{A}$ *and the condition (C3) holds;*

(3) $\mathrm{sub}\,\mathcal{C} = \mathcal{A}$ *and the condition (C2) holds.*

Proof. We will only show the equivalence "(1) $\Leftrightarrow$ (2)". The implication "(1) $\Rightarrow$ (2)" is trivial. For the converse, let $M \in \mathcal{A}$. Since $\mathrm{fac}\,\mathcal{F} = \mathcal{A}$ we may take a short exact sequence $0 \to M' \to F \to M \to 0$ with $F \in \mathcal{F}$. Applying the condition (C3) to M' we

---

[2]The cotorsion pair introduced here is also referred as a *complete* cotorsion pair; see [28, 32].

get a short exact sequence $0 \to M' \to C' \to F' \to 0$ with $C' \in \mathcal{C}$ and $F' \in \mathcal{F}$. Consider the following pushout diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M' & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C' & \dashrightarrow & E & \dashrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & F' & = & F' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

By $\mathcal{F} = {}^{\perp_1}\mathcal{C}$ the subcategory $\mathcal{F}$ is closed under extensions. Consider the short exact sequence in the middle column. We infer that $E \in \mathcal{F}$. Then the short exact sequence in the middle row proves the condition (C2) for M. $\qquad\square$

We will recall a remarkable result due to Auslander and Buchweitz on cotorsion pairs.

Let $\mathcal{X} \subseteq \mathcal{A}$ be a full additive subcategory and let $n \geq 0$. Set $\mathcal{X}^n$ to the full subcategory of $\mathcal{A}$ consisting of objects M with an exact sequence $0 \to X^{-n} \to \cdots \to X^{-1} \to X^0 \to M \to 0$ such that each $X^{-i} \in \mathcal{X}$. Note that $\mathcal{X}^0 = \mathcal{X}$. Set $\mathcal{X}^{-1} = 0$. Denote by $\widehat{\mathcal{X}}$ the union of all these $\mathcal{X}^n$'s.

Consider a full additive subcategory $\omega \subseteq \mathcal{X}$. We say that ω *cogenerates* $\mathcal{X}$ provided that each object X fits into a short exact sequence $0 \to X \to W \to X' \to 0$ with $W \in \omega$ and $X' \in \mathcal{X}$. In this case ω is said to be a *cogenerator* of $\mathcal{X}$.

The following is contained in [4, Theorem 1.1]. Let us remark that it is proved directly by using induction on n and taking suitable pushout of short exact sequences.

Proposition A.0.11. (Auslander-Buchweitz's decomposition theorem) *Let $\mathcal{X} \subseteq \mathcal{A}$ be a full additive subcategory which is closed under extensions. Let ω be a cogenerator of $\mathcal{X}$ and let $n \geq 0$. Then for each $C \in \mathcal{X}^n$, there are short exact sequences*

$$
0 \longrightarrow Y_C \longrightarrow X_C \longrightarrow C \longrightarrow 0,
$$
$$
0 \longrightarrow C \longrightarrow Y^C \longrightarrow X^C \longrightarrow 0,
$$

such that $X_C, X^C \in \mathcal{X}$, $Y^C \in \omega^{n-1}$ and $Y_C \in \omega^n$. $\qquad\square$

We say that a cogenerator ω of $\mathcal{X}$ is *Ext-injective* provided that $\operatorname{Ext}_{\mathcal{A}}^n(X, W) = 0$ for all $n \geq 1$, $X \in \mathcal{X}$ and $W \in \omega$. This implies by dimension-shift that $\operatorname{Ext}_{\mathcal{A}}^1(X, C) = 0$ for $X \in \mathcal{X}$ and $C \in \widehat{\omega}$.

The proof of the following result is contained in the one of [4, Proposition 3.6].

Theorem A.0.12. (Auslander-Buchweitz) *Let $\omega \subseteq \mathcal{X} \subseteq \mathcal{A}$ be full additive subcategories such that ω is closed under taking direct summands and $\mathcal{X}$ is closed under extensions and taking direct summands. Suppose that $\widehat{\mathcal{X}} = \mathcal{A}$ and that ω is an Ext-injective cogenerator of $\mathcal{X}$. Then $(\mathcal{X}, \widehat{\omega})$ is a cotorsion pair in $\mathcal{A}$.*

Proof. We have observed that $\widehat{\omega} \subseteq \mathcal{X}^{\perp_1}$. In view of the proposition above, it suffices to show that $\widehat{\omega}$ is closed under taking direct summands. In fact one shows that $\widehat{\omega} = \mathcal{X}^{\perp_1}$. It suffices to show that $\widehat{\omega} \supseteq \mathcal{X}^{\perp_1}$. Let $C \in \mathcal{X}^{\perp_1}$. Consider the short exact sequence $0 \to Y \to X \to C \to 0$ with $Y \in \widehat{\omega}$ and $X \in \mathcal{X}$. Note that $Y \in \widehat{\omega} \subseteq \mathcal{X}^{\perp_1}$ and then we have $X \in \mathcal{X}^{\perp_1}$, since $\mathcal{X}^{\perp_1}$ is closed under extensions. Note that X fits into a short exact sequence $0 \to X \to W \to X' \to 0$ with $W \in \omega$ and $X' \in \mathcal{X}$. By $X \in \mathcal{X}^{\perp_1}$ we infer that the sequence splits. Recall that ω is closed under taking direct summands. We deduce that $X \in \omega$ and then $C \in \widehat{\omega}$. $\square$

Remark A.0.13. *Assume that $\mathcal{X}^n = \mathcal{A}$. Then we have $\widehat{\omega} = \omega^n$ by [4, Proposition 3.6]. Then the cotorsion pair is given by $(\mathcal{X}, \omega^n)$.*

We will recall an important result which generates abundance of cotorsion pairs. Recall that an abelian category $\mathcal{A}$ *has enough projective objects* means that each object is a factor object of a projective object.

The following important result is contained in [37, Theorem 2.4]; compare [30, Theorem 10].

Theorem A.0.14. (Eklof-Trlifaj, Hovey) *Let $\mathcal{A}$ be a Grothendieck category with enough projective objects and let $\mathcal{S} \subseteq \mathcal{A}$ be a set of objects. Set $\mathcal{C} = \mathcal{S}^{\perp_1}$ and $\mathcal{F} = {}^{\perp_1}\mathcal{C}$. Then $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in $\mathcal{A}$.* $\square$

The cotorsion pair above is called the cotorsion pair *cogenerated* by the set $\mathcal{S}$ of objects.

Finally we discuss resolving subcategories. Let $\mathcal{A}$ be an abelian category with enough projective objects. A full additive subcategory $\mathcal{X}$ is a *resolving subcategory* provided that it contains all the projective objects and it is closed under extensions, taking kernels of epimorphisms and direct summands ([3, p.99]). A typical example of a resolving subcategory is given by ${}^{\perp}\mathcal{Y}$ where $\mathcal{Y}$ is a subcategory of $\mathcal{A}$ and ${}^{\perp}\mathcal{Y} = \{M \in \mathcal{A} \mid \operatorname{Ext}_{\mathcal{A}}^i(M, Y) = 0, \text{ for all } i \geq 1, Y \in \mathcal{Y}\}$. Dually if $\mathcal{A}$ has enough

injective objects, we have the notion of *coresolving subcategory* and a full subcategory of the form $\mathcal{X}^{\perp}$ is coresolving.

The following observation is rather easy; see [32, Lemma 2.2.10].

Proposition A.0.15. *Let $\mathcal{A}$ be an abelian category with enough projective and injective objects. Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair in $\mathcal{A}$. Then $\mathcal{X}$ is resolving if and only if $\mathcal{Y}$ is coresolving. In this case we have $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$.* $\qquad\square$

In the case of this proposition the cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be *hereditary*.

Appendix B

A Proof of Beligiannis's Theorem

In this section we will sketch a proof of Beligiannis's Theorem, which claims that for an artin algebra the stable category of Gorenstein-projective modules modulo projective modules is a compactly generated triangulated category; see Theorem 3.1.1. Our proof follows the one in [38]. We will present a result due to Beligiannis and Reiten on cotorsion pairs in the category of modules induced by the subcategory of Gorenstein-projective modules.

For an additive category $\mathfrak{a}$ denote by $\mathbf{K}(\mathfrak{a})$ the homotopy category of complexes in $\mathfrak{a}$ and by $\mathbf{K}^+(\mathfrak{a})$ (resp. $\mathbf{K}^-(\mathfrak{a})$ and $\mathbf{K}^b(\mathfrak{a})$) the subcategory consisting of bounded below (resp. bounded above and bounded) complexes. Recall that each of these homotopy categories has a canonical triangulated structure.

Throughout A will be an artin R-algebra where R is a commutative artinian ring. Denote by A-Mod the category of left A-modules and by A-Proj (resp. A-Inj) the full subcategory consisting of projective (resp. injective) A-modules. For an A-module X denote by $DX = \operatorname{Hom}_R(X, E)$ its Matlis dual where E is the minimal injective cogenerator for R. Note that DX is a right A-module and it is viewed as a left A^{op}-module. For an A-module X, set $X^* = \operatorname{Hom}_A(X, A)$ which has a natural left A^{op}-module structure.

Recall that a complex $P^\bullet$ of A-modules is *homotopically projective* provided that any chain map from $P^\bullet$ to an acyclic complex is homotopic to zero ([60]); if in addition the complex $P^\bullet$ consists of projective modules it is called *semi-projective*. For example, a bounded above complex of projective modules is semi-projective. For each complex $X^\bullet$ there is a *semi-projective resolution*, that is, a quasi-isomorphism $P^\bullet \to X^\bullet$ with $P^\bullet$ semi-projective; this semi-projective resolution is unique up to homotopy. Denote the complex $P^\bullet$ by $\mathbf{p}X^\bullet$. This gives rise to a triangle functor $\mathbf{p}\colon \mathbf{K}(A\text{-Mod}) \to \mathbf{K}(A\text{-Proj})$ which is called the *semi-projective resolution functor*. For details, see [39, Chapter 8].

Dually we have the notions of *homotopically injective* and *semi-injective* complex-

es. For every complex $X^\bullet$ there is a unique semi-injective resolution, that is, a quasi-isomorphism $X^\bullet \to \mathbf{i}X^\bullet$ with $\mathbf{i}X^\bullet$ semi-injective. This gives rise to the *semi-injective resolution functor* $\mathbf{i}\colon \mathbf{K}(A\text{-Mod}) \to \mathbf{K}(A\text{-Inj})$; for details, see [39, Chapter 8].

We begin with the following fact.

Lemma B.0.16. *Let $X^\bullet$ be a complex of A-modules. Then we have a natural isomorphism $D\mathbf{p}X^\bullet \simeq \mathbf{i}DX^\bullet$; if the complex $X^\bullet$ is bounded below, then we have $D\mathbf{i}X^\bullet \simeq \mathbf{p}DX^\bullet$.*

Proof. Applying D to the quasi-isomorphism $\mathbf{p}X^\bullet \to X^\bullet$ we get a quasi-isomorphism $DX^\bullet \to D\mathbf{p}X^\bullet$. Note that the complex $D\mathbf{p}X^\bullet$ consists of injective modules. To show the first isomorphism it suffices to show that $D\mathbf{p}X^\bullet$ is homotopically injective. For an acyclic complex $N^\bullet$, we have $\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(N^\bullet, D\mathbf{p}X^\bullet) \simeq \mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(\mathbf{p}X^\bullet, DN^\bullet) = 0$ since $\mathbf{p}X^\bullet$ is homotopically projective and $DN^\bullet$ is acyclic. The second isomorphism is easy to prove. $\qquad\square$

Recall that the *Nakayama functor* $\nu = DA \otimes_A\colon A\text{-Mod} \to A\text{-Mod}$ has a right adjoint $\nu^- = \mathrm{Hom}_A(DA, -)\colon A\text{-Mod} \to A\text{-Mod}$. Note that the Nakayama functor induces an equivalence $\nu\colon A\text{-Proj} \to A\text{-Inj}$ whose quasi-inverse is given by ν^-.

Applying the above to complexes, we have an equivalence $\nu\colon \mathbf{K}(A\text{-Proj}) \to \mathbf{K}(A\text{-Inj})$ with quasi-inverse given by ν^-.

Denote by A-mod (resp. A-proj, A-inj) the category of finitely generated (resp. projective, injective) A-modules. We note the following fact.

Lemma B.0.17. *Let $X^\bullet \in \mathbf{K}^-(A\text{-mod})$. Then we have a natural isomorphism $\nu^-\mathbf{i}X^\bullet \simeq (\mathbf{p}DX^\bullet)^*$.*

Proof. By Lemma B.0.16 we have $\mathbf{p}DX^\bullet \simeq D\mathbf{i}X^\bullet$. Observe that we may assume that $\mathbf{i}X^\bullet$ lies in $\mathbf{K}^-(A\text{-inj})$. Note that for a module $I \in A\text{-inj}$ we have a natural isomorphism $\nu^-I \simeq (DI)^*$. From this we infer that $(D\mathbf{i}X^\bullet)^* \simeq \nu^-\mathbf{i}X^\bullet$. We are done. $\qquad\square$

Denote by $\mathbf{K}^{-,b}(A\text{-proj})$ the full subcategory of $\mathbf{K}^-(A\text{-proj})$ consisting of complexes with only finitely many nonzero cohomologies. For the notions of compact objects and compactly generated triangulated categories, we refer to [52, 53].

The following result is due to Jørgensen ([43, Theorem 2.4]); also see Krause ([41, Example 2.6]) and Neeman ([56, Propositions 7.12 and 7.14]).

Lemma B.0.18. *Let A be an artin algebra. Then the homotopy category $\mathbf{K}(A\text{-Proj})$ is compactly generated; moreover, a complex is compact if and only if it is isomorphic to a complex of the form $(P^\bullet)^*$ for $P^\bullet \in \mathbf{K}^{-,b}(A^{\mathrm{op}}\text{-proj})$.* $\qquad\square$

Denote by $\mathbf{K}_{\mathrm{tac}}(A\text{-Proj})$ the full subcategory of $\mathbf{K}(A\text{-Proj})$ consisting of totally a-cyclic complexes. It is a triangulated subcategory. Denote by A-GProj the category of Gorenstein-projective A-modules; it is a Frobenius exact category such that its projective objects are equal to projective A-modules. Denote by A-$\underline{\mathrm{GProj}}$ the *stable category* of A-GProj modulo projective modules; it has a canonical triangulated structure. For details, see Chapter 2.

The following result is well known; see [41, Lemma 7.3].

Lemma B.0.19. *There is a triangle equivalence* A-$\underline{\mathrm{GProj}} \xrightarrow{\sim} \mathbf{K}_{\mathrm{tac}}(A\text{-Proj})$ *sending a Gorenstein-projective module to its complete resolution, the quasi-inverse of which is given by the functor* $Z^0(-)$ *of taking the zeroth cocycles.* $\qquad\square$

Recall that for a complex $X^\bullet$ of A-modules we denote by $H^n(X^\bullet)$ its n-th cohomology for each $n \in \mathbb{Z}$.

Lemma B.0.20. *Let* $X^\bullet \in \mathbf{K}(A\text{-Mod})$ *and* $I^\bullet \in \mathbf{K}(A\text{-Inj})$. *For each* $n \in \mathbb{Z}$, *we have the following natural isomorphisms*

$$\mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(A, X^\bullet[n]) \simeq H^n(X^\bullet) \text{ and } \mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(\mathbf{i}A, I^\bullet[n]) \simeq H^n(I^\bullet).$$

Proof. The first isomorphism is well known and the second follows from the first one and [41, Lemma 2.1]. $\qquad\square$

For a subset $\mathcal{S}$ of objects in a triangulated category $\mathcal{T}$, consider its *right orthogonal subcategory* $\mathcal{S}^\perp = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(S, X[n]) = 0, \text{ for all } n \in \mathbb{Z}, S \in \mathcal{S}\}$; it is a triangulated subcategory of $\mathcal{T}$.

We need the following important result; see [52, Theorem 2.1] and [38, Proposition 1.7(1)].

Lemma B.0.21. *Let* $\mathcal{T}$ *be a compactly generated triangulated category and let* $\mathcal{S}$ *be a set of compact objects in* $\mathcal{T}$. *Then the right orthogonal subcategory* $\mathcal{S}^\perp$ *is compactly generated.* $\qquad\square$

Now we are in the position to prove Beligiannis's Theorem.

Theorem B.0.22. (Beligiannis) *Let* A *be an artin algebra. Then the triangulated category* A-$\underline{\mathrm{GProj}}$ *is compactly generated.*

Proof. Recall that a complex $P^\bullet$ of projective A-modules is totally acyclic if and only if $P^\bullet$ and $\nu P^\bullet$ are both acyclic; see Lemma 2.2.9. Note that

$$H^n(\nu P^\bullet) \simeq \mathrm{Hom}_{\mathbf{K}(A\text{-Inj})}(\mathbf{i}A, \nu P^\bullet[n]) \simeq \mathrm{Hom}_{\mathbf{K}(A\text{-Proj})}(\nu^-\mathbf{i}A, P^\bullet[n]),$$

where the first isomorphism is by Lemma B.0.20 and the second follows from the e-quivalence $\nu\colon \mathbf{K}(A\text{-Proj}) \xrightarrow{\sim} \mathbf{K}(A\text{-Inj})$. By Lemma B.0.17 we have $\nu^{-}\mathbf{i}A \simeq (\mathbf{p}DA)^*$ and by Lemma B.0.18 it is compact in $\mathbf{K}(A\text{-Proj})$. Then it is direct to conclude that in $\mathbf{K}(A\text{-Proj})$ we have $\mathbf{K}_{\mathrm{tac}}(A\text{-Proj}) = \{A, \nu^{-}\mathbf{i}(A)\}^{\perp}$. Then the result follows from Lemmas B.0.19 and B.0.21. $\qquad\square$

We note the following consequence of Beligiannis's Theorem.

Corollary B.0.23. *Let A be an artin algebra. Then each complex $P^{\bullet}$ of projective A-modules fits into a triangle*

$$P_1^{\bullet} \longrightarrow P^{\bullet} \longrightarrow P_2^{\bullet} \to P_1^{\bullet}[1]$$

such that $P_1^{\bullet} \in \mathbf{K}_{\mathrm{tac}}(A\text{-Proj})$ and $P_2^{\bullet} \in \mathbf{K}_{\mathrm{tac}}(A\text{-Proj})^{\perp}$.

Proof. Consider the inclusion functor $\mathbf{K}_{\mathrm{tac}}(A\text{-Proj}) \hookrightarrow \mathbf{K}(A\text{-Proj})$; it preserves coproducts. By Beligiannis's Theorem and Lemma B.0.19 the category $\mathbf{K}_{\mathrm{tac}}(A\text{-Proj})$ is compactly generated. We apply Brown representability theorem to get a right adjoint of this inclusion ([53, Theorem 4.1]). The adjoint yields for each complex $P^{\bullet}$ such a triangle; see [56, Chapter 9]. $\qquad\square$

For a class $\mathcal{S}$ of A-modules, set $\mathcal{S}^{\perp} = \{X \in A\text{-Mod} \mid \mathrm{Ext}_A^i(S, X) = 0 \text{ for all } i \geq 1, S \in \mathcal{S}\}$. The following result is contained in [15, Chapter X, Theorem 2.4(iv)]; also see [14, Proposition 3.4]. Observe that it is stronger than Corollary 3.1.2.

Theorem B.0.24. (Beligiannis-Reiten) *Let A be an artin algebra. Then the pair $(A\text{-GProj}, (A\text{-GProj})^{\perp})$ is a cotorsion pair in A-Mod.*

Proof. Note that both A-GProj and $(A\text{-GProj})^{\perp}$ are closed under taking direct summands. Then it suffices to show that for an A-module M, there are short exact sequences $0 \to Y \to G \to M \to 0$ and $0 \to M \to Y' \to G' \to 0$ such that $G, G' \in A\text{-GProj}$ and $Y, Y' \in (A\text{-GProj})^{\perp}$.

We apply Corollary B.0.23 to a projective resolution $\mathbf{p}M$ of M. We get a triangle $P_1^{\bullet} \to \mathbf{p}M \to P_2^{\bullet} \to P_1^{\bullet}[1]$ with $P_1^{\bullet} \in \mathbf{K}_{\mathrm{tac}}(A\text{-Proj})$ and $P_2^{\bullet} \in \mathbf{K}_{\mathrm{tac}}(A\text{-Proj})^{\perp}$. Note that $P_1^{\bullet}$ is acyclic and then $H^n(P_2^{\bullet}) = 0$ for $n \neq 0$.

By rotating the triangle and adding some null-homotopic complexes to $P_1^{\bullet}$ and $P_2^{\bullet}[1]$, we may assume that we have a short exact sequence $0 \to P_2^{\bullet}[-1] \to P_1^{\bullet} \to \mathbf{p}M \to 0$ of complexes. For each complex $X^{\bullet}$ denote by $C^0(X^{\bullet})$ the cokernel of d_X^{-1}. Applying $C^0(-)$ to the sequence, we get a short exact sequence $0 \to Y \to G \to M \to 0$ of modules. Since $P_1^{\bullet}$ is totally acyclic, the module G is Gorenstein-projective. We claim that $Y \in$

$(A\text{-GProj})^{\perp}$. Note that the brutally truncated complex $\sigma^{\leq 0}(P_2^{\bullet}[-1])$ is a projective resolution of Y. We write $\mathbf{p}Y = \sigma^{\leq 0}(P_2^{\bullet}[-1])$. Take G' to be a Gorenstein-projective module and $P^{\bullet}$ its complete resolution. Then we have the following isomorphisms

$$\begin{aligned}
\underline{\mathrm{Hom}}_A(G, Y) &\simeq \mathrm{Hom}_{\mathbf{K}(A\text{-Mod})}(P^{\bullet}[-1], Y) \\
&\simeq \mathrm{Hom}_{\mathbf{K}(A\text{-Proj})}(P^{\bullet}[-1], \mathbf{p}Y) \\
&= \mathrm{Hom}_{\mathbf{K}(A\text{-Proj})}(P^{\bullet}[-1], \sigma^{\leq 0}(P_2^{\bullet}[-1])) \\
&\simeq \mathrm{Hom}_{\mathbf{K}(A\text{-Proj})}(P^{\bullet}[-1], P_2^{\bullet}[-1]) = 0,
\end{aligned}$$

where the first isomorphism is easy to see, the second follows from the dual of [41, Lemma 2.1] and the fourth follows from that fact that all chain morphisms from a totally acyclic complex to a bounded below complex of projective modules is null-homotopic. Here $\underline{\mathrm{Hom}}_A(-, -)$ means the morphism spaces in the stable category $A\text{-}\underline{\mathrm{Mod}}$ of $A\text{-Mod}$ modulo projective modules. Then we are done with the claim by Lemma 3.1.14.

We have shown the first sequence. For the second on, we may assume that there is a short exact sequence $0 \to \mathbf{p}M \to P_2^{\bullet} \to P_1^{\bullet}[1] \to 0$ of complexes. Similar as above we are done by applying the functor $C^0(-)$ to this sequence. $\qquad\square$

We end this section with an immediate consequence of Beligiannis-Reiten's Theorem.

Corollary B.0.25. *Let M be an A-module. Then there exists a proper GP-resolution*

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow G \longrightarrow M \longrightarrow 0$$

such that each P^{-i} is projective and G is Gorenstein-projective.

Proof. We apply Beligiannis-Reiten's Theorem and Lemma 3.1.14. $\qquad\square$

Appendix C

Open Problems

In this section we will list some open problems in Gorenstein homological algebra of artin algebras. They are mainly on CM-finite artin algebras.

Let A be an artin algebra. Recall that A is CM-finite provided that up to isomorphism there are only finitely many indecomposable finitely generated Gorenstein-projective A-modules. Observe that algebras of finite representation type are CM-finite. Recall that a remarkable result due to Auslander states that an artin algebra A is of finite representation type if and only if every (not necessarily finitely generated) A-module is a direct sum of finitely generated ones.

The following analogue of Auslander's result for Gorenstein-projective modules is asked in [20].

Problem A. *Is it true that an artin algebra A is CM-finite if and only if every Gorenstein-projective A-module is a direct sum of finitely generated ones?*

An affirmative answer is given for the case where A is Gorenstein ([20]). Recall that an artin algebra A is virtually Gorenstein if $(A\text{-GProj})^{\perp} = {}^{\perp}(A\text{-GInj})$ and that Gorenstein algebras are virtually Gorenstein. In fact, an affirmative answer to Problem A is given even for the case where A is virtually Gorenstein ([13]).

Based on the results in [14], Problem A is equivalent to the following one.

Problem B. *Is it true that a CM-finite artin algebra A is virtually Gorenstein?*

Let us remark that an affirmative answer to Problem B is given in [12, Example 8.4(2)], while the argument there is incorrect[1]. Hence Problem B stays open at present.

Recall that an artin algebra A is CM-free provided that A-Gproj $= A$-proj. Closedly

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[1]I would like to thank Professor Apostolos Beligiannis for a private communication concerning this remark. The argument in the second to last sentence in [12, Example 8.4(2)] is incomplete.

related to Problem A is the following.

Problem C. *For a CM-free artin algebra A, do we have A-GProj = A-Proj?*

For a CM-finite artin algebra A, take G to be an additive generator of A-Gproj. We call the algebra $\Gamma = \mathrm{End}_A(G)$ the *CM-Auslander* algebra of A. Recall that there is, up to Morita equivalence, a one-to-one correspondence between algebras of finite representation type and algebras having global dimension at most 2 and dominant dimension at least 2; this correspondence is called the *Auslander correspondence*.

Problem D. *What kinds of artin algebras is the CM-Auslander algebra of a CM-finite artin algebra? Is there an analogue of Auslander correspondence relating CM-finite artin algebras with their CM-Auslander algebras?*

We call that an artin algebra A is *CM-bounded* provided that the dimensions of all indecomposable finitely generated Gorenstein-projective A-modules are uniformly bounded. Recall that a famous theorem due to Roiter states that an artin algebra A is of finite representation type if the dimensions of all indecomposable finitely generated A-modules are uniformly bounded; see [58].

The following question then is natural.

Problem E. *Is a CM-bounded artin algebra necessarily CM-finite?*

An affirmative answer to this problem is known in the case where A is a 1-Gorenstein algebra.

Recall that the stable category A-Gproj of A-Gproj modulo projective modules is a triangulated category. However the information carried by this category is not clear yet[2].

Problem F. *What is the Grothendieck group $K_0(A\text{-}\underline{\mathrm{Gproj}})$? What about other invariants of the algebra A given by A-Gproj?*

Recall that $\mathbf{D}_{\mathrm{GP}}(A)$ is the Gorenstein-projective derived category of A. In our point of view, the properties and the structure of this category are far from clear.

Problem G. *Does $\mathbf{D}_{\mathrm{GP}}(A)$ always have arbitrary coproducts? For what kinds of algebras A the category $\mathbf{D}_{\mathrm{GP}}(A)$ is compactly generated?*

Note that for a self-injective algebra A we have $\mathbf{D}_{\mathrm{GP}}(A) = \mathbf{K}(A\text{-Mod})$; in this case, $\mathbf{D}_{\mathrm{GP}}(A)$ is compactly generated if and only if A is of finite representation type ([61,

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[2]I would like to thank Dr. Guodong Zhou for discussions on this problem.

Proposition 2.6]). Moreover for a CM-finite Gorenstein algebra A, by combining the results in [20] and [22] one infers that the category $\mathbf{D}_{\mathrm{GP}}(A)$ is compactly generated.

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